SUBCLASSES OF CLOSE-TO-CONVEX FUNCTIONS

HARJINDER SINGH AND B. S. MEHROK

Abstract. We introduce some subclasses of close-to-convex functions and obtain sharp results for coefficients, distortion theorems and argument theorems from which results of several authors follows as special cases.

1. Introduction and Definitions

Principle of Subordination ([9], [13]). Let \( f(z) \) and \( F(z) \) be two functions analytic in the open unit disc \( E = \{ z; |z| < 1 \} \). Then \( f(z) \) is subordinate to \( F(z) \) in \( E \) if there exists a function \( w(z) \) analytic in \( E \) and satisfying the conditions \( w(0) = 0 \) and \( |w(z)| < 1 \) such that \( f(z) = F(w(z)) \). If \( F(z) \) is univalent in \( E \), the above definition is equivalent to \( f(0) = F(0) \) and \( f(E) \subset F(E) \).

Bounded Functions. By \( \mathcal{U} \), we denote the class of analytic functions of the form

\[ w(z) = \sum_{n=1}^{\infty} c_n z^n, \quad z \in E, \tag{1.1} \]

which satisfy the conditions \( w(0) = 0 \) and \( |w(z)| < 1 \).

Let \( \mathcal{A} \) denote the class of functions of the form

\[ f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.2} \]

which are analytic in the unit disc \( E = \{ z; |z| < 1 \} \). The subclass of univalent functions in \( \mathcal{A} \) is denoted by \( S \).

\( S^* \) and \( C \) represent the classes of functions in \( \mathcal{A} \) which satisfy, respectively, the conditions

\[ \text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > 0, \tag{1.3} \]

Received December 26, 2011, accepted December 18, 2012.
Communicated by Chung-Tsun Shieh.
2010 Mathematics Subject Classification. Primary 30C45; Secondary 30C50.
Key words and phrases. Subordination, bounded functions, univalent functions, starlike functions, convex functions and close-to-convex functions.
Corresponding author: Harjinder Singh.
A function \( f(z) \) in \( \mathcal{A} \) is said to be close-to-convex if there exists a function

\[
g(z) = z + \sum_{n=2}^{\infty} b_n z^n
\]

in \( S^* \) such that

\[
\text{Re} \left\{ \frac{zf'(z)}{g(z)} \right\} > 0.
\]

The class of functions \( f(z) \) in \( \mathcal{A} \) with the condition (1.6) is denoted by \( K \) and called the class of close-to-convex functions. The class \( K \) was introduced by Kaplan [7] it was shown by him that all close-to-convex functions are univalent.

If \( g \in \mathcal{C} \), the class of functions in \( \mathcal{A} \) subject to the condition (1.6) may be denoted by \( K_1 \) which is the subclass of \( K \).

\( S^*(A,B) \) and \( C(A,B) \) are the classes of functions in \( \mathcal{A} \) which satisfy, respectively, the conditions

\[
\text{Re} \left\{ \frac{zf'(z)}{g(z)} \right\} < \frac{1 + Az}{1 + Bz}, \quad g \in S^*, \quad -1 \leq B < A \leq 1.
\]

(1.7)

\[
\text{Re} \left\{ \frac{(zf'(z))'}{g'(z)} \right\} < \frac{1 + Az}{1 + Bz}, \quad g \in C, \quad -1 \leq B < A \leq 1.
\]

(1.8)

In particular, \( S^*(1,-1) \equiv S^* \) and \( C(1,-1) \equiv C \).

The class \( S^*(A,B) \) was introduced and study by Janowski [6] and also by Goel and Mehrok [4]. It is obvious that \( g \in C(A,B) \) implies that \( zg'(z) \in S^*(A,B) \).

\( K(C,D) \) represent the class of functions \( f(z) \) in \( \mathcal{A} \) for which

\[
\text{Re} \left\{ \frac{zf'(z)}{g(z)} \right\} < \frac{1 + Cz}{1 + Dz}, \quad g \in S^*, \quad -1 \leq D < C \leq 1.
\]

(1.9)

If \( g \in \mathcal{C} \), the corresponding class may be denoted by \( K_1(C,D) \).

The class \( K^*(A,B) \) consists of functions \( f(z) \) in \( \mathcal{A} \) such that

\[
\text{Re} \left\{ \frac{zf'(z)}{g(z)} \right\} > 0, \quad g \in S^*(A,B), \quad -1 \leq B < A \leq 1.
\]

(1.10)

If \( g \in C(A,B) \), the corresponding class may be denoted by \( K_1^*(A,B) \).

