SINGULAR RIGHT FOCAL BOUNDARY VALUE PROBLEM
WITH GIVEN MAXIMAL VALUES

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Abstract. In this paper, we prove existence results for the singular problem \((−1)^{n−p}(Φ_m x^{(n−1)})′(t) = µf(t, x(t), . . . , x^{(n−1)}(t)),\)
for \(0 < t < 1, x^{(i)}(0) = 0, i = 0, 1, . . . , p − 1, x^{(i)}(1) = 0, i = p, \)
\(p + 1, . . . , n − 1, \max\{x(t) : t ∈ [0, 1]\} = A. \) The paper presents
conditions which guarantee that for any \(A > 0\) there exists \(µ_A > 0\) such that the
above problem with \(µ = µ_A\) has a solution \(x ∈ C^{n−1}([0, 1])\) which is positive
on \((0, 1).\) Here the positive Carathéodory function \(f\) may be singular at the
zero value of all its phase variables. Proofs are based on the Leray-Schauder
degree and Vitali’s convergence theorem.

1. Introduction

The right focal boundary value problems has been widely studied by a number of
authors in recent years. For details, see [1, 7, 8, 9, 10, 15] and the references therein.
However the boundary value problems treated in the above mentioned references are
not allowable to process singularity. For studies about higher-order singular boundary value
problem, we refer to [2, 3, 4, 5, 6, 17].

Agarwal, O’Regan and Lakshmikantham studied the existence of solutions for right
focal boundary value problem in [3]:
\[
\begin{align*}
(-1)^{n-p} y^{(n)} &= \phi(t)f(t, y, . . . , y^{(n-1)}), \quad n \geq 2, \ t \in (0, 1), \\
y^{(i)}(0) &= 0, \quad 0 \leq i \leq p - 1, \\
y^{(i)}(1) &= 0, \quad p \leq i \leq n - 1,
\end{align*}
\]
(1.1)
where \(f \in C([0, 1] × (0, ∞)^p, (0, ∞)), f(t, y_0, . . . , y_{n−1})\) may be singular at \(y_i = 0, 0 \leq \)
i \(p - 1, \phi \in C(0, 1)\) with \(\phi > 0\) on \((0, 1)\) and \(\phi \in L^1[0, 1], \phi\) may be singular at \(t = 0\)
and/or 1. However, by assuming that \(f\) has the following increasing condition
\[
\sum_{i=0}^{p-1} h_i(u_i) \leq f(t, u_0, . . . , u_{p-1}) \leq \sum_{i=0}^{p-1} g_i(u_i) + r(u)
\]
(1.2)
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on \([0, 1] \times (0, \infty)^p\) with \(h_i > 0\) continuous and non-increasing on \((0, \infty)\) for each \(i = 0, \ldots, p - 1\), \(g_i > 0\) continuous and non-increasing on \((0, \infty)\) for each \(i = 0, \ldots, p - 1\), and \(r \geq 0\) continuous, nondecreasing on \([0, \infty)\), here \(|u| = \max\{u_0, u_1, \ldots, u_{n-1}\}\).

\[
\int_0^1 \phi(s) g_i(k_i s^{p-1}) ds < \infty \text{ for each } i = 0, \ldots, p - 1, \quad \text{(1.3)}
\]

here \(k_i > 0(i = 0, \ldots, p - 1)\) are constant, and

\[
\text{if } z > 0 \text{ satisfies } z \leq a_0 + b_0 r(z) \text{ for constants } a_0 \geq 0 \text{ and } b_0 \geq 0, \quad \text{(1.4)}
\]

then there exists a constant \(K\) (which may depend only on \(a_0\) and \(b_0\)) with \(z \leq K\).

The authors obtain an existence result. In fact, condition (1.4) implies the degree of variable \(u\) in the term \(r(u)\) must be smaller than 1.

