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ON A SUBCLASS OF p-HARMONIC MAPPINGS

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Abstract. The purpose of the present paper is to introduce two new classes $HS_p(\alpha)$ and $HC_p(\alpha)$ of p-harmonic mappings together with their corresponding subclasses $HS_p^0(\alpha)$ and $HC_p^0(\alpha)$. We prove that the mappings in $HS_p(\alpha)$ and $HC_p(\alpha)$ are univalent and sense-preserving in U and obtain extreme points of $HS_p^0(\alpha)$ and $HC_p^0(\alpha)$, $HS_p(\alpha)\cap T_p$ and $HC_p(\alpha)\cap T_p$ are determined, where T_p denotes the set of p-harmonic mapping with non negative coefficients. Finally, we establish the existence of the neighborhoods of mappings in $HC_p(\alpha)$. Relevant connections of the results presented here with various known results are briefly indicated.

1. Introduction

A $p(\geq 1)$ times continuously differentiable complex-valued function $F = u + iv$ in a domain $D \subseteq C$ is p-harmonic if F satisfies the p-harmonic equation $\underbrace{\Delta\ldots\Delta}_{p} F = 0$, where Δ represents the complex Laplacian operator

$$\Delta = \frac{4\partial^2}{\partial z\partial\overline{z}} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

A mapping F is p-harmonic in a simply connected domain D if and only if F has the following representation

$$F(z) = \sum_{k=1}^{p} |z|^{2(k-1)} G_{p-k+1}(z),$$

where each $G_{p-k+1}(z)$ is harmonic, i.e. $\Delta G_{p-k+1}(z) = 0$ for $k \in \{1,2,\ldots,p\}$ (cf. [8, Proposition 2.1]).

It should be noted that, if we take $p = 1$ and $p = 2$, then F is harmonic and biharmonic, respectively.

The properties of harmonic, biharmonic and p-harmonic mappings have been investigated by many researchers (see [1], [2], [3], [4], [5], [7], [9], [10], [13], [17], [18], [19], [27]).

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Let $U_r = \{z \in C : |z| < r\} (r > 0)$. In particular, we use U to denote the unit disc U_1.

In 2002, Öztürk and Yalcin [22] introduced and studied two new subclasses $HS(\alpha)$ and $HC(\alpha)$ of harmonic univalent mappings. The class $HS(\alpha)$ denote the function of the form

$$\begin{aligned} f(z) &= h(z) + \overline{g(z)} \\ &= z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b_n z^n} \end{aligned}$$

that satisfy the condition

$$\sum_{n=2}^{\infty} (n-\alpha)(|a_n| + |b_n|) \leq (1-\alpha)(1-|b_1|) \qquad (0 \leq \alpha < 1,\ 0 \leq |b_1| < 1)$$

and $HC(\alpha)$ the class of all mappings in $HS(\alpha)$ subject to the condition

$$\sum_{n=2}^{\infty} n(n-\alpha)(|a_n| + |b_n|) \leq (1-\alpha)(1-|b_1|) \qquad (0 \leq \alpha < 1,\ 0 \leq |b_1| < 1).$$

The corresponding subclasses of $HS(\alpha)$ and $HC(\alpha)$ with $b_1 = 0$ are denoted by $HS^0(\alpha)$ and $HC^0(\alpha)$, respectively. They proved that the image domains of U under the mappings in $HS^0(\alpha)$ and $HC^0(\alpha)$ are starlike and convex. The results for these subclasses were improved and generalized by Dixit and Porwal in [11], (see also [23]).

For $\alpha = 0$, the classes $HS(\alpha)$, $HC(\alpha)$, $HS^0(\alpha)$ and $HC^0(\alpha)$ are reduced to HS, HC, HS^0 and HC^0, respectively.

2. Preliminaries

Suppose F is a p-harmonic mapping with the following expression

$$F(z) = \sum_{k=1}^{p} |z|^{2(k-1)} G_{p-k+1}(z), \tag{2.1}$$

where for each $k \in \{1, 2, \dots p\}$, the harmonic mapping G_{p-k+1} has the expression

$$G_{p-k+1} = h_{p-k+1} + \overline{g_{p-k+1}},$$

where both h_{p-k+1} and g_{p-k+1} are analytic and satisfy the following conditions

$$h_{p-k+1}(z) = \sum_{j=1}^{\infty} a_{j,p-k+1} z^j \qquad \text{with } a_{1,p} = 1$$

and

$$g_{p-k+1}(z) = \sum_{j=1}^{\infty} b_{j,p-k+1} z^j.$$

We use J_F to denote the Jacobian of F, that is

$$J_F = |F_z|^2 - |F_{\overline{z}}|^2.$$

Then it is known that F is sense-preserving and locally univalent if $J_F > 0$.