For \( -1 \leq D \leq B < A \leq C \leq 1 \), let \( K(A,B;C,D) \) be the subclass of \( K \) satisfying

\[
\text{Re} \left\{ \frac{zf'(z)}{g(z)} \right\} < \frac{1 + Cz}{1 + Dz}, \quad g \in S^*(A,B).
\]

(1.11)

If \( g \in C(A,B) \), the corresponding class may be denoted by \( K_1(A,B;C,D) \).

Throughout the paper, we take \( -1 \leq D \leq B < A \leq C \leq 1 \), \( w(z) \in \mathcal{W} \) and \( z \in \mathcal{E} \). From the above definitions, we have the following observations.
(i) $K(1, -1; C, D) \equiv K(C, D)$ and $K_1(1, -1; C, D) \equiv K_1(C, D)$;
(ii) $K(A, B; 1, -1) \equiv K^*(A, B)$ and $K_1(A, B; 1, -1) \equiv K^*_1(A, B)$;
(iii) $K(1, -1; 1, -1) \equiv K$ and $K_1(1, -1; 1, -1) \equiv K_1$.

2. Preliminary lemmas

**Lemma 2.1** ([3]). Let $P(z) = \frac{1 + Cw(z)}{1 + Dw(z)} = 1 + \sum_{n=1}^{\infty} p_n z^n$, then

$$|p_n| \leq (C - D).$$

Result is sharp for the functions $P_n(z) = \frac{1 + C^n - K}{1 + D^n - K}$, $|\delta| = 1$ and $n \geq 1$.

**Lemma 2.2** ([4]). Let $g \in S^*(A, B)$, then, for $A - (n - 1)B \geq (n - 2)$, $(n \geq 3)$,

$$|b_n| \leq \frac{1}{(n - 1)!} \prod_{k=2}^{n} (A - (k - 1)B).$$

*Equality holds for the function $g_0(z)$ defined by*

$$g_0(z) = z(1 + B\delta)^{(A-B)/B}, \quad |\delta| = 1.$$

Since $g(z) \in C(A, B)$ implies that $zg'(z) \in S^*(A, B)$, we have the following

**Lemma 2.3.** Let $g \in C(A, B)$, then, for $A - (n - 1)B \geq (n - 2)$, $(n \geq 3)$,

$$|b_n| \leq \frac{1}{n!} \prod_{k=2}^{n} (A - (k - 1)B).$$

*Result is sharp for the function $g_1(z)$ defined by*

$$g_1(z) = (1 + B\delta)^{(A-B)/B}, \quad |\delta| = 1.$$

**Lemma 2.4** ([5]). Let $g \in S^*(A, B)$, then, for $|s| \leq 1$, $|t| \leq 1$, $(s \neq t)$

$$\frac{tg(sz)}{sg(tz)} = \begin{cases} (1 + Bs z)^{(A-B)/B}, & B \neq 0; \\ \exp A(s-t)z, & B = 0. \end{cases}$$

**Lemma 2.5.** If $g \in S^*(A, B)$, then, for $|z| = r < 1$,

$$r(1 - Br)^{(A-B)/B} \leq |g(z)| \leq r(1 + Br)^{(A-B)/B}, \quad B \neq 0; \quad (2.1)$$

$$r \exp(-Ar) \leq |g(z)| \leq r \exp(Ar), \quad B = 0; \quad (2.2)$$

$$\left| \arg \frac{g(z)}{z} \right| \leq \frac{(A-B)}{B} \sin^{-1}(Br), \quad B \neq 0; \quad (2.3)$$
Equality sign in these bounds is attained by the function $g_0(z)$ defined by

$$g_0(z) = \begin{cases} 
  z(1 + B \delta z)^{(A-B)/B}, & B \neq 0; \\
  z \exp(A \delta z), & B = 0, |\delta| = 1.
\end{cases}$$

**Proof.** Letting $s \to 1$ and $t \to 0$ in the Lemma 2.4, we obtain

$$\frac{g(z)}{z} < (1 + B z)^{(A-B)/B}, \quad B \neq 0; \quad (2.5)$$

$$\frac{g(z)}{z} < \exp(A z), \quad B = 0. \quad (2.6)$$

(2.5) implies that

$$\frac{g(z)}{z} = (1 + B w(z))^{(A-B)/B}, \quad B \neq 0. \quad (2.7)$$

Case (i) $B > 0$.

$$\left| (1 + B w(z))^{(A-B)/B} \right| = \left| \exp \left\{ \frac{(A-B)}{B} \log(1 + B w(z)) \right\} \right|$$

$$= \exp \left\{ \frac{(A-B)}{B} \log|1 + B w(z)| \right\}$$

$$= |1 + B w(z)|^{(A-B)/B}$$

$$\leq (1 + B r)^{(A-B)/B}.$$ 

Case (ii) $B < 0$.