In [6], the singular problem \((-1)^n x^{(2n)}(t) = \mu f(t, x, \ldots, x^{(2n-2)}), x^{(2j)}(0) = x^{(2j)}(T) = 0, (0 \leq j \leq n - 1), \max|x(t)|: 0 \leq t \leq T| = A \text{ depending on the parameter } \mu\) is considered. The existence of at least one positive solution was obtained under the assumption

\[
f(t, x_0, \ldots, x_{n-2}) \leq \phi(t) + \sum_{j=0}^{2n-2} q_j(t) \omega_j(|x_j|) + \sum_{j=0}^{2n-2} h_j(t)|x_j|^\alpha_j
\]

for a.e. \(t \in J\) and for each \((x_0, \ldots, x_{2n-2}) \in D\), where \(\phi, h_j \in L_1(J)\) and \(q_j \in L_\infty(J)\) are nonnegative, \(\omega_j: R_+ \to R_+\) are non-increasing, \(\alpha_j \in (0, 1)\).

Motivated by the above results, we consider the right focal boundary value problem in the following form

\[
(-1)^{n-p}(\Phi_m(x^{(n-1)}))' (t) = \mu f(t, x(t), \ldots, x^{(n-1)}(t)), \quad 0 < t < 1, \quad \text{(1.5)}
\]

\[
x^{(i)}(0) = 0, \quad i = 0, 1, \ldots, p - 1, \quad x^{(i)}(1) = 0, \quad i = p, p + 1, \ldots, n - 1. \quad \text{(1.6)}
\]

Together with the boundary conditions (1.6), we discuss the condition

\[
\max\{|x(t)|: t \in J\} = A, \quad \text{(1.7)}
\]

where \(\Phi_m x := |x|^{m-2}x, m > 1, \Phi_m\) is the inverse operator of \(\Phi_m\), where \(\frac{1}{m_1} + \frac{1}{m_2} = 1, n \geq 2\).

Let \(J = [0, 1], R_- = (-\infty, 0), R_+ = (0, \infty), R_0 = R \setminus \{0\},\)

\[
D = \begin{cases}
\prod_{i=0}^{p} R_+ \times \cdots \times R_+ \times R_- \times \cdots \times R_+, & n - p = 2k + 1, \\
\prod_{i=0}^{p} R_+ \times \cdots \times R_+ \times R_- \times \cdots \times R_-, & n - p = 2k.
\end{cases}
\]

Nonlinearity term \(f\) satisfies local Carathéodory conditions on \(J \times D(f \in \text{Car}(J \times D))\) and may be singular at the zero value of all its phase variables. By using Leray-Schauder
Let $A \in \mathbb{R}^+$. By a solution of BVP (1.5)-(1.7) we understand a function $x \in AC^{n-1}(J)$ (i.e., $x$ has an absolutely continuous $(n-1)$st derivative on $J$) such that 
(i) $x^{(i)}(t) > 0$ on $(0,1)$ for $i = 0,\ldots,p-1$ and $(-1)^{2n-p-1}x^{(i)}(t) > 0$ on $[0,1)$ for $i = p,\ldots,n-1$. 
(ii) $x$ satisfies boundary conditions (1.6), (1.7), 
(iii) there exists $\mu_A \in R_+$ such that $x$ fulfills (1.5) with $\mu = \mu_A$ for a.e. $t \in J$.

By a solution of BVP (1.5), (1.6) we understand a function $x \in AC^{2n-1}(J)$ such that $x^{(i)}(t) > 0$ on $(0,1)$ for $i = 0,\ldots,p-1$ and $(-1)^{n-i+1}x^{(2n-p-1)}(t) > 0$ on $[0,1)$ for $i = p,\ldots,n-1$, $x$ satisfies boundary conditions (1.6) and (1.5) holds a.e. $t \in J$.

The purpose of this paper is to give conditions which guarantee the existence of a solution of BVP (1.5)-(1.7) for each given $A \in R_+$.

From now on, $\|x\| = \max\{|x(t)| : t \in J\}$, $\|x\|_1 = \int_0^1 |x(t)|dt$ and $\|x\|_\infty = ess \sup\{ |x(t)| : 0 \leq t \leq 1\}$ stands for the norm in $C^0(J)$, $L_1(J)$, and $L_\infty(J)$, respectively. For any measurable set $M \subset R$, $\mu(M)$ denotes the Lebesgue measure of $M$.

