THE ISOMORPHISMS AND THE CENTER OF WEAK QUANTUM ALGEBRAS \( \omega_{sl_q}(2) \)

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Abstract. The aim of this paper is to describe the centre as well as the structure of \( \omega_{sl_q}(2) \) by applying the deformation of Harish-Chandra homomorphism.

Introduction

Throughout, the basic field is the complex number field \( \mathbb{C} \). All algebras, modules and vector spaces are over \( \mathbb{C} \) unless otherwise specified. Let \( q \) be a parameter which is not a root of unity.

F. Li and S. Duplij [9] constructed a quantum algebra \( \omega_{sl_q}(2) \). By definition, the quantum algebra \( \omega_{sl_q}(2) \) is generated by the four variables \( E, F, K, \overline{K} \) with the relations:

\[
\begin{align*}
K \overline{K} &= \overline{K}K = J \\
JK &= KJ = \overline{K} \\
KE &= q^2EK, \quad \overline{K}E = q^{-2}E\overline{K} \\
KF &= q^{-2}FK, \quad \overline{K}F = q^2F\overline{K} \\
EF - FE &= \frac{K - \overline{K}}{q - q^{-1}}
\end{align*}
\]

This is an interesting example of weak Hopf algebras in the sense of [7]. In the paper [9], the authors gave a detail description of the structure theory of \( \omega_{sl_q}(2) \), such as its basis, group-like elements, regular quasi-R matrix and so on.

As a continuation of the paper [9], we will study the isomorphisms among these weak quantum algebras and their centre. Several people have considered the problems of Hopf algebra automorphisms. For example, [1, 4, 11]. In [3] the isomorphisms among quantum algebras \( U_{r,s}(sl_n) \) with different parameters \( r, s \) were investigated. However, nobody has considered the same problem for the weak quantum algebra \( \omega_{sl_p}(2) \). By applying the idea of [3] and some known facts, we can yield the group of automorphisms of weak Hopf algebra \( \omega_{sl_q}(2) \). It is shown that \( \varphi : \omega_{sl_q}(2) \to \omega_{sl_p}(2) \) is a weak Hopf
algebra isomorphism if and only if \( p = \pm q \). If this is the case, we will determine all such isomorphisms. Let \( U_q(sl_2) \) be the quantum group corresponding to three dimensional semisimple Lie algebra \( sl_2 \). As is known, one of many beautiful results for \( U_q(sl_2) \) can be described by the Harish-Chandra homomorphism (see [6]). Similar to the case of \( U_q(sl_2) \), we would like to study the centre of \( \omega_{sl_2} q(2) \) and give the analogous statements by applying the modification of Harish-Chandra homomorphism. Let 

\[
Y = \{ E^i F^j (1 - J) \mid i \geq 0, j \geq 0 \},
\]

where we set \( K^0 = K^0 = J, a_i, b_j \in \mathbb{C} (i \geq 0, j \geq 0) \). Let \( Z_q \) be a polynomial algebra generated by the element \( C_q \) and \( J \). It is shown that the centre of \( \omega_{sl_2} q(2) \) is \( Z_q \oplus Y \) and the restriction of the Harish-Chandra homomorphism to \( Z_q \) is an isomorphism onto the sub-algebra of \( P[K, K] \) generated by \( qK + q^{-1}K \).

This paper is organized as follows. Some basic facts and concepts are reviewed in Section 1. Then we attempt to get the isomorphism theorem for \( \omega_{sl_2} q(2) \) in Section 2. Finally we devote to get the statements about the centre of \( \omega_{sl_2} q(2) \) in the last section.

1. Preliminaries

There are at least two generalizations of a Hopf algebra, which are called weak Hopf algebras. One of them was introduced and studied in [7, 8]. In this sense the weak Hopf algebra \( (H, \mu, \eta, \Delta, \varepsilon) \) is just both bialgebra and there exists a so-called weak antipode \( T \in \text{Hom}_k(H, H) \) of \( H \) such that \( T \ast I \ast T = T, I \ast T \ast I = I \), where \( I \) is an identity map of \( H \). Another definition of a weak Hopf algebra was introduced in [2]. The earlier proposals of face algebras [5], generalized Kac algebras [12] are weak Hopf algebras in this sense. However, the above two definitions of weak Hopf algebras are not included in each other.

