ON THE ARITHMETIC–GEOMETRIC MEAN INEQUALITY

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Abstract. We obtain some refinements of the Arithmetic–Geometric mean inequality. As an application, we find the maximum value of a multi-variable function.

1. Introduction

We assume that \( a_1, a_2, \ldots, a_n \) are \( n \) positive real numbers, and as usual, we define their arithmetic and geometric means, respectively by

\[
A = \frac{1}{n} \sum_{i=1}^{n} a_i, \quad \text{and} \quad G = \left( \prod_{i=1}^{n} a_i \right)^{\frac{1}{n}}.
\]

We consider the functions \( g(x) = e^x - x^e \) and \( h(x) = x^{1/x} \) over \((0, \infty)\). The function \( h \) has an absolute maximum at \( x = e \). Thus, if \( x > 0 \) then \( e^{1/e} \geq x^{1/x} \), or equivalently \( g(x) \geq 0 \), with equality if and only if \( x = e \). For \( i = 1, 2, \ldots, n \), we take \( x = a_i/e \) in \( e^x \geq x^e \), and then we multiply the resulting inequalities to get

\[
e^{\frac{a_i}{e}} \geq \left( \prod_{i=1}^{n} \frac{a_i}{G} \right)^{\frac{1}{n}} = \left( e^n G^n \right)^{\frac{1}{n e}} = e^{n e},
\]

from which we obtain \( A \geq G \), with equality if and only if \( a_i/e = e \) for \( i = 1, 2, \ldots, n \), or equivalently for when \( a_1 = a_2 = \cdots = a_n \).

The above argument for obtaining the Arithmetic–Geometric mean inequality is due to Schaumberger [1]. In this note we replace \( g(x) \) by a smaller positive function to get some refinements of the this inequality. More precisely, we obtain the following result.

Theorem 1.1. Assume that \( a_1, a_2, \ldots, a_n \) are \( n \) positive real numbers with arithmetic and geometric means \( A \) and \( G \), respectively. Then, we have

\[
A \geq G + \mathcal{R} \geq G,
\]

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where
\[ R = \frac{G}{ne} \log \left( 1 + \frac{1}{e^{ne}} \prod_{i=1}^{n} \left( \frac{a_i}{G} - 1 \right)^2 \right) \geq 0, \]
with equality if and only if \( a_1 = a_2 = \cdots = a_n \).

2. Proof of Theorem 1.1

**Lemma 2.1.** For \( x > 0 \) we define
\[ f(x) = e^x - x^e - \frac{1}{e^2} (x - e)^2. \]
The inequality \( f(x) \geq 0 \) is valid for \( x > 0 \), with equality if and only if \( x = e \). Moreover, \( \frac{1}{e^2} \) is the best possible constant for which the above inequality is valid.

**Proof.** As Figure 1 shows, \( f(x) \) takes its minimum value equal to 0 at \( x = e \). Also, we have \( \lim_{x \to 0^+} f(x) = 0 \), which proves optimal choose of the constant \( \frac{1}{e^2} \). This completes the proof. □

![Graphs of functions](image)

Figure 1: Graphs of the functions \( f(x) = e^x - x^e - \frac{1}{e^2} (x - e)^2 \) and \( g(x) = e^x - x^e \) over the intervals \((0, 3)\) and \((0, 5)\).

**Proof of Theorem 1.1.** We apply Lemma 2.1 by taking \( x = a_i e / G \) in \( f(x) \), from which we obtain
\[ e^{\sum_{i=1}^{n} a_i} \geq \left( \frac{a_i e}{G} \right)^e + \frac{1}{e^2} \left( \frac{a_i e}{G} - e \right)^2 = \left( \frac{a_i e}{G} \right)^e + \left( \frac{a_i}{G} - 1 \right)^2 \quad \text{(for } i = 1, 2, \ldots, n). \]

We multiply these inequalities to get
\[ e^{\sum_{i=1}^{n} a_i A} = e^{\sum_{i=1}^{n} \frac{a_i e}{G}} \geq \left( \prod_{i=1}^{n} \frac{a_i e}{G} \right)^e + \prod_{i=1}^{n} \left( \frac{a_i}{G} - 1 \right)^2 = e^{ne} + \prod_{i=1}^{n} \left( \frac{a_i}{G} - 1 \right)^2. \]