Let $B = -B', B' > 0$. Then

$$\left| (1 + B w(z))^{(A-B)/B} \right| = \left| \left( (1 - B' w(z))^{-1} \right)^{(A-B)/B'} \right|$$

$$= \left| (1 - B' w(z))^{-1} \right|^{(A-B)/B'}$$

$$\leq \left( \frac{1}{1 - B' r} \right)^{(A-B)/B'}$$

$$= (1 + B r)^{(A-B)/B}.$$ 

Combining the cases (i) and (ii), (2.1) follows from (2.7). Similarly, we get (2.2) from (2.6).

Again from (2.5), we obtain (2.3) as follows

$$\left| \arg \frac{g(z)}{z} \right| \leq \frac{(A-B)}{B} |\arg(1 + B w(z))| \leq \frac{(A-B)}{B} \sin^{-1}(B r).$$

Similarly (2.4) directly follows from (2.6).

On the same lines we can prove the following
Lemma 2.6. If \( g \in C(A, B) \), then, for \( |z| = r < 1 \),

\[
\frac{1}{A} |1 - (1 - Br)^{A/B}| \leq |g(z)| \leq \frac{1}{A} |(1 + Br)^{A/B} - 1|, \quad B \neq 0;
\]

\[
\frac{1}{A} |1 - \exp(-Ar)| \leq |g(z)| \leq \frac{1}{A} |\exp(Ar) - 1|, \quad B = 0;
\]

\[
|\arg \frac{g(z)}{z}| \leq \frac{A}{B} \sin^{-1}(Br), \quad B \neq 0;
\]

\[
|\arg \frac{g(z)}{z}| \leq Ar, \quad B = 0.
\]

Lemma 2.7 ([2]). Let \( f \) and \( g \) are analytic functions and \( h \) be convex univalent function in \( E \) such that \( f \prec h \) and \( g \prec h \). Then \( (1 - \lambda)f + \lambda g \prec h \), \((0 \leq \lambda \leq 1)\).

3. Coefficient estimates

Theorem 3.1. Let \( f \in K(A, B; C, D) \). Then, for \( A - (n - 1)B \geq (n - 2), (n \geq 3) \),

\[
|a_n| \leq \frac{1}{n!} \sum_{k=2}^{n} \{A - (k - 1)B\} + \frac{(C - D)}{n} \left(1 + \sum_{k=2}^{n-1} \frac{1}{k!} \prod_{j=2}^{k} \{A - (j - 1)B\}\right). \tag{3.1}
\]

Bound (3.1) is sharp.

Proof. By definition of \( K(A, B; C, D) \),

\[
\frac{zf'(z)}{g(z)} = \frac{1 + Cw(z)}{1 + Dw(z)} = P(z).
\]

Expanding the series,

\[
(z + 2a_2z^2 + \cdots + na_nz^n + \cdots)
\]

\[
= (z + b_2z^2 + \cdots + b_{n-1}z^{n-1} + b_nz^n + \cdots)(1 + p_1z + p_2z^2 + \cdots + p_{n-1}z^{n-1} + \cdots). \tag{3.2}
\]

Equating the coefficients of \( z^n \) in (3.2),

\[
n a_n = b_n + p_1 b_{n-1} + p_2 b_{n-2} + \cdots + p_{n-2} b_2 + p_{n-1}.
\]

Applying triangular inequality and Lemma 2.1, we get

\[
n|a_n| \leq |b_n| + (C - D) \left(1 + \sum_{k=2}^{n-1} |b_k|\right). \tag{3.3}
\]

Using Lemma 2.2 in (3.3), we obtain

\[
n|a_n| \leq \frac{1}{(n - 1)!} \prod_{k=2}^{n} \{A - (k - 1)B\} + (C - D) \left(1 + \sum_{k=2}^{n-1} \frac{1}{k!} \prod_{j=2}^{k} \{A - (j - 1)B\}\right)
\]
which yields (3.1). The bound (3.1) is sharp for the function \( f_0(z) \) defined by

\[
  f_0(z) = \left( \frac{1 + C\delta_1 z}{1 + D\delta_1 z} \right)^{(A-B)/B}, \quad |\delta_1| = |\delta_2| = 1.
\]