The assumptions imposed upon the function $f$ in (1.5) are listed as follows:

(H1) $f \in Car(J \times D)$ and there exists nonnegative functions $\phi \in L_1(J), q_i \in L_\infty(J)$, and continuous functions $g_i : [0,1] \times R^n \rightarrow R^+$ $(i = 0,\ldots,n-1)$ and non-increasing nonnegative continuous function $\omega_i : R_+ \rightarrow R_+$ such that for $(t,x) \in J \times D$,
\[
f(t,x_0,\ldots,x_{n-1}) = \phi(t) + \sum_{i=0}^{n-1} q_i(t)\omega_i(|x_i|) + \sum_{i=0}^{n-1} g_i(t,x_i),
\]
where
\[
\lim_{|x_i| \to -\infty} \sup_{t \in [0,1]} \frac{g_i(t,x_i)}{(\Phi_m(|x_i|))^{k_i}} = \alpha_i \geq 0, \quad k_i \text{ are any constants in } (0,1), i = 0,\ldots,p-1,
\]
and
\[
\lim_{|x_i| \to -\infty} \sup_{t \in [0,1]} \frac{g_i(t,x_i)}{(\Phi_m(|x_i|))^{k_i}} = \beta_i \geq 0, \quad i = p,\ldots,n-1,
\]
and $\omega_i$ satisfies
\[
\int_0^1 \omega_i(s^{p-i})ds < \infty, \quad 0 \leq i \leq p-1, \quad \int_0^1 \omega_i(P_i(s))ds < \infty, \quad p \leq i \leq n-1,
\]
where
\[
P_i(t) = \frac{1}{(n-2-i)!} \int_0^1 (\theta - t)^{n-2-i} \Phi_{m'} \left( \int_0^1 \phi(r)dr \right) d\theta,
\]
and there exists $\lambda > 0$ such that
\[
\omega_i(xy) \leq \lambda \omega_i(x)\omega_i(y) \text{ for } x, y \in (0,\infty).
\]
The paper is organized as follows. Section 2 presents the \( \textit{priori} \) bound of BVP (1.5)-(1.7). Besides, we prove that some sets of functions containing solutions of our auxiliary regular BVPs are uniformly absolutely continuous on \( J \). Section 3 deals with auxiliary regular BVPs of problem (1.5), (1.6), (1.7). First we prove the existence of solution by applying the Borsuk antipodal theorem and the Leray-Schauder degree (see, e.g. [12]). Then we prove the existence of solution for problem (1.5), (1.6), (1.7). Proof is based on the Arzelà-Ascoli theorem and the Vitali’s convergence theorem, see, e.g. [11, 12, 14].

2. Auxiliary Results

Lemma 2.1. If \( y \) is a solution of BVP (1.5), (1.6), then \( y(t) \) is a fixed point of the operator

\[
(Ty)(t) = (-1)^{n-p-1} \int_0^1 G(t,s) \Phi_m' \left( \int_s^1 f(\theta, y(\theta), \ldots, y^{(n-1)}(\theta)) \, d\theta \right) \, ds, \tag{2.1}
\]

where \( G(t,s) \) is the Green’s function of the following BVP

\[
\begin{aligned}
& x^{(n-1)}(t) = 0, \quad t \in (0,1), \\
& x^{(i)}(0) = 0, \quad i = 0, \ldots, p-1, \quad x^{(i)}(1) = 0, \quad i = p, \ldots, n-2,
\end{aligned}
\]

and \( G(t,s) \) can be expressed as

\[
G(t,s) = \frac{1}{(n-2)!} \begin{cases} 
\sum_{i=0}^{p-1} \binom{n-2}{i} t^i (-s)^{n-i-2}, & 0 \leq s \leq t \leq 1; \\
- \sum_{i=p}^{n-2} \binom{n-2}{i} t^i (-s)^{n-i-2}, & 0 \leq t \leq s \leq 1.
\end{cases}
\]

Furthermore,

\[
(-1)^{n-p-1} \frac{\partial^i}{\partial t^i} G(t,s) \geq 0, \quad i = 0, \ldots, p-1,
\]

\[
(-1)^{n-i-1} \frac{\partial^i}{\partial t^i} G(t,s) \geq 0, \quad i = p, \ldots, n-2, \quad (t,s) \in J \times J. \tag{2.2}
\]

Proof. By integrating the equation in (1.5) from \( t \in [0,1) \) to 1 and using \( x^{(n-1)}(1) = 0 \), we obtain that

\[
(-1)^{n-p} \Phi_m \left( y^{(n-1)}(t) \right) = - \int_t^1 f(\theta, y(\theta), \ldots, y^{(n-1)}(\theta)) \, d\theta,
\]

i.e.

\[
y^{(n-1)}(t) = (-1)^{n-p} \Phi_m' \left( \int_t^1 f(\theta, y(\theta), \ldots, y^{(n-1)}(\theta)) \, d\theta \right).
\]
By [15] we have the result is true.