Denote by $HS_p(\alpha)$ the class of all mappings of the form (2.1) satisfying the condition

$$\sum_{k=1}^{p}\sum_{j=2}^{\infty}(2(k-1)+j-\alpha)(|a_{j,p-k+1}|+|b_{j,p-k+1}|) \leq 1-|b_{1,p}|-\sum_{k=2}^{p}(2k-1)(|a_{1,p-k+1}|+|b_{1,p-k+1}|)$$

with

$$0 \leq |b_{1,p}|+\sum_{k=2}^{p}(2k-1)(|a_{1,p-k+1}|+|b_{1,p-k+1}|) < 1 \tag{2.2}$$

and the subclass

$HS_p^0(\alpha) = \{F \in HS_p(\alpha) : b_{1,p} = 0 \text{ and for } k \in \{2,3,\dots,p\}, a_{1,p-k+1} = b_{1,p-k+1} = 0\}$.

Denote by $HC_p(\alpha)$ the class of p-harmonic mappings F subject to the condition

$$\sum_{k=1}^{p}\sum_{j=2}^{\infty}\left(2(k-1)+\frac{j(j-\alpha)}{1-\alpha}\right)(|a_{j,p-k+1}|+|b_{j,p-k+1}|) \leq 1-|b_1,p|-\sum_{k=2}^{p}(2k-1)(|a_{1,p-k+1}|+|b_{1,p-k+1}|)$$

with

$$0 \leq |b_1,p|+\sum_{k=2}^{p}(2k-1)(|a_{1,p-k+1}|+|b_{1,p-k+1}|) < 1. \tag{2.3}$$

The corresponding subclass of $HC_p(\alpha)$ with $b_{1,p} = a_{1,p-k+1} = b_{1,p-k+1} = 0$ for all $k \in \{2,3,\dots,p\}$ is denoted by $HC_p^0(\alpha)$.

(i) For $\alpha = 0$, the classes $HS_p(\alpha)$ and $HC_p(\alpha)$ reduce to the classes HS_p and HC_p studied by Qiao and Wang [24].

(ii) For $p = 1$, the classes $HS_p(\alpha) \equiv HS(\alpha)$ and $HC_p(\alpha) \equiv HC(\alpha)$ were studied by Öztürk and Yalcin [22].

(iii) For $p = 1$, $\alpha = 0$, the classes $HS_p(\alpha) \equiv HS$ and $HC_p(\alpha) \equiv HC$ were studied by Avci and Zlotkiewicz [6].

Suppose F is a p-harmonic mapping with the expression (2.1). Following Ruscheweyh [25], we use $N_\delta^\alpha(F)$ to denote the δ-neighborhood of F in p-harmonic mappings, i.e.,

$$N_\delta^\alpha(F) = \Bigg\{F^* : |b_{1,p}-B_{1,p}|+\sum_{k=2}^{p}(2k-1)(|a_{1,p-k+1}-A_{1,p-k+1}|+|b_{1,p-k+1}-B_{1,p-k+1}|)$$

$$+\sum_{k=1}^{p}\sum_{j=2}^{\infty}\left(2(k-1)+\frac{j-\alpha}{1-\alpha}\right)(|a_{j,p-k+1}-A_{j,p-k+1}|+|b_{j,p-k+1}-B_{j,p-k+1}|)\le\delta\Big\}$$

where

$$F^*(z)=z+\sum_{j=2}^{\infty}A_{j,p}z^j+\sum_{j=1}^{\infty}\overline{B}_{j,p}\overline{z}^j+\sum_{k=2}^{p}|z|^{2(k-1)}\left(\sum_{j=1}^{\infty}A_{j,p-k+1}z^j+\sum_{j=1}^{\infty}\overline{B}_{j,p-k+1}\overline{z}^j\right).$$

3. Main Results

First we discuss that the mappings in $HS_p(\alpha)$ and $HC_p(\alpha)$ are univalent and sense-preserving.

Theorem 3.1. *Each mapping in $HS_p(\alpha)$ is univalent and sense-preserving.*

Proof. Let $F\in HS_p(\alpha)$ and $z_1\ne z_2\in U$ with $|z_1|\le|z_2|$.