One knows that the quantum algebra \( \omega_{sl_2} q(2) \) is a weak Hopf algebra in the sense of [7]. The comultiplication \( \Delta \), the counit \( \varepsilon \) and the weak antipode \( T \) are given by the following formulas

\[
\Delta(E) = 1 \otimes E + E \otimes K, \quad \Delta(F) = F \otimes 1 + K \otimes F, \\
\Delta(K) = K \otimes K, \quad \Delta(K) = K \otimes K, \\
\varepsilon(E) = \varepsilon(F) = 0, \quad \varepsilon(K) = \varepsilon(K) = 1, \\
T(E) = -E K, \quad T(F) = -K F, \quad T(K) = K, \quad T(K) = K.
\]
It is noticed that $J \neq 0$. If $J = 1$, $\omega sl_q(2)$ is isomorphic to $U_q(sl_2)$. Recall that the quantum algebra $U_q(sl_2)$, is generated by $E', F', K', K'^{-1}$ with the relations:
\[
K'^{-1}K' = K'K'^{-1} = 1,
K'E'K'^{-1} = q^2E', K'F'K'^{-1} = q^{-2}F',
E'F' - F'E' = \frac{K' - K'^{-1}}{q - q^{-1}}.
\]

$U_q(sl_2)$ is of Hopf algebra structure, the comultiplication and antipode are
\[
\Delta(E') = 1 \otimes E' + E' \otimes K', \quad \Delta(F') = F' \otimes 1 + K'^{-1} \otimes F',
\Delta(K') = K' \otimes K', \quad \Delta(K') = K' \otimes K',
\epsilon(E') = \epsilon(F') = 0, \quad \epsilon(K') = \epsilon(K'^{-1}) = 1,
S(E') = -E'K'^{-1}, \quad S(F) = -K'^{-1}F, \quad S(K') = K'^{-1}, \quad S(K'^{-1}) = K'.
\]

Accordingly, we always assume that $J \neq 0$ and $J \neq 1$.

Let $W = \omega sl_q(2)J$ and $Y = \omega sl_q(2)(1 - J)$.

**Lemma 1.1.** ([9, Theorem 4]) As ideals of $\omega sl_q(2)$ we have $\omega sl_q(2) = W \oplus Y$. Moreover, $W \cong U_q(sl_2)$ as Hopf algebras. The basis of $W$ is
\[
\{E_iF_jK^l, E_iF_jK^m, E_iF_jJ \mid i \geq 0, j \geq 0, l > 0, m > 0\}
\]
and the basis of $Y$ is
\[
\{E_iF_j(1 - J) \mid i \geq 0, j \geq 0\}.
\]

**Proof.** We sketch the proof as follows.

It is easy to see that $J$ is a central idempotent. Therefore, $\omega sl_q(2)J$ as well as $\omega sl_q(2)(1 - J)$ are ideals of $\omega sl_q(2)$. Hence,
\[
\omega sl_q(2) = \omega sl_q(2)J \oplus \omega sl_q(2)(1 - J)
\]
as ideals. One can see that $W$ is of the basis
\[
\{E_iF_jK^l, E_iF_jK^m, E_iF_jJ \mid i \geq 0, j \geq 0, l > 0, m > 0\}
\]
and $Y$ has the basis $\{E_iF_j(1 - J) \mid i \geq 0, j \geq 0\}$. In fact, $W$ is a Hopf algebra (the identity of $W$ is $J$), the co-multiplication $\Delta$ is
\[
\Delta(EJ) = J \otimes EJ + EJ \otimes K, \quad \Delta(FJ) = FJ \otimes J + K \otimes FJ, \quad \Delta(K) = K \otimes K, \quad \Delta(K) = K \otimes K.
\]
The counit $\epsilon$ is
\[
\epsilon(EJ) = \epsilon(FJ) = 0, \quad \epsilon(K) = \epsilon(K) = 1.
\]
and the antipode is
\[ T(EJ) = -EK, \ T(FJ) = -KF, \ T(K) = K, \ T(K) = K. \]

Now let \( \rho \) be the algebra morphism from \( U_q(sl_2) \) to \( W \) subjecting to
\[
\rho(E') = EJ, \ \rho(F') = FJ, \ \rho(K') = K, \ \rho(K'^{-1}) = K.
\]

It is straightforward to see that \( \rho \) is a Hopf algebra isomorphism.