Similarly we can prove

**Theorem 3.2.** Let \( f \in K_1(A, B; C, D) \). Then, for \( A - (n - 1)B \geq (n - 2), (n \geq 3), \)

\[
|a_n| \leq \frac{1}{n} \left[ \frac{1}{n!} \prod_{k=2}^{n} |A - (k - 1)B| + (C - D) \left( 1 + \sum_{k=2}^{n-1} \frac{1}{k!} \prod_{j=2}^{k} |A - (j - 1)B| \right) \right].
\]

(3.4)

The bound \( (3.4) \) is sharp for the function \( f_1(z) \) given by

\[
  f_1(z) = \left( \frac{1 + C\delta_1 z}{1 + D\delta_1 z} \right)^{(A-B)/B} - 1, \quad |\delta_1| = |\delta_2| = 1.
\]

**Remark 3.1.** (i) If \( f \in K(1, -1; C, D) \equiv K(C, D) \), \( |a_n| \leq 1 + \frac{(n-1)(C-D)}{2} \) which is the result due to Mehrok [10].

(ii) If \( f \in K_1(1, -1; C, D) \equiv K_1(C, D) \), \( |a_n| \leq \frac{1}{n} [1 + (n - 1)(C - D)] \), a result due to Mehrok and Singh [11].

**Remark 3.2.** (i) If \( f \in K(A, B; 1, -1) \equiv K^*(A, B) \), for \( A - (n - 1)B \geq (n - 2), (n \geq 3), \)

\[
|a_n| \leq \frac{1}{n(n!)} \prod_{k=2}^{n} |A - (k - 1)B| + \frac{2}{n} \left( 1 + \sum_{k=2}^{n-1} \frac{1}{k!} \prod_{j=2}^{k} |A - (j - 1)B| \right).
\]

This result was proved by Goel and Mehrok [5].

(ii) If \( f \in K_1(A, B; 1, -1) \equiv K_1^*(A, B) \), for \( A - (n - 1)B \geq (n - 2), (n \geq 3), \)

\[
|a_n| \leq \frac{1}{n(n!)} \prod_{k=2}^{n} |A - (k - 1)B| + \frac{2}{n} \left( 1 + \sum_{k=2}^{n-1} \frac{1}{k!} \prod_{j=2}^{k} |A - (j - 1)B| \right).
\]

**Remark 3.3.** (i) If \( f \in K(1, -1; 1, -1) \equiv K \), then \( |a_n| \leq n \). The result due to Reade [13].

(ii) If \( f \in K_1(1, -1; 1, -1) \equiv K_1 \), then \( |a_n| \leq 2 - \frac{1}{n} \). This result was obtained by Silverma and Telage [15].

4. Distortion theorems

**Theorem 4.1.** Let \( f \in K(A, B; C, D) \), then

\[
\left( \frac{1 - Cr}{1 - Dr} \right)^{(A-B)/B} \leq |f'(z)| \leq \left( \frac{1 + Cr}{1 + Dr} \right)^{(A-B)/B}, \quad B \neq 0; \tag{4.1}
\]

\[
\left( \frac{1 - Cr}{1 - Dr} \right) \exp(-Ar) \leq |f'(z)| \leq \left( \frac{1 + Cr}{1 + Dr} \right) \exp(AR), \quad B = 0; \tag{4.2}
\]
\[
\int_0^r \left( \frac{1-Cu}{1-Du} \right) (1-Bu)^{(A-B)/B} du \leq |f(z)| \leq \int_0^r \left( \frac{1+Cu}{1+Du} \right) (1+Bu)^{(A-B)/B} du, \quad B \neq 0; \quad (4.3)
\]
\[
\int_0^r \left( \frac{1-Cu}{1-Du} \right) \exp(-Au) du \leq |f(z)| \leq \int_0^r \left( \frac{1+Cu}{1+Du} \right) \exp(Au) du, \quad B = 0. \quad (4.4)
\]

All these bounds are sharp.