Then

$$\begin{aligned}|F(z_1)-F(z_2)| &= |\sum_{k=1}^{p}(|z_1|^{2(k-1)}G_{p-k+1}(z_1)-|z_2|^{2(k-1)}G_{p-k+1}(z_2))|\\
&\ge |z_1-z_2|\Big\{1-\left|\sum_{j=2}^{\infty}a_{j,p}\frac{z_1^j-z_2^j}{z_1-z_2}+\sum_{j=1}^{\infty}\overline{b}_{j,p}\frac{\overline{z}_1^j-\overline{z}_2^j}{z_1-z_2}\right|\\
&\quad-\Big|\sum_{k=2}^{p}\Big(\sum_{j=1}^{\infty}a_{j,p-k+1}\frac{|z_1|^{2(k-1)}z_1^j-|z_2|^{2(k-1)}z_2^j}{z_1-z_2}\\
&\quad+\sum_{j=1}^{\infty}\overline{b}_{j,p-k+1}\frac{|z_1|^{2(k-1)}\overline{z}_1^j-|z_2|^{2(k-1)}\overline{z}_2^j}{z_1-z_2}\Big)\Big|\Big\}\\
&\ge |z_1-z_2|(1-|b_{1,p}|-|z_2|\sum_{j=2}^{\infty}j(|a_{j,p}|+|b_{j,p}|)\\
&\quad-|z_2|\sum_{k=2}^{p}\sum_{j=1}^{\infty}(2(k-1)+j)(|a_{j,p-k+1}|+|b_{j,p-k+1}|)\\
&\ge |z_1-z_2|(1-|b_{1,p}|-|z_2|\sum_{j=2}^{\infty}\frac{j-\alpha}{1-\alpha}(|a_{j,p}|+|b_{j,p}|)\\
&\quad-|z_2|\sum_{k=2}^{p}\sum_{j=1}^{\infty}\left(2(k-1)+\frac{j-\alpha}{1-\alpha}\right)(|a_{j,p-k+1}|+|b_{j,p-k+1}|))\\
&\ge |z_1-z_2|(1-|b_{1,p}|)(1-|z_2|)\\
&>0.\end{aligned}$$

Hence each mapping in $HS_p(\alpha)$ is univalent.

The sense-preserving property of elements in $HS_p(\alpha)$ easily follows from the following chain of inequalities about the Jacobian of F:

$$
\begin{aligned}
J_F(z) &= |F_z(z)|^2 - |F_{\overline{z}}(z)|^2 \\
&= (|F_z(z)| + |F_{\overline{z}}(z)|)(|F_z(z)| - |F_{\overline{z}}(z)|) \\
&= (|F_z(z)| + |F_{\overline{z}}(z)|)\Bigg[\Bigg|1 + \sum_{j=2}^{\infty} j a_{j,p} z^{j-1} + \sum_{k=2}^{p}\sum_{j=2}^{\infty} |z|^{2(k-1)} j a_{j,p-k+1} z^{j-1} \\
&\quad + \sum_{k=2}^{p} |z|^{2(k-1)} a_{1,p-k+1} + \sum_{k=2}^{p} (k-1)|z|^{2(k-1)} \left(\sum_{j=1}^{\infty} a_{j,p-k+1} z^{j-1} + \frac{\overline{z}}{z}\sum_{j=1}^{\infty} \overline{b}_{j,p-k+1}\overline{z}^{j-1}\right)\Bigg| \\
&\quad - \Bigg|\sum_{j=1}^{\infty} j\overline{b}_{j,p}\overline{z}^{j-1} + \sum_{k=2}^{p}\sum_{j=2}^{\infty} |z|^{2(k-1)} j\overline{b}_{j,p-k+1}\overline{z}^{j-1} \\
&\quad + \sum_{k=2}^{p} |z|^{2(k-1)}\overline{b}_{1,p-k+1} + \sum_{k=2}^{p} (k-1)|z|^{2(k-1)} \left(\frac{z}{\overline{z}}\sum_{j=1}^{\infty} a_{j,p-k+1} z^{j-1} + \sum_{j=1}^{\infty} \overline{b}_{j,p-k+1}\overline{z}^{j-1}\right)\Bigg|\Bigg] \\
&\geq (|F_z(z)| + |F_{\overline{z}}(z)|)[1 - |b_{1,p}| - \sum_{k=2}^{p} (2k-1)(|a_{1,p-k+1}| + |b_{1,p-k+1}|) \\
&\quad - |z| \sum_{k=1}^{p}\sum_{j=2}^{\infty} (2(k-1) + j)(|a_{j,p-k+1}| + |b_{j,p-k+1}|)] \\
&\geq (|F_z(z) + F_{\overline{z}})(1 - |b_{1,p}|)(1 - |z|) \\
&\geq (|F_z(z)| + |F_{\overline{z}}(z)|)[1 - |b_{1,p}| - \sum_{k=2}^{p} (2k-1)(|a_{1,p-k+1}| + |b_{1,p-k+1}|) \\
&\quad - |z| \sum_{k=1}^{p}\sum_{j=2}^{\infty} \left(2(k-1) + \frac{j-\alpha}{1-\alpha}\right)(|a_{j,p-k+1}| + |b_{j,p-k+1}|)] \\
&\geq (|F_z(z)| + |F_{\overline{z}}(z)|)(1 - |b_{1,p}|)(1 - |z|) \\
&> 0
\end{aligned}
$$