Let \( [m] = \frac{q^m - q^{-m}}{q - q^{-1}} \) for \( m \geq 0 \) and

\[
[m]! = [1][2] \cdots [m], \ [0]! = 1, \ \left( \begin{array}{c} n \\ l \end{array} \right) = \frac{[n]!}{[l]![n-l]!}.
\]

We have
\[
EF^m = F^mE + [m]E^{m-1}q^{-(m-1)}K - q^{m-1}K q - q^{-1}.
\]

Let \( V \) be a \( \omega_{sl_q}(2) \)-module and \( 0 \neq v \in V \). If \( Kv = \lambda v \) and \( Knv = \lambda v \) for \( \lambda, \bar{\lambda} \in \mathbb{C} \), we can conclude that if \( \lambda \neq 0 \), \( \bar{\lambda} = \lambda^{-1} \) and if \( \lambda = 0 \), \( \bar{\lambda} = 0 \). We fix such a number \( \bar{\lambda} \) which is corresponded to \( \lambda \). We denote by \( V^\lambda \) the subspace of all vectors \( v \) in \( V \) such that \( Kv = \lambda v \). The scalar \( \lambda \) is called a weight of \( V \) if \( V^\lambda \neq 0 \). An element \( v \neq 0 \) of \( V \) is said to be a highest weight vector of weight \( \lambda \) if it is generated by a highest weight vector of \( \lambda \).

Given a \( \lambda \in \mathbb{C} \), we consider an infinite-dimensional vector space \( V(\lambda) \) with basis \( \{v_i\}_{i \in \mathbb{N}} \). For \( p \geq 0 \), we set
\[
Kv_p = \lambda q^{-2p}v_p, \ Knv_p = \lambda q^{2p}v_p, \quad (6)
\]
\[
Ev_p = q^{-p} - 1 q^{p-1} \lambda v_{p-1}, \quad (7)
\]
\[
Fv_{p-1} = [p]v_p, \ E\varepsilon_0 = 0. \quad (8)
\]

**Lemma 1.2.** Relations (6)-(8) define a \( \omega_{sl_q}(2) \)-modules structure on \( V(\lambda) \). The element \( v_0 \) generates \( V(\lambda) \) as a \( \omega_{sl_q}(2) \)-module and is a highest weight vector of weight \( \lambda \) such that \( V(\lambda) \) is the highest weight module.

**Proof.** Let \( l = \lambda \bar{\lambda} \). It is noticed that \( \lambda \neq 0 \) if and only if \( \lambda \neq 0 \). Also if \( \lambda \neq 0 \) then \( l = 1 \) if \( \lambda = 0 \), then \( l = 0 \). Therefore, either \( \lambda \neq 0 \) or \( \lambda = 0 \), we have \( \lambda^2 \bar{\lambda} = \lambda \) and
\( \lambda^q = \overline{\lambda} \). Immediate computations yield

\[
\begin{align*}
KKv_p &= KKKv_p = \lambda v_p, \\
KEv_p &= q^2 EKv_p, \\
KFv_p &= q^{-2} FKv_p,
\end{align*}
\]

\[
[E,F]v_p = ([p+1]q^p \lambda - q^q \overline{\lambda} - [p]q^{(p-1)}\lambda - q^{p-1} \overline{\lambda})v_p
\]

\[
= \frac{q^{-2p} \lambda - q^{2p} \overline{\lambda}}{q - q^{-1}} v_p = \frac{K - \overline{K}}{q - q^{-1}} v_p.
\]

This shows that the relations (6)-(8) define a \( \omega_{\mathfrak{sl}_q}(2) \)-module structure on \( V(\lambda) \). On the other hand, we have \( K^0v_0 = \lambda v_0 \) and \( E^0v_0 = 0 \), which means that \( v_0 \) is a highest weight vector of weight \( \lambda \). Finally, (8) implies that \( v_p = \frac{1}{[p]!} F^p v_0 \) for all \( p \), which proves that \( V(\lambda) \) is generated by \( v_0 \).

The highest weight \( \omega_{\mathfrak{sl}_q}(2) \)-module \( V(\lambda) \) is called the Verma module of highest weight \( \lambda \). We will apply the Verma module \( V(\lambda) \) to give a description of the centre of \( \omega_{\mathfrak{sl}_q}(2) \).

### 2. Isomorphisms Among Weak Quantum Algebras

We now investigate the isomorphisms among weak quantum algebras.

Let \( U_p(\mathfrak{sl}_2) \) be the algebra generated by \( E', F', K', K'^{-1} \) and the relations as that of \( U_q(\mathfrak{sl}_2) \) where \( q \) is replaced by \( p \). It is also a Hopf algebra with the same comultiplications as \( U_q(\mathfrak{sl}_2) \).