**Remark 2.1.** It follows from (2.1) (2.2) that
\[
\begin{cases}
y^{(i)}(t) > 0, & i = 0, \ldots, p - 1, \ t \in (0, 1) \\
-(-1)^{i-p}y^{(i)}(t) > 0, & i = p, \ldots, n - 1, \ t \in [0, 1).
\end{cases}
\tag{2.3}
\]

As in [5], for each \( m \in \mathbb{N} \), define \( X_m, \varphi_m \in C^0(\mathbb{R}), R_m \subset \mathbb{R} \) and \( f_m \in \text{Car}(J \times \mathbb{R}^n) \) by the formulas
\[
X_m(u) = \begin{cases}
u, & \text{for } u \geq \frac{1}{m}, \\
\frac{1}{m}, & \text{for } u < \frac{1}{m},
\end{cases}
\quad \varphi_m(u) = \begin{cases}-\frac{1}{m}, & \text{for } u > -\frac{1}{m}, \\
u, & \text{for } u \leq -\frac{1}{m},
\end{cases}
\]
\[
\tau_m(u) = \begin{cases}X_m(u), & \text{for } n - p = 2k + 1, \\
\varphi_m(u), & \text{for } n - p = 2k,
\end{cases}
\]
and
\[
f_m(t, x_0, \ldots, x_{n-1}) = \phi(t) + \sum_{i=0}^{p-1} q_i(t)\omega_i(|X_m(x_i)|) + q_p(t)\omega_p(|X_m(x_p)|) + q_{p+1}(t)\omega_{p+1}(|\varphi_m(x_{p+1})|)
\]
\[+ \cdots + q_{n-1}(t)\omega_{n-1}(|\tau_m(x_{n-1})|) + \sum_{i=0}^{n-1} g_i(t, x_i)
\]
for \((t, x_0, \ldots, x_{n-1}) \in J \times \mathbb{R}^n\). Hence
\[
0 < \phi(t) \leq f_m(t, x_0, \ldots, x_{n-1})
\leq \phi(t) + \sum_{i=0}^{n-1} q_i(t)\omega_i(|x_i|) + \sum_{i=0}^{n-1} g_i(t, x_i)
\tag{2.4}
\]
for a.e. \( t \in J \) and each \((x_0, \ldots, x_{n-1}) \in \mathbb{R}^n\).

Consider auxiliary regular differential equation
\[
(\Phi_m x^{(n-1)})'(t) = \mu f_m(t, x(t), \ldots, x^{(n-1)}(t))
\tag{2.5}
\]
depending on the parameters \( \mu \in \mathbb{R} \) and \( m \in \mathbb{N} \).

**Lemma 2.2.** Let \( m \in \mathbb{N} \), then
\[
x^{(i)}(t) \geq t^{p-i-1}\Gamma, \ i = 0, \ldots, p - 1; \quad (-1)^{2n-p-i}x^{(i)}(t) \geq P_i(t), \ i = p, \ldots, n - 1, \tag{2.6}
\]
on \( J \) for any solution \( x \) of BVP (2.6), (1.6), where \( \Gamma = (-1)^{n-p-1} \int_0^1 G(1, s)\Phi_m \left( \int_s^1 \phi(\theta)d\theta \right) ds \).
Proof. By [2] we have
\[ x^{(i)}(t) \geq t^{p-i}x^{(1)}(1) \text{ for } t \in J, \ i = 0, \ldots, p - 1. \]  
(2.7)

Applying the inequality \( \|x^{(i)}\| \geq \|x\|, i = 0, \ldots, n - 1 \), and (2.1), (2.2) to (2.7) we get
\[ x^{(i)}(t) \geq t^{p-i}x(1) \geq -(1)^{n-p-1}t^{p-i} \int_0^1 G(1, s)\Phi_{m'} \left( \int_s^1 \mu \phi(\theta)d\theta \right) ds \]
\[ = \Phi_{m'}(\mu) t^{p-i} \Gamma \]
for \( t \in J, \ i = 0, \ldots, p - 1 \).