**Proof.** Since \( f \in K(A, B; C, D) \), it follows that
\[
\frac{zf'(z)}{g(z)} = \frac{1+Cw(z)}{1+ Dw(z)}
\]
which maps \(|w(z)| \leq r\) onto the circle
\[
\left| \frac{(zf'(z)}{g(z)} - \frac{1-CDr^2}{1-Dr^2} \right| \leq \frac{(C-D)r}{(1-D^2 r^2)}.
\]

This yields
\[
\frac{1-Cr}{1-Dr} \leq \left| \frac{zf'(z)}{g(z)} \right| \leq \frac{1+Cr}{1+Dr}
\]
which further implies that
\[
\frac{1-Cr}{1-Dr} |g(z)| \leq |zf'(z)| \leq \frac{1+Cr}{1+Dr} |g(z)|. \quad (4.5)
\]

Using (2.1) and (2.2) along with (4.5), we obtain (4.1) and (4.2).

Now
\[
|f(z)| = \int_0^r f'(z) dz 
\]
\[
\leq \int_0^r |f'(z)| dr 
\]
\[
\leq \begin{cases} 
\int_0^r \left( \frac{1+Cr}{1+Dr} \right) (1+Br)^{(A-B)/B} dr, & B \neq 0; \\
\int_0^r \left( \frac{1+Cr}{1+Dr} \right) \exp(Ar) dr, & B = 0.
\end{cases}
\]

Let \( z_0, |z_0| = 1 \), be so chosen that \(|f(z_0)| \leq |f(z)|\) for all \( z, |z| = r \). If \( L(z_0) \) is the pre-image of the segment \([0, f(z_0)]\) in \( E \), then
\[
|f(z)| = \int_{L(z_0)} |f'(z)| dz 
\]
\[
\leq \begin{cases} 
\int_0^r \left( \frac{1-Cr}{1-Dr} \right) (1-Br)^{(A-B)/B} dr, & B \neq 0; \\
\int_0^r \left( \frac{1-Cr}{1-Dr} \right) \exp(-Ar) dr, & B = 0.
\end{cases}
\]
Equality signs in (4.1), (4.2), (4.3) and (4.4) are attained by the function \( f_2(z) \) defined by

\[
 f'_2(z) = \begin{cases} 
 1 + C \delta_1 z 
 & \frac{1}{1 + D \delta_1 z} (1 + B \delta_2 r)^{(A-B)/B}, \quad B \neq 0; \\
 1 + C \delta_1 z 
 & \frac{1}{1 + D \delta_1 z} \exp(A \delta_2 z), \quad B = 0, \quad |\delta_1| = |\delta_2| = 1. 
\end{cases}
\]

Similarly, by using Lemma 2.6, we can prove

**Theorem 4.2.** Let \( f \in K_1(A, B; C, D) \), then

\[
 \begin{align*}
 \frac{1}{A} \left( \frac{1-Cr}{1-Dr} \right) \left| \frac{1}{1-(1-Br)^{A/B}} \right| \leq |f'(z)| & \leq \frac{1}{A} \left( \frac{1+Cr}{1+Dr} \right) (1+(Br)^{A/B} - 1), \quad B \neq 0; \\
 \frac{1}{A} \left( \frac{1-Cr}{1-Dr} \right) \exp(-Ar) \leq |f'(z)| & \leq \frac{1}{A} \left( \frac{1+Cr}{1+Dr} \right) \exp(Ar), \quad B = 0; \\
 \frac{1}{A} \int_0^r \left( \frac{1-Cu}{1-Du} \right) \left| (1-(1-Bu)^{A/B}) du \right| & \leq \frac{1}{A} \int_0^r \left( \frac{1+Cu}{1+Du} \right) (1+(Bu)^{A/B} - 1) du, \quad B \neq 0; \\
 \frac{1}{A} \int_0^r \left( \frac{1-Cu}{1-Du} \right) \exp(-Au) du & \leq |f'(z)| \leq \frac{1}{A} \int_0^r \left( \frac{1+Cu}{1+Du} \right) \exp(Au) du, \quad B = 0.
\end{align*}
\]

All these bounds are sharp and extremal function is given by \( f_3(z) \) defined by

\[
 f'_3(z) = \begin{cases} 
 1 + C \delta_1 z 
 & \frac{1}{A} \left( \frac{1+C \delta_1 z}{1+D \delta_1 z} \right) (1+B \delta_2 z)^{A/B} - 1), \quad B \neq 0; \\
 1 + C \delta_1 z 
 & \frac{1}{A} \left( \frac{1+C \delta_1 z}{1+D \delta_1 z} \right) \exp(A \delta_2 z), \quad B = 0, \quad |\delta_1| = |\delta_2| = 1. 
\end{cases}
\]