The following lemma gives a condition that \( U_p(\mathfrak{sl}_2) \cong U_q(\mathfrak{sl}_2) \) as Hopf algebras.

**Lemma 2.1.** \( U_p(\mathfrak{sl}_2) \cong U_q(\mathfrak{sl}_2) \) as Hopf algebras if and only if \( p = \pm q^\pm 1 \).

**Proof.** For convenience, we replace the generators \( E', F', K', K'^{-1} \) of \( U_q(\mathfrak{sl}_2) \) by \( E, F, K, K^{-1} \). The abuse notations are used in the proof.

Let \( \phi : U_p(\mathfrak{sl}_2) \rightarrow U_q(\mathfrak{sl}_2) \) be a bialgebra isomorphism, then we have

\[
\Delta(\phi(E')) = (\phi \otimes \phi)(\Delta(E')) = 1 \otimes \phi(E') + \phi(E') \otimes \phi(K').
\]

(9)

Note that \( \phi(K') \) is necessarily a group-like element. Therefore, \( \phi(E') \) is a skew-primitive element in \( U_q(\mathfrak{sl}_2) \).

By Theorem 5.4.1, Lemma 5.5.5, the subsequent comments in [10], and [4, Theorem A], we can assume that

\[
\phi(K') = K, \phi(E') = aE + bFK + c(1 - K).
\]

Then (9) automatically holds. Applying \( \phi \) to the equation \( K'E' = p^2 E'K' \), we yield that

\[
K(aE + bFK + c(1 - K)) = p^2(aE + bFK + c(1 - K))K.
\]
Consequently, we get
\[ 0 = a(q^2 - p^2) = b(p^2 - q^{-2}) = c(1 - p^2). \]
It follows that \( c = 0 \) since \( p \) is not a root of unity.

If \( a \neq 0 \), then \( p^2 = q^2 \) and \( b = 0 \). In this case, we get that \( \phi(E) = aE \) and \( p = \pm q \).
If this is the case, we get that \( \phi(F) = \pm a^{-1}F, K \rightarrow K, K^{-1} \rightarrow K^{-1}. \)

If \( b \neq 0 \), then \( p^2 = q^{-2} \) and \( a = 0 \). In this case, we get that \( \phi(E) = bFK \) and \( p = \pm q^{-1} \). Similarly, \( \phi(F) = \pm b^{-1}K^{-1}E \).
Conversely, if \( p = \pm q \), it is obvious that \( \psi : U_p(sl_2) \cong U_q(sl_2) \) defined by
\[
\psi(E') = E, \ \psi(F') = \pm F, \ \psi(K') = K, \ \psi(K'^{-1}) = K^{-1}
\]
is a Hopf algebra isomorphism.

If \( p = \pm q^{-1} \), then \( \psi : U_p(sl_2) \cong U_q(sl_2) \) defined by
\[
\psi(E') = FK, \ \psi(F') = \pm K^{-1}E, \ \psi(K') = K, \ \psi(K'^{-1}) = K^{-1}
\]
is a Hopf algebra isomorphism.

Let \( \omega sl_2(2) \) be the algebra generated by \( E, F, K, K^{-1} \) and the relations as that of \( \omega sl_q(2) \) where \( q \) is replaced by \( p \). It is a weak algebra with the same comultiplications as \( \omega sl_q(2) \).

**Lemma 2.2.** Let \( x \in Y \) and \( b \neq 0 \). If \( \Delta(x) = b(1 - J) \otimes EJ + 1 \otimes x + x \otimes K \), then \( x = bE(1 - J) \).