for $z \neq 0$ and the obvious fact

$$J_F(0) = 1 - |b_{1,p}|^2 > 0.$$

Thus the proof of Theorem 3.1 is established. □

Next, we discuss the geometric properties of mappings belonging to $HS_p^0(\alpha)$ and $HC_p^0(\alpha)$, respectively.

Theorem 3.2. *Each mapping in $HS_p^0(\alpha)$ maps U onto a domain starlike with respect to the origin.*

Proof. Let $r \in (0,1)$ be a fixed number and

$$
\begin{aligned}
F_r(z) &= \sum_{k=1}^{p} r^{2(k-1)} G_{p-k+1}(z) \\
&= z + \sum_{j=2}^{\infty}\left(\sum_{k=1}^{p} r^{2(k-1)} a_{j,p-k+1}\right) z^j + \sum_{j=2}^{\infty}\left(\sum_{k=1}^{p} r^{2(k-1)}\overline{b}_{j,p-k+1}\right)\overline{z}^j.
\end{aligned}
$$

Obviously, F_r is a harmonic mapping. Since

$$\begin{aligned}
&\sum_{j=2}^{\infty} j\left|\sum_{k=1}^{p} r^{2(k-1)} a_{j,p-k+1}\right| + \sum_{j=2}^{\infty} j\left|\sum_{k=1}^{p} r^{2(k-1)} b_{j,p-k+1}\right| \\
&\leq \sum_{j=2}^{\infty}\sum_{k=1}^{p} j(|a_{j,p-k+1}|+|b_{j,p-k+1}|) \\
&\leq \sum_{j=2}^{\infty}\sum_{k=1}^{p}\left(2(k-1)+\frac{j-\alpha}{1-\alpha}\right)(|a_{j,p-k+1}|+|b_{j,p-k+1}|) \\
&\leq 1,
\end{aligned}$$

it follows that $F_r \in HS^0$. By [[6], Theorem 2], we know that F_r maps U onto a domain starlike with respect to the origin. That is, for each $r_1 \in (0,1)$,

$$\frac{\partial}{\partial\theta}\arg F_r(r_1 e^{i\theta}) > 0$$

for $0 \leq \theta < 2\pi$. Letting $r_1 = r$ yields

$$\frac{\partial}{\partial\theta}\arg F_r(r_1 e^{i\theta}) > 0$$

for $0 \leq \theta < 2\pi$. That fact

$$\frac{\partial}{\partial\theta}\arg F_r(r_1 e^{i\theta}) = \frac{\partial}{\partial\theta}\arg F(re^{i\theta})$$

show that F is starlike with respect to the origin. □

Theorem 3.3. *Each mapping in $HC_p^0(\alpha)$ maps U_r $(r \in (0,1))$ onto a convex domain.*

Proof. The proof of above theorem is similar to Theorem 3.2. So we omit details involved. □

Next we determine the extreme points of $HS_p^0(\alpha)$ and $HC_p^0(\alpha)$, respectively.