5. Argument theorems

**Theorem 5.1.** Let \( f \in K(A, B; C, D) \), then

\[
 \begin{align*}
 |\arg f'(z)| & \leq \frac{(A-B)}{B} \sin^{-1}(Br) + \sin^{-1} \left\{ \frac{(C-D)r}{(1-CDr^2)} \right\}, \quad B \neq 0; \quad (5.1) \\
 |\arg f'(z)| & \leq Ar + \sin^{-1} \left\{ \frac{(C-D)r}{(1-CDr^2)} \right\}, \quad B = 0. \quad (5.2)
\end{align*}
\]

The results are sharp.

**Proof.** From (1.11), we have \( \frac{z f'(z)}{g(z)} = \frac{1+Cw(z)}{1+ Dw(z)} \). Since the transformation \( \frac{z f'(z)}{g(z)} = \frac{1+Cw(z)}{1+ Dw(z)} \) maps \( |w(z)| \leq r \) onto the circle

\[
 \left| \frac{z f'(z)}{g(z)} - \frac{1-CDr^2}{1-D^2r^2} \right| \leq \frac{(C-D)r}{(1-D^2r^2)},
\]

therefore

\[
 \left| \arg \frac{z f'(z)}{g(z)} \right| \leq \sin^{-1} \left\{ \frac{(C-D)r}{(1-CDr^2)} \right\}.
\]

This implies that

\[
 |\arg f'(z)| \leq \left| \arg \frac{g(z)}{z} \right| + \sin^{-1} \left\{ \frac{(C-D)r}{(1-CDr^2)} \right\}. \quad (5.3)
\]
Inequality (5.3) together with (2.3) and (2.4) yield (5.1) and (5.2) respectively. Equality sign in (5.1) and (5.2) holds for the function \( f_2(z) \) defined by (4.6) in which

\[
\delta_1 = \frac{r}{z} \left[ \frac{-(C + D)r + i((1 - C^2 r^2)(1 - D^2 r^2))^{1/2}}{(1 + CD r^2)} \right] \quad (5.4)
\]

and

\[
\delta_1 = \frac{r}{z} \left[ -Br + i(1 - B^2 r^2)^{1/2} \right]. \quad (5.5)
\]

Similarly, by using Lemma 2.6, we have

**Theorem 5.2.** Let \( f \in K_1(A, B; C, D) \), then

\[
|\arg f'(z)| \leq \frac{A}{B} \sin^{-1}(Br) + \sin^{-1}\left\{ \frac{(C - D)r}{(1 - CD r^2)} \right\}, \quad B \neq 0;
\]

\[
|\arg f'(z)| \leq Ar + \sin^{-1}\left\{ \frac{(C - D)r}{(1 - CD r^2)} \right\}, \quad B \neq 0.
\]

The results are sharp for the function \( f_3(z) \) defined in (4.7) where \( \delta_1 \) and \( \delta_2 \) are given by (5.4) and (5.5), respectively.

**Remark 5.4.** Taking \( A = 1 \) and \( B = -1 \) in the Theorem 4.1, we get the result proved by Mehrok [10].

**Remark 5.5.** On taking \( C = 1 \) and \( D = -1 \) in the Theorems 4.1 and 5.1, we get the results due to Goel and Mehrok [5].

**Remark 5.6.** Letting \( A = C = 1 \) and \( B = D = -1 \) in the Theorems 4.1 and 5.1, we obtain the results proved by Ogawa [12] and Krzyz [8] for the class \( K \).

**Remark 5.7.** For \( C = 1 \) and \( D = -1 \) in Theorems 4.2 and 5.2, we get the results established by Gawad and Thomas [1].

### 6. Convex set of functions

**Theorem 6.1.** If \( f \) and \( h \in K(A, B; C, D) \), then

\[(1 - \lambda)f + \lambda h \in K(A, B; C, D), \quad (0 \leq \lambda \leq 1).\]

**Proof.** Since \( \frac{1+Cb}{1+D^2} \) is convex univalent in \( E \), the theorem follows by Lemma 2.7 definition of \( K(A, B; C, D) \).
References


Department of Mathematics, Govt. Rajindra College, Bathinda, Punjab - 151001, India.

E-mail: harjindpreet@gmail.com

#643E, Bhai Randhir Singh Nagar, Ludhiana, Punjab - 141001, India.

E-mail: beantsingh.mehrok@gmail.com