On the other hand, by (2.3) we have \((-1)^{n-p}x^{(n)}(t) \geq \phi(t), t \in J\). Integrating the above inequality from \( t \) to 1, we get step by step
\[ (-1)^{2n-p-1}x^{(i)}(1) \geq \int_t^1 \frac{(\theta - t)^{n-i-2} \Phi_{m'} \left( \int_0^1 \phi(r)dr \right) d\theta}{(n - i - 2)!} = P_i(t), \ i = p, \ldots, n - 1. \]

Lemma 2.3. Suppose that assumption \((H_1)\) is satisfied, \( m \in \mathbb{N} \) and \( A \in \mathbb{R}^+ \). Denote \( \mu_* = \Phi_m \left( \frac{1}{\Gamma} \right) \). Then there is no solution in BVP (2.5), (1.6), (1.7) for \( \mu > \mu_* \).

Proof. Suppose \( x(t) \) is a solution of BVP (2.5), (1.6), (1.7). By (2.1) and (2.4) we have
\[ x(t) = (-1)^{n-p-1} \int_0^1 G(t, s)\Phi_{m'} \left( \mu \int_s^1 f_m(r, x(r), \ldots, x_{n-1}(r))dr \right) ds \]
\[ \geq (-1)^{n-p-1} \int_0^1 G(t, s)\Phi_{m'} \left( \mu \int_s^1 \phi(r)dr \right) ds, \]
i.e.
\[ \max \{x(t) : t \in J\} \geq (-1)^{n-p-1} \Phi_{m'}(\mu) \max_{t \in J} \int_0^1 G(t, s)\Phi_{m'} \left( \int_s^1 \phi(r)dr \right) ds \]
\[ > (-1)^{n-p-1} \Phi_{m'}(\mu_*) \int_0^1 G(1, s)\Phi_{m'} \left( \int_s^1 \phi(r)dr \right) ds \]
\[ = \Phi_{m'}(\mu_*) \Gamma = A, \]
which contradicts to (1.7).

Lemma 2.4. Suppose \( 0 < u \in L^1[0, T], 0 \leq \psi \in L^\infty[0, T] \) and
\[ u(t) \leq K + \int_t^T u(s)\psi(s)ds, t \in [0, T], K > 0. \]

Then \( u(t) \leq K \exp \int_t^T \psi(s)ds, \forall t \in [0, T]. \)
Proof. Let $G(t) = K + \int_t^T u(s) \psi(s) ds$, then $G'(t) = -u(t)\psi(t) \geq -\psi(t)G(t)$, i.e.

\[
\frac{G'(t)}{G(t)} \geq -\psi(t).
\]

Integrating the above inequality from $t$ to $T$ we have

\[
\ln K - \ln G(t) \geq -\int_t^T \psi(s) ds,
\]

i.e. $G(t) \leq K \exp \int_t^T \psi(s) ds$. Then

\[
u(t) \leq G(t) \leq K \exp \int_t^T \psi(s) ds.
\]

Lemma 2.5. Let assumption $(H_1)$ be satisfied and $A \in \mathbb{R}^+$. Then there exists a positive constant $P$ depending only on $A$ such that for any solution $x$ of BVP (2.5), (1.6) with a $\mu \in \mathbb{R}^+ \supseteq \Phi_m(\lambda)\mu_*$ and so $\mu \leq \mu_*$.

Following we will show $\|x^{(j)}\| \leq P$, $j = 0, \ldots, n - 1$. We finish the proof by three steps.

Step 1. It follows from boundary condition that

\[
x^{(i)}(t) = \int_0^t \frac{(t-s)^{p-i-1}}{(p-i-1)!} \left( \int_s^1 \frac{(\theta-s)^{n-p-2}}{(n-p-2)!} |x^{(n-1)}(\theta)| d\theta \right) ds, \quad i = 0, \ldots, p - 1. (2.10)
\]

\[
(-1)^2n-p-i x^{(i)}(t) = \int_t^1 \frac{(\theta-t)^{n-i-2}}{(n-i-2)!} |x^{(n-1)}(\theta)| d\theta, \quad i = p, \ldots, n - 2. (2.11)
\]