**Proof.** Let \( x = \sum_{s,t} \xi(s,t)E^sF^t(1 - J) \). By the assumption, we have
\[
\Delta(x) = \sum_{s,t} \xi(s,t)E^sF^t(1 - J) \otimes K + \sum_{s,t} \xi(s,t)1 \otimes E^sF^t(1 - J) + b(1 - J) \otimes EJ.
\]
On the other hand, if \( j > 0 \), then \( K^j J = K^j \) and \( K^jJ = K^j \). One easily sees that
\[
\Delta(x) = \left( \sum_{s,t,i,j} \xi(s,t)q^{i(s-i)}q^{j(t-j)} \begin{bmatrix} s \cr i \end{bmatrix} \begin{bmatrix} t \cr j \end{bmatrix} E^i F^t - j \otimes E^{s-i}K^iF^j \right)
- \left( \sum_{s,t,i,j} \xi(s,t)q^{i(s-i)}q^{j(t-j)} \begin{bmatrix} s \cr i \end{bmatrix} \begin{bmatrix} t \cr j \end{bmatrix} E^i F^t - j \otimes E^{s-i}K^iJ \right)
= \left( \sum_{s,t,i,j \neq 0} \xi(s,t)q^{j(t-j)} \begin{bmatrix} t \cr j \end{bmatrix} F^{t-j} \otimes E^{s-i}F^j(1 - J) \right)
+ \left( \sum_{s,t,i \neq 0} \xi(s,t)q^{i(s-i)} \begin{bmatrix} s \cr i \end{bmatrix} E^i F^t (1 - J) \otimes E^{s-i}K^i \right)
+ \left( \sum_{s,t} \xi(s,t)F^t \otimes E^s \right) - \left( \sum_{s,t} \xi(s,t)F^t J \otimes E^s J \right)
Comparing the above two equality for $\Delta(x)$, all $t = 0$ and $s = 1$. Hence we can assume that $x = aE(1 - J)$ and we get

$$\Delta(x) = a(1 - J) \otimes E(1 - J) + 1 \otimes x + x \otimes K.$$ 

It follows that $a = b$ and $x = bE(1 - J)$.

The same argument shows that there is no element $x \in Y$ such that $\Delta(x) = x \otimes K + 1 \otimes x + b(1 - J) \otimes FK$ and $x \in Y$ where $b \neq 0$.

The main result of this section is as follows.

**Theorem 2.3.** $\omega sl_q(2) \cong \omega sp_q(2)$ as weak Hopf algebras if and only if $p = \pm q$.

**Proof.** Let $\gamma : \omega sl_q(2) \cong \omega sp_q(2)$ be a weak Hopf algebra isomorphism. One knows that $\gamma$ sends group-likes to group-likes, now it is easy to see that $\gamma(J) = J$.

According to Lemma 1.1, $\omega sl_q(2) = W \oplus Y$, $W \cong U_q(sl_2)$ as Hopf algebras; $\omega sp_q(2) = W' \oplus Y'$, $W' \cong U_p(sl_2)$ as Hopf algebras, where $Y$, $Y'$ are spanned respectively by the same set $\{EF^i(1 - J) \mid i \geq 0, j \geq 0\}$ as an ideal of $\omega sl_q(2)$ and $\omega sp_q(2)$.

Let $\text{inj}_q : W \rightarrow \omega sl_q(2)$ be the inclusion defined by

$$P \rightarrow J, \ EJ \rightarrow EJ, \ FJ \rightarrow FJ, \ K \rightarrow K, \ \overline{K} \rightarrow \overline{K},$$

and then extend it by linearity. It is easy to see that $\text{inj}_q$ is a weak Hopf algebra injection. Indeed, $\text{inj}_q$ is an algebra homomorphism. For the relation (3),

$$\text{inj}_q(K)\text{inj}_q(EJ) = KEJ = q^2EJK = q^2\text{inj}_q(EJ)\text{inj}_q(K).$$

The rest of (3) and the relations (4) are similar. For the relation (5),

$$\text{inj}_q(EJ)\text{inj}_q(FJ) - \text{inj}_q(FJ)\text{inj}_q(EJ) = (EF - FE)J = \frac{\text{inj}_q(K) - \text{inj}_q(\overline{K})}{q - q^{-1}}.$$

The map $\text{inj}_q$ is also a coalgebra map. Indeed,

$$\Delta(\text{inj}_q(EJ)) = \Delta(EJ) = J \otimes EJ + EJ \otimes K$$

and

$$\Delta(\text{inj}_q(EJ)) = (\text{inj}_q \otimes \text{inj}_q)(J \otimes EJ + EJ \otimes K) = J \otimes EJ + EJ \otimes K.$$ 