Theorem 3.4. *The extreme points of $HS_p^0(\alpha)$ are the mappings with the following forms*

$$\begin{aligned}
F_k(z) &= z + |z|^{2(k-1)} a_{n,p-k+1} z^n \quad \text{or} \\
F_k^*(z) &= z + |z|^{2(k-1)} \overline{b}_{m,p-k+1} \overline{z}^m,
\end{aligned}$$

where

$$k \in \{1,2,\ldots,p\}, |a_{n,p-k+1}| = \frac{1}{2(k-1)+\frac{n-\alpha}{1-\alpha}} \qquad (n \geq 2)$$

and

$$|b_{m,p-k+1}| = \frac{1}{2(k-1)+\frac{m-\alpha}{1-\alpha}} \qquad (m \geq 2).$$

Proof. Assume that F is an extreme point of $HS_p^0(\alpha)$ and let

$$\begin{aligned} F(z) &= \sum_{k=1}^{p} |z|^{2(k-1)} G_{p-k+1}(z) \\ &= z + \sum_{k=1}^{p} |z|^{2(k-1)} \left(\sum_{j=2}^{\infty} a_{j,p-k+1} z^j + \sum_{j=2}^{\infty} \overline{b}_{j,p-k+1} \overline{z}^j \right). \end{aligned}$$

Obviously, the coefficients of F satisfy the following equality

$$\sum_{k=1}^{p} \sum_{j=2}^{\infty} \left(2(k-1) + \frac{j-\alpha}{1-\alpha} \right) (|a_{j,p-k+1}| + |b_{j,p-k+1}|) = 1.$$

we claim that there exist at most one coefficient $a_{q_1,p-k+1}$ or $b_{q_2,p-k+1}$ for some $k \geq 2$ of F which is not 0.

We prove this claim by contradiction. Suppose that there exist some $k_1 \geq 2$ and $k_2 \geq 2$ such that $a_{q_1,p-k_1+1} \neq 0$ and $b_{q_2,p-k_2+1} \neq 0$ or $a_{q_1,p-k_1+1} \neq 0$ and $a_{q_2,p-k_2+1} \neq 0$ or $b_{q_1,p-k_1+1} \neq 0$ and $b_{q_2,p-k_2+1} \neq 0$. Without loss of generality, we assume the first case, i.e., both $a_{q_1,p-k_1+1}$ and $b_{q_2,p-k_2+1}$ are not 0 for some $k_1 \geq 2$ and $k_2 \geq 2$.

Choosing $\lambda > 0$ small enough and x, y with $|x| = |y| = 1$ properly, leaving all coefficients of F but $a_{q_1,p-k_1+1}, b_{q_2,p-k_2+1}$ by $a_{q_1,p-k_1+1} + \frac{\lambda x}{2(k_1-1) + \frac{q_1-\alpha}{1-\alpha}}$ and $b_{q_2,p-k_2+1} - \frac{\lambda y}{2(k_2-1) + \frac{q_2-\alpha}{1-\alpha}}$ or $a_{q_1,p-k_1+1} - \frac{\lambda x}{2(k_1-1)+\frac{q_1-\alpha}{1-\alpha}}$ and $b_{q_2,p-k_2+1} + \frac{\lambda y}{2(k_2-1)+\frac{q_2-\alpha}{1-\alpha}}$ respectively, we obtain two mappings F_1 and F_2.

Obviously, F_1 and $F_2 \in HS_p^0(\alpha)$ and $F = \frac{1}{2}(F_1 + F_2)$. This is the desired contradiction. Our claim is proved. Therefore any extreme point $F \in HS_p^0(\alpha)$ must have the form

$$F_k(z) = z + |z|^{2(k-1)} a_{n,p-k+1} z^n$$

or

$$F_k^*(z) = z + |z|^{2(k-1)} \overline{b}_{m,p-k+1} \overline{z}^m$$

with

$$|a_{n,p-k+1}| = \frac{1}{2(k-1) + \frac{n-\alpha}{1-\alpha}} (n \geq 2)$$

and

$$|b_{m,p-k+1}| = \frac{1}{2(k-1) + \frac{m-\alpha}{1-\alpha}} (m \geq 2).$$

Now we come to prove that for any $F \in HS_p^0(\alpha)$ with the above form must be an extreme point of $HS_p^0(\alpha)$. It suffices to prove the case of F_k since the proof for the case of F_k^* is similar.