It follows from (2.11) that

\[
|x^{(i)}(t)| \leq \frac{(1-t)^{n-i-1}}{(n-i-1)!} |x^{(n-1)}(t)|, \quad i = p, \ldots, n - 1. (2.12)
\]

From (2.10) we have

\[
\|x^{(i)}\| \leq \frac{1}{(p-i-1)!(n-p-1)!} \|x^{(n-1)}\|, \quad i = 0, \ldots, p - 1. (2.13)
\]

Step 2. Prove $|x^{(n-1)}(t)| \leq P$, $t \in [0, 1]$. 

\[
\mu \leq \mu_* \text{ and } \|x^{(j)}\| \leq P \text{ for } 0 \leq j \leq n - 1, (2.9)
\]

where $\mu_*$ is defined in Lemma 2.3.
For any small $\varepsilon > 0$, there is $\delta > 0$ so that
\[ |g_i(t, x_i)| < (\alpha_i + \varepsilon)(\Phi_m(|x_i|))^{k_i} \]
uniformly for $t \in [0, 1]$, $k_i \in (0, 1)$ and $|x_i| > \delta, i = 0, \ldots, p - 1$,
and
\[ |g_i(t, x_i)| < (\beta_i + \varepsilon)\Phi_m(|x_i|) \]
uniformly for $t \in [0, 1]$, and $|x_i| > \delta, i = p, \ldots, n - 1$.

Let, for $i = 0, \ldots, n - 1$,
\[
\Delta_{1,i} = \{ t : t \in [0, 1], |x_i(t)| \leq \delta \},
\]
\[
\Delta_{2,i} = \{ t : t \in [0, 1], |x_i(t)| > \delta \},
\]
\[ g_{\delta,i} = \max_{t \in [0, 1], |x_i(t)| \leq \delta} g_i(t, x_i). \]

For some $m > 0, t \in [0, 1],$
\[
(-1)^{n-p}(\Phi_m x^{(n-1)}(t)) = \mu f_m(t, x(t), \ldots, x_{n-1}(t)). \number{2.14}
\]

Integrating the above equality from $t$ to 1, noticing Lemma 2.2, (1.9), (1.10), (2.4) and (2.12) (2.13) we have
\[
\Phi_m(|x^{(n-1)}(t)|) \leq \mu* \int_0^1 \left[ \phi(s) + \sum_{i=0}^{p-1} q_i(s)\omega_i(s^{p-1}\Gamma) + \sum_{i=p}^{n-1} q_i(s)\omega_i(P_i(s)) \right] ds
\]
\[ + \sum_{i=0}^{n-1} \int_{\delta_i, i \cap [0,1]} g_i(s, x(s))ds + \sum_{i=0}^{n-1} \int_{\delta_i, i \cap [0,1]} g_i(s, x^{(i)}(s))ds \]
\[ \leq \mu* \left[ A + \sum_{i=0}^{n-1} g_{\delta,i} + \sum_{i=0}^{n-1} \int_{\delta_i, i \cap [0,1]} g_i(s, x^{(i)}(s))ds \right] \]
\[ \leq \mu* \left[ A + \sum_{i=0}^{n-1} g_{\delta,i} + \sum_{i=0}^{p-1} (\alpha_i + \varepsilon) \left( \Phi_m \left( \frac{|x^{(n-1)}(0)|}{(p-i-1)!(n-p-1)!} \right) \right)^{k_i} \right. \]
\[ + \int_0^1 \sum_{i=p}^{n-1} (\beta_i + \varepsilon)\Phi_m \left( \left( \frac{1-s}{n-i-1} \right)^{n-i-1} \right) \Phi_m(|x^{(n-1)}(s)|)ds \]
\[ \left. + \int_0^1 \sum_{i=0}^{p-1} ||q_i||_{\infty} \lambda_i (s^{p-1}) \omega_i(\Gamma) + \sum_{i=p}^{n-1} ||q_i||_{\infty} \omega_i(P_i(s)) \right] ds, \]
i.e.
\[
|\Phi_m(x^{(n-1)}(t))| \leq \left( C + D \left( \Phi_m(|x^{(n-1)}(0)|) \right)^{k_i} \right) + \int_0^1 E(s)\Phi_m(|x^{(n-1)}(s)|)ds.
\]