Similarly, we have $\Delta(\text{inj}_q(X)) = (\text{inj}_q \otimes \text{inj}_q)\Delta(X)$ where $X = FJ, K, \overline{K}$ or $J$. It is easy to see that $W' = \text{im}(\gamma \circ \text{inj}_q)$ since $\gamma(J) = J$. This implies that if $\gamma : \omega sl_q(2) \rightarrow \omega sp_q(2)$ is a weak Hopf algebra isomorphism, then $U_p(sl_2) \cong U_q(sl_2)$ as Hopf algebras. By Lemma 2.1, $p = \pm q^{\pm 1}$. However, if $p = \pm q^{-1}$, we must have

$$\gamma(EJ) = b(FJ)K, \ \gamma(FJ) = \pm b^{-1}\overline{K}(EJ), \ \gamma(K) = K, \ \gamma(\overline{K}) = \overline{K}.$$
for some \( b \neq 0 \). If there is a way to extend it to \( \omega_{sl}(2) \) such that \( \gamma \) is a weak Hopf algebra isomorphism, we assume that \( \gamma(E(1 - J)) = x \), then \( 0 \neq x \in Y \) and \( \gamma(E) = \gamma(EJ + E(1 - J)) = b(FJ)K + x \). Since \( \gamma \) is a weak Hopf algebra isomorphism, we have
\[
\Delta(b(FJK) + x) = (b(FJ)K + x) \otimes K + 1 \otimes (b(FJ)K + x).
\]
Hence, \( \Delta(x) = x \otimes K + 1 \otimes x + b(1 - J) \otimes FK \). It is impossible, so \( p = \pm q \).

Conversely, if \( p = \pm q \), we set
\[
\gamma(E) = E, \gamma(F) = \pm F, \gamma(K) = K, \gamma(\bar{K}) = \bar{K}.
\]
It is easy to see that \( \gamma \) is a weak Hopf algebra isomorphism.

The proof is completed.

Now we can determine all such isomorphisms. Indeed, if \( \gamma : \omega_{sl}(2) \cong \omega_{sl}(2) \) is an isomorphism of weak Hopf algebra, then \( p = \pm q \). Furthermore, \( \gamma \circ \text{inj}_q \) is an isomorphism of Hopf algebras between \( W \) and \( W' \), defined by
\[
J \rightarrow J, EJ \rightarrow aEJ, FJ \rightarrow \pm a^{-1}FJ, K \rightarrow K, \bar{K} \rightarrow \bar{K}
\]
by Lemma 2.1. The map \( \gamma \) restricted to \( W \) must be of this form. To get the map \( \gamma \), we assume that \( \gamma(E(1 - J)) = x \), it is easy to see that \( \gamma(E) = aEJ + x \) and \( x \in Y \). Since \( \gamma \) is a weak Hopf algebra isomorphism, we then get that \( \Delta(x) = 1 \otimes x + x \otimes K + a(1 - J) \otimes EJ \). By Lemma 2.2, we have \( x = aE(1 - J) \). Similarly, we also have \( \gamma(F(1 - J)) = \pm a^{-1}F(1 - J) \). This implies that \( \gamma \) has to be \( J \rightarrow J, E \rightarrow aE, F \rightarrow \pm a^{-1}F, K \rightarrow K, \bar{K} \rightarrow \bar{K} \) and extended linearity.

3. The Centre of \( \omega_{sl}(2) \)

In [13], the authors introduce a new quantum algebra \( U_q(f(H, K)) \), which generalizes the quantum group \( U_q(sl_2) \). Then they obtained statements about its centre by applying the Harish-Chandra homomorphism. In this section, we give the similar description about the centre of \( \omega_{sl}(2) \). Recall that
\[
P[K, \bar{K}] = \left\{ a_0J + \sum_{i > 0} a_iK^i + \sum_{j > 0} b_j\bar{K}^j \mid J = K\bar{K} = \bar{K}K, K = KJ, J\bar{K} = J \right\}.
\]
We set \( K^0 = J = \bar{K}^0 \) for convenience.

Keeping all notations as the previous sections. Let \( Z_q \) denote the centre of \( W \) and \( \omega \) the centre of \( \omega_{sl}(2) \). To state our main result, several lemmas are needed as follows.

Lemma 3.1. \( Y \subseteq Z_\omega \).

Proof. It is noticed that
\[
Y = \{ E^iF^j(1 - J) \mid i \geq 0, j \geq 0 \}.
\]
Since

\[ E(E^i F^j (1 - J)) = E^i (F^j E + [j] F^{j-1} \frac{q^{-1} (q^{-1} - K - q^{j-1} K)}{q - q^{-1}})(1 - J) \]

\[ = E^i F^j (1 - J) E, \]

\[ F(E^i F^j (1 - J)) = (E^i F - [i] E^{i-1} \frac{q^{-1} (q^{-1} - K - q^{j-1} K)}{q - q^{-1}}) F^j (1 - J) \]

\[ = E^i F^j (1 - J) F, \]

\[ K(E^i F^j (1 - J)) = q^{2i-2j} E^i F^j (1 - J) K = 0 = E^i F^j (1 - J) K, \]

\[ \overline{K}(E^i F^j (1 - J)) = q^{2i-2j} E^i F^j (1 - J) \overline{K} = 0 = E^i F^j (1 - J) \overline{K}. \]

The result follows.