Suppose there exist two functions F_3 and $F_4 \in HS_p^0(\alpha)$ such that $F_k = tF_3 + (1-t)F_4$ with some $t \in (0,1)$. For $q = 3,4$, let

$$F_q(z) = z + \sum_{k=1}^{p} |z|^{2(k-1)} \left(\sum_{j=2}^{\infty} a_{j,p-k+1}^{(q)} z^j + \sum_{j=2}^{\infty} \overline{b}_{j,p-k+1}^{(q)} \overline{z}^j \right).$$

Then

$$\begin{aligned} |ta_{n,p-k+1}^{(3)} + (1-t)a_{n,p-k+1}^{(4)}| &= |a_{n,p-k+1}| \\ &= \frac{1}{2(k-1)+n}. \end{aligned}$$

Since all coefficients of $F_q (q = 3,4)$ satisfy

$$|a_{j,p-k+1}^{(q)}| \le \frac{1}{2(k_0-1) + \frac{j-\alpha}{1-\alpha}} \text{ and } |b_{j,p-k+1}^{(q)}| \le \frac{1}{2(k_0-1) + \frac{j-\alpha}{1-\alpha}},$$

where $j \ge 2$ and $k_0 \in \{1,2,\dots,p\}$, (4.5) implies

$$a_{n,p-k+1}^{(3)} = a_{n,p-k+1}^{(4)}$$

all all other coefficients of F_3, F_4 are 0. Thus $F_k = F_3 = F_4$, which shows that F_k is an extreme point of $HS_p^0(\alpha)$. □

Theorem 3.5. *The set of extreme points of $HC_p^0(\alpha)$ consists of mappings with the forms*

$$F_k(z) = z + |z|^{2(k-1)} a_{n,p-k+1} z^n$$

or

$$F_k^*(z) = z + |z|^{2(k-1)} \overline{b}_{m,p-k+1} \overline{z}^m$$

where $k = 1,2,\dots,p$,

$$|a_{n,p-k+1}| = \frac{1}{2(k-1) + \frac{n(n-\alpha)}{1-\alpha}} \qquad (n \ge 2)$$

and

$$|b_{m,p-k+1}| = \frac{1}{2(k-1) + \frac{m(m-\alpha)}{1-\alpha}} \qquad (m \ge 2)$$

.

Proof. The proof of this theorem is much akin to that of Theorem 3.4. Therefore we omit details. □

The classes of analytic and harmonic functions with nonnegative (or negative) coefficients possess many interesting properties and many references have been in literature, see, for example, [[12], [15], [16], [20], [21], [26]], (see also [14]). In the following, we consider the p-harmonic mappings with non negative coefficients. Let

$$T_p = \{F : F(z) = \sum_{k=1}^{p} |z|^{2(k-1)} \sum_{j=1}^{\infty} (a_{j,p-k+1} z^j + \overline{b}_{j,p-k+1} \overline{z}^j)$$

$$\text{with } a_{1,p} = 1, a_{j,p-k+1} \geq 0, b_{j,p-k+1} \geq 0 \text{ for } j \geq 1, k = 1,2,\ldots,p\}.$$

Theorem 3.6. *Suppose F is p-harmonic in U. Then $F \in HS_p(\alpha) \cap T_p$ if and only if*

$$F(z) = \sum_{k=1}^{p} \sum_{j=1}^{\infty} (X_{kj} h_{kj}(z) + Y_{kj} g_{kj}(z)),$$

where

$$h_{kj}(z) = z + |z|^{2(k-1)} \frac{z^j}{2(k-1) + \frac{j-\alpha}{1-\alpha}} (2 \leq k \leq p, j \geq 1),$$

$$g_{kj}(z) = z + |z|^{2(k-1)} \frac{\overline{z}^j}{2(k-1) + \frac{j-\alpha}{1-\alpha}} (2 \leq k \leq p, j \geq 1),$$

$$h_{11}(z) = z,\ h_{1j}(z) = z + \frac{z^j}{\frac{j-\alpha}{1-\alpha}} (j \geq 2),$$

$$g_{1j}(z) = z + \frac{\overline{z}^j}{\frac{j-\alpha}{1-\alpha}} (j \geq 1),$$

and

$$\sum_{k=1}^{p} \sum_{j=1}^{\infty} (X_{kj} + Y_{kj}) = 1, (X_{kj} \geq 0,\ Y_{kj} \geq 0).$$

In particular, the extreme points of $HS_p(\alpha) \cap T_p$ are $\{h_{kj}\}$ and $\{g_{kj}\}$.