Thus, we have
\[ e^{\sum_{i=1}^{n} a_i A} \geq e^{ne} \left( 1 + \frac{1}{e^{ne}} \prod_{i=1}^{n} \left( \frac{a_i}{G} - 1 \right)^2 \right). \]
Finally, we take logarithm and we divide the resulting inequality by $ne$ to obtain
\[
\frac{A}{G} \geq 1 + \frac{1}{ne} \log \left( 1 + \frac{1}{e^{ne}} \prod_{i=1}^{n} \left( \frac{a_i}{G} - 1 \right)^2 \right),
\]
with equality if and only if $a_1 = a_2 = \cdots = a_n$. This completes the proof. \qed

3. Some applications

One may rewrite the Arithmetic–Geometric mean inequality in the forms
\[
A - G \geq 0, \quad \text{and} \quad \frac{A}{G} - 1 \geq 0.
\]
As the first application of Theorem 1.1, we obtain the following refinement of the above mentioned inequalities.

**Theorem 3.1.** Assume that $a_1, a_2, \ldots, a_n$ are $n$ positive real numbers which are not simultaneously equal, with arithmetic and geometric means $A$ and $G$, respectively. Then, we have
\[
A - G \geq \frac{Ge^{ne(\frac{A}{G} - 1)}}{ne(e^{ne(\frac{A}{G} - 1)} - e^{ne})} \prod_{i=1}^{n} \left( \frac{a_i}{G} - 1 \right)^2 \geq 0,
\]
or equivalently
\[
\frac{A}{G} - 1 \geq \frac{e^{ne(\frac{A}{G} - 1)}}{ne^{ne+1}(e^{ne(\frac{A}{G} - 1)} - 1)} \prod_{i=1}^{n} \left( \frac{a_i}{G} - 1 \right)^2 \geq 0.
\]

**Proof.** Assume that $a_1, a_2, \ldots, a_n$ are $n$ positive real numbers which are not simultaneously equal, so that $A > G$. By using the result of Theorem 1.1, we have $R \leq A - G$, which is equivalent to
\[
\frac{1}{e^{ne}} \prod_{i=1}^{n} \left( \frac{a_i}{G} - 1 \right)^2 \leq e^{ne(\frac{A}{G} - 1)} - 1.
\]
On the other hand, for $0 \leq x \leq \beta$, we have $\log(1 + x) \geq \frac{\log(1 + \beta)}{\beta} x$ because $\frac{\log(1+x)}{x}$ is decreasing on $(0, \beta]$. We use this inequality by putting $x = \frac{1}{e^{ne}} \prod_{i=1}^{n} \left( \frac{a_i}{G} - 1 \right)^2 \geq 0$ and $\beta = e^{ne(\frac{A}{G} - 1)} - 1$ to get
\[
R = \frac{G}{ne} \log \left( 1 + \frac{1}{e^{ne}} \prod_{i=1}^{n} \left( \frac{a_i}{G} - 1 \right)^2 \right) \geq \frac{G}{ne} \left( \frac{e^{ne(\frac{A}{G} - 1)}}{e^{ne(\frac{A}{G} - 1)} - e^{ne}} \prod_{i=1}^{n} \left( \frac{a_i}{G} - 1 \right)^2 \right).
\]
This completes the proof. \qed

As the second application of Theorem 1.1, we observe that it allows us to find the maximum value of a multi-variable function, without using partial derivative tests.
Theorem 3.2. We have

$$\max_{a_i > 0} \frac{G}{n} e \log \left( 1 + \frac{1}{e^n} \prod_{i=1}^{n} \left( \frac{a_i}{G} - 1 \right)^2 \right) = A - G.$$ 

Remark 3.3. We assume that $a_i > 0$, and then we replace $a_i$ by $1/a_i$, from which the inequality $A \geq G$ implies validity of the well-known Geometric-Harmonic mean inequality, asserting that $G \geq H$, where $H$ refers to the harmonic mean of the positive real numbers $a_1, a_2, \ldots, a_n$. We observe that the replacement $a_i \rightarrow 1/a_i$ gives the replacements $A \rightarrow 1/H$ and $G \rightarrow 1/G$. By applying this fact, one may rewrite all of the above results concerning the means $A$ and $G$, to obtain similar results concerning the means $G$ and $H$.

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References


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