Let

\[ C_q = EFJ + \frac{q^{-1} K + q \overline{K}}{(q - q^{-1})^2} = FEJ + \frac{qK + q^{-1} \overline{K}}{(q - q^{-1})^2}. \]

(10)

It is called the \( J \)-quantum Casimir element.

Let \( W^K \) be the sub-algebra of \( W \) consisting of all elements commuting with \( K \). For any \( x \in W^K \), then \( xK = Kx \) and \( xJ = Jx = x \). It follows that

\[ \overline{K} x J = \overline{K} J x = J x \overline{K}. \]

Hence \( \overline{K} x = x \overline{K} \) and the elements of \( W^K \) commute with \( \overline{K} \).

Let \( I = WE \cap W^K \), it is a left ideal of \( W^K \).

The following three lemmas are very similar to [6, Lemma VI.4.2-Lemma VI. 4.3] and their proofs are more or less the same.

**Lemma 3.2.** The element \( C_q \in Z_\omega \).

**Lemma 3.3.** Any element of \( W \) belongs to \( W^K \) if and only if it is of the form \( \sum_{i \geq 0} F^i P_i E^i \), where \( P_0, P_1, \ldots \) are elements of \( \mathbb{P}[K, \overline{K}] \).

**Lemma 3.4.** We have \( I = FW \cap W^K \) and \( W^K = \mathbb{P}[K, \overline{K}] \oplus I \).

It results from \( I = FW \cap W^K \) that \( I \) is a two-sided ideal and that the projection \( \varphi \) from \( W^K \) onto \( \mathbb{P}[K, \overline{K}] \) is a morphism of algebras. The map \( \varphi \) is called the Harish-Chandra homomorphism. It permits one to express the action of the centre \( Z_q \) of \( W \) on a highest weight module.

The following lemmas are similar to [6, Lemma VI.4.4-Lemma VI. 4.7], but details in the proofs have to be changed to suit for our cases. For completeness, we write them down here.

**Lemma 3.5.** Let \( V \) be a highest weight \( \omega_{sl_q(2)} \)-module with highest weight \( \lambda \). Then, for any central element \( z \) of \( W \) and any \( v \in V \), we have \( zv = \varphi(z)(\lambda, \overline{\lambda})v \), where \( \varphi(z) \) is element of \( \mathbb{P}[K, \overline{K}] \) and that \( \varphi(z)(\lambda, \overline{\lambda}) \) is its value at \( \lambda, \overline{\lambda} \).
Proof. Let $v_0$ be a highest weight vector generating $V$ and $z$ is a central element of $W$, the element $z$ can be written in the form
\[ z = \varphi(z) + \sum_{i>0} F_i P_i E^i. \]
Since $E v_0 = 0$ and $K v_0 = \lambda v_0$, $K v_0 = \lambda v_0$, we get $z v_0 = \varphi(z)(\lambda, \overline{\lambda}) v_0$. If $v$ is an arbitrary element of $V$, we have $v = z x v_0$ for some $x \in \omega s l_2(2)$. It is noticed that $x = x_1 + x_2$ where $x_1 \in W$ and $x_2 \in Y$. Since $Y W = W Y = 0$, $z x_2 = x_2 z = 0$ and $zx = xz$. Hence
\[ z v = z x v_0 = x z v_0 = \varphi(z)(\lambda, \overline{\lambda}) x v_0 = \varphi(z)(\lambda, \overline{\lambda}) v. \]
The result follows.

We now consider the restriction of the Harish-Chandra homomorphism to $Z_q$.

Lemma 3.6. Let $z \in Z_q$ and if $\varphi(z) = 0$, then $z = 0$.