Proof. Since

$$\begin{aligned} F(z) &= \sum_{k=1}^{p} \sum_{j=1}^{\infty} (X_{kj} h_{kj}(z) + Y_{kj} g_{kj}(z)) \\ &= z + \sum_{k=2}^{p} |z|^{2(k-1)} \sum_{j=1}^{\infty} \left(\frac{X_k}{2(k-1) + \frac{j-\alpha}{1-\alpha}} z^j + \frac{Y_{kj}}{2(k-1) + \frac{j-\alpha}{1-\alpha}} \overline{z}^j \right) + \sum_{j=2}^{\infty} \frac{X_{1j}}{\frac{j-\alpha}{1-\alpha}} z^j + \sum_{j=1}^{\infty} \frac{Y_{1j}}{\frac{j-\alpha}{1-\alpha}} \overline{z}^j \end{aligned}$$

and

$$\begin{aligned} &\sum_{k=1}^{p} \sum_{j=2}^{\infty} \left(2(k-1) + \frac{j-\alpha}{1-\alpha} \right) \left(\left| \frac{X_{kj}}{2(k-1) + \frac{j-\alpha}{1-\alpha}} \right| + \left| \frac{Y_{kj}}{2(k-1) + \frac{j-\alpha}{1-\alpha}} \right| \right) \\ &\quad + |Y_{11}| + \sum_{k=2}^{p} (2k-1) \left(\left| \frac{X_{k1}}{2k-1} \right| + \left| \frac{Y_{k1}}{2k-1} \right| \right) \\ &\leq \sum_{k=1}^{p} \sum_{j=2}^{\infty} (X_{kj} + Y_{kj}) + \sum_{k=2}^{p} (X_{k1} + Y_{k1}) + Y_{11} \end{aligned}$$

$$\le 1 - X_{11}$$
$$\le 1,$$

we see that $F \in HS_p(\alpha)$.

Conversely, assuming that $F \in HS_p(\alpha) \cap T_p$ and setting

$$\begin{aligned}
X_{kj} &= \left(2(k-1) + \frac{j-\alpha}{1-\alpha}\right) a_{j,p-k+1} \ (2 \le k \le p, j \ge 1),\\
X_{1j} &= \frac{j-\alpha}{1-\alpha} a_{j,p} \quad (j \ge 2),\\
Y_{kj} &= \left(2(k-1) + \frac{j-\alpha}{1-\alpha}\right) b_{j,p-k+1} \ (1 \le k \le p, j \ge 1),
\end{aligned}$$

and

$$X_{11} = 1 - \sum_{k=1}^{p} \sum_{j=2}^{\infty} (X_{kj} + Y_{kj}) - \sum_{k=2}^{p} (X_{k1} + Y_{k1}) - Y_{11},$$

we obtain

$$F(z) = \sum_{k=1}^{p} \sum_{j=1}^{\infty} (X_{kj} h_{kj}(z) + Y_{kj} g_{kj}(z)),$$

The proof is complete. □

Theorem 3.7. *Suppose F is p-harmonic in U. Then $F \in HC_p(\alpha) \cap T_p$ if and only if*

$$F(z) = \sum_{k=1}^{p} \sum_{j=1}^{\infty} (X_{kj} h_{kj}(z) + Y_{kj} g_{kj}(z)),$$

where

$$\begin{aligned}
h_{kj}(z) &= z + |z|^{2(k-1)} \frac{z^j}{2(k-1) + \frac{j(j-\alpha)}{1-\alpha}} (2 \le k \le p, j \ge 1),\\
g_{kj}(z) &= z + |z|^{2(k-1)} \frac{\overline{z}^j}{2(k-1) + \frac{j(j-\alpha)}{1-\alpha}} (2 \le k \le p, j \ge 1),\\
h_{11}(z) &= z,\ h_{1j}(z) = z + \frac{z^j}{\frac{j(j-\alpha)}{1-\alpha}} (j \ge 2),\\
g_{1j}(z) &= z + \frac{\overline{z}^j}{\frac{j(j-\alpha)}{1-\alpha}} (j \ge 1),
\end{aligned}$$

and

$$\sum_{k=1}^{p} \sum_{j=1}^{\infty} (X_{kj} + Y_{kj}) = 1, (X_{kj} \ge 0,\ Y_{kj} \ge 0).$$

In particular, the extreme points of $HC_p(\alpha) \cap T_p$ are $\{h_{kj}\}$ and $\{g_{kj}\}$.

Proof. The proof of above theorem is similar to that of Theorem 3.6. So we omit details involved. □

4. Neighborhoods

Theorem 3.1. *Assume that*

$$F_1(z) = z + \sum_{j=2}^{\infty} a_{j,p} z^j + \sum_{j=1}^{\infty} \overline{b}_{j,p} \overline{z}^j + \sum_{k=2}^{p} |z|^{2(k-1)} \left(\sum_{j=1}^{\infty} a_{j,p-k+1} z^j + \sum_{j=1}^{\infty} \overline{b}_{j,p-k+1} \overline{z}^j \right)$$

belongs to $HC_p(\alpha)$. *If*

$$\delta \le (1-c_0)(1-|b_{1,p}| - \sum_{k=2}^{p} (2k-1)(|a_{1,p-k+1}| + |b_{1,p-k+1}|),$$

then $N_\delta^\alpha(F_1) \subset HS_p(\alpha)$, *where*

$$c_0 = \frac{2(p-1)(1-\alpha) + (2-\alpha)}{2[(p-1)(1-\alpha) + (2-\alpha)]}.$$

Proof. The δ-neighborhood of F_1 is the set

$$\begin{aligned} N_\delta^\alpha(F_1) = \Big\{ F_2 : \sum_{k=1}^{p} \sum_{j=2}^{\infty} \left(2(k-1) + \frac{j-\alpha}{1-\alpha} \right) (|a_{j,p-k+1} - A_{j,p-k+1}| \\ + |b_{j,p-k+1} - B_{j,p-k+1}|) + |b_{1,p} + B_{1,p}| + \sum_{k=2}^{p} (2k-1)(|a_{1,p-k+1} - A_{1,p-k+1}| \\ + |b_{1,p-k+1} - B_{1,p-k+1}|) \le \delta \Big\}, \end{aligned}$$

where

$$F_2(z) = z + \sum_{j=2}^{\infty} A_{j,p} z^j + \sum_{j=1}^{\infty} \overline{B}_{j,p} \overline{z}^j + \sum_{k=2}^{\infty} |z|^{2(k-1)} \left(\sum_{j=1}^{\infty} A_{j,p-k+1} z^j + \sum_{j=1}^{\infty} \overline{B}_{j,p-k+1} \overline{z}^j \right).$$

If

$$\delta \le (1-c_0)(1-|b_{1,p}| - \sum_{k=2}^{p} (2k-1)(|a_{1,p-k+1}| + |b_{1,p-k+1}|),$$

then we have

$$\begin{aligned} \sum_{j=2}^{\infty} \frac{j-\alpha}{1-\alpha} |A_{j,p}| + \sum_{j=1}^{\infty} \frac{j-\alpha}{1-\alpha} |B_{j,p}| + \sum_{k=2}^{p} \sum_{j=2}^{\infty} \left(2(k-1) + \frac{j-\alpha}{1-\alpha} \right) (|A_{j,p-k+1}| + |B_{j,p-k+1}|) \\ \le \sum_{k=2}^{p} (2k-1)(|a_{1,p-k+1} - A_{1,p-k+1}| + |b_{1,p-k+1} - B_{1,p-k+1}| + |b_{1,p} - B_{1,p}|) \\ + \sum_{k=1}^{p} \sum_{j=2}^{\infty} \left(2(k-1) + \frac{j-\alpha}{1-\alpha} \right) (|a_{j,p-k+1} - A_{j,p-k+1}| + |b_{j,p-k+1} - B_{j,p-k+1}|) \end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=2}^{p}(2k-1)(|a_{1,p-k+1}|+|b_{1,p-k+1}|)+|b_{1,p}| \\
&+\sum_{k=1}^{p}\sum_{j=2}^{\infty}\left(2(k-1)+\frac{j-\alpha}{1-\alpha}\right)(|a_{j,p-k+1}|+|b_{j,p-k+1}|) \\
\le\ & \delta+\sum_{k=2}^{p}(2k-1)(|a_{1,p-k+1}|+|b_{1,p-k+1}|)+|b_{1,p}| \\
&+c_0\sum_{k=1}^{p}\sum_{j=2}^{\infty}\left(2(k-1)+\frac{j-\alpha}{1-\alpha}\right)(|a_{j,p-k+1}|+|b_{j,p-k+1}|) \\
\le\ & \delta+c_0+(1-c_0)\left(\sum_{k=2}^{p}(2k-1)(|a_{1,p-k+1}|+|b_{1,p-k+1}|)+|b_{1,p}|\right) \\
\le\ & 1,
\end{aligned}
$$

whence $F_2 \in HS_p(\alpha)$. □

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