Proof. Let $z$ be an element in the centre of $W$ such that $\varphi(z) = 0$. Assume $z$ non-zero, it can be written as $z = \sum_{i=k}^l F_i P_i E^i$ where $0 < k \leq l$ are integers and $P_i$, $\cdots$, $P_l$ are non-zero elements of $P[K, \overline{K}]$. Consider the Verma module $V(\lambda)$ whose highest weight is neither a power of $q$ or $0$ (therefore, $\overline{\lambda} = \lambda^{-1}$). The relations (6)-(8) show that $E v_0 = 0$ if and only if $p = 0$. We apply $z$ to the vector $v_k$ of $V(\lambda)$, on one hand, Lemma 3.6 implies that $z v_k = \varphi(z)(\lambda, \overline{\lambda}) v_k = 0$, on the other hand, we get $z v_k = F_k P_k E^k v_k = c P_k(z, \overline{\lambda}) v_k$ where $c$ is a non-zero constant. It follows that $P_k(\lambda, \overline{\lambda}) = 0$. As a consequence, we have a non-zero polynomial with infinitely many roots. It is a contradiction.

For any element $Q$ of $P[K, \overline{K}]$, denoted by $\tilde{Q}$ the polynomial defined by the change of variable $Q(\lambda, \overline{\lambda}) = \tilde{Q}(q^{-1} \lambda, q \lambda)$.

Lemma 3.7. For any element $z$ in $Z_q$, we have $\tilde{\varphi}(z)(\lambda, \overline{\lambda}) = \varphi(z)(\lambda, \overline{\lambda})$.

Proof. If $\lambda = 0$, then $\overline{\lambda} = 0$, the result is obvious. The following is under the assumption that $\lambda \neq 0$. Therefore, $\overline{\lambda} = \lambda^{-1}$. For any integer $n > 0$ consider the Verma module $V(q^n)$. By the formula (7), we have
\[ E v_n = \frac{q^{-(n-1)} q^{n-1} - q^{n-1} q^{-(n-1)}}{q - q^{-1}} v_{n-1} = 0. \]
Thus, $v_n$ is a highest weight vector of weight $q^{n-1-2n} = q^{-n-1}$. By Lemma 3.5, a central element $z$ acts on the module generated by $v_n$ as the multiplication by the scalar $\varphi(z)(q^{-n-1}, q^{n+1})$, but since $v_n$ is in $V(q^{-n})$, then
\[ z v_n = \varphi(z)(q^{-n-1}, q^{n+1}) v_n. \]
In other words, we have
\[ \tilde{\varphi}(z)(q^n, q^{-n}) = \varphi(z)(q^{-n}, q^n). \]
The lemma follows.

**Lemma 3.8.** Any polynomial of $P[K, \overline{K}]$ satisfying the relation $Q(\lambda, \overline{\lambda}) = Q(\overline{\lambda}, \lambda)$ is a polynomial in $k[K + \overline{K}]$.

**Proof.** We proceed by induction on the degree of the polynomial on $K$. If the degree is 0, the statement holds trivially. Suppose that the lemma is proved for all degrees $< n$ and let $Q$ be element of degree $n$ for $K$ such that $Q(\lambda, \overline{\lambda}) = Q(\overline{\lambda}, \lambda)$. Then we may write $Q$ in the form
\[
Q = c(K^n + \overline{K}^n) + \text{ (terms of degree } < n) .
\]
Now
\[
K^n + \overline{K}^n = (K + \overline{K})^n + \text{ (terms of degree } < n),
\]
where we set $(K + \overline{K})^0 = J$, $J^2 = J = K\overline{K}$. One concludes by applying the induction hypothesis.

We are ready to prove our main theorem.

**Theorem 3.9.** When $q$ is not a root of unity, the centre of $\omega sl_q(2)$ is $Z_q \oplus Y$, where $Z_q$ is a polynomial algebra generated by the element $C_q$ and $J$. The restriction of the Harish-Chandra homomorphism to $Z_q$ is an isomorphism onto the sub-algebra of $P[K, \overline{K}]$ generated by $qK + q^{-1}\overline{K}$.

**Proof.** We have already known that the restriction of $\varphi$ to the $Z_q$ is injective by Lemma 3.6. We are left to determine its image. By Lemma 3.7 and Lemma 3.8, the latter is contained in the sub-algebra of $P[K, \overline{K}]$ generated by $qK + q^{-1}\overline{K}$. Consider the central element $C_q$ defined by (10), we know that
\[
\varphi(C_q) = \frac{1}{(q - q^{-1})^2}(qK + q^{-1}\overline{K}), \quad \varphi(K\overline{K}) = K\overline{K},
\]
which proves that the image of $Z_q$ is the whole sub-algebra and that $C_q$ and $J$ generate $Z_q$. The latter is a polynomial algebra generated by $C_q$ and $J$. By Lemma 1.1, $\omega sl_q(2) = W \oplus Y$. It follows that $Z_\omega = Z_q \oplus Y$.

**References**


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