where
\[
\Lambda = \int_0^1 \left[ \phi(s) + \sum_{i=0}^{p-1} ||q_i||_{\infty} \lambda_i (s^{p-1}) \omega_i(\Gamma) + \sum_{i=p}^{n-1} ||q_i||_{\infty} \omega_i(P_i(s)) \right] ds,
\]
\[ C = 2\mu_* \left[ \Lambda + \sum_{i=0}^{n-1} g_\delta,i \right], \]

\[ D = 2\mu_* \sum_{i=0}^{p-1} (\alpha_i + \varepsilon) \left( \Phi_m \left( \frac{1}{(p-i-1)!(n-p-1)!} \right) \right)^{k_i}, \]

\[ E(t) = 2\mu_* \sum_{i=p}^{n-1} (\beta_i + \varepsilon) \Phi_m \left( \frac{(1-t)^{n-i-1}}{(n-i-1)!} \right). \]

By Lemma 2.4 and keep in mind \( k_i \in (0, 1) \), so there exists \( P \) (which does not independent on \( \lambda \)) such that \[ |x^{(n-1)}(0)| = \|x^{(n-1)}\| \leq P. \]

**Step 3.** Prove \( \|x^{(i)}\| \leq P \) for \( i = 0, 1, \ldots, n - 1 \).

By (2.12) (2.15) and Step 2, we have

\[ \|x^{(i)}\| \leq \frac{P}{(n-i-1)!} \leq P, \quad \text{for} \quad i = p, \ldots, n - 2, \]

\[ \|x^{(i)}\| \leq \frac{P}{(n-p-1)!(p-i-1)!} \leq P, \quad \text{for} \quad i = 0, \ldots, p - 1. \]

Thus \( \|x^{(i)}\| \leq P \) for \( i = 0, 1, \ldots, n - 1 \).

**Lemma 2.6.** Let assumption \( (H_1) \) be satisfied and \( A \in R_+ \). Let BVP (2.5), (1.6), (1.7) has a solution \( x_m \) for each \( m \in N \) with \( \mu = \mu_m \) in (2.5). Then the sequence \[ \{\mu_m f_m(t, x_m(t), \ldots, x_m^{(n-1)}(t))\} \subset L_1(J) \]

is uniformly absolutely continuous on \( J \), that is for each \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that

\[ \mu_m \int_{\mathcal{M}} f_m(t, x_m(t), \ldots, x_m^{(n-1)}(t))dt < \varepsilon \]

for any measurable set \( \mathcal{M} \subset J, \mu(\mathcal{M}) < \delta. \)

**Proof.** With respect to (2.5) and properties of measurable sets, it is sufficient to verify that for every \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that for any at most countable set \( \{(a_j, b_j)\}_{j \in J} \) of mutually disjoint intervals \( \{(a_j, b_j)\}_{j \in J} \) with \( \sum_{j \in J} (b_j - a_j) < \delta \), we have for each \( m \in N \),

\[ \sum_{j \in J} \int_{a_j}^{b_j} \left[ \phi(t) + \sum_{i=0}^{n-1} q_i(t)\omega_i(|x_m^{(i)}|) + \sum_{i=0}^{n-1} g_i(t, x_m^{(i)}(t)) \right] dt < \varepsilon. \quad (2.15) \]

By Lemma 2.2 we have

\[ x_m^{(i)}(t) \geq t^{p-i-1} \Phi(t), \quad i = 0, \ldots, p - 1, \quad t \in J, \]

\[ |x_m^{(i)}(t)| \geq P_i(t), \quad i = p, \ldots, n - 1, \quad t \in J. \quad (2.16) \]
In addition by Lemma 2.4
\[ \|x_m^{(j)}\| \leq P, \quad i = 0, \ldots, p - 1. \] (2.17)

From (1.12), (2.16), (2.17) we have
\[ \sum_{j \in J} \int_{a_j}^{b_j} \left[ \phi(t) + \sum_{i=0}^{n-1} q_i(t) \omega_i(|x_m^{(j)}|) + \sum_{i=0}^{n-1} g_i(t, x^{(i)}(t)) \right] dt \]
\[ \leq \sum_{j \in J} \int_{a_j}^{b_j} \left[ \phi(t) + \sum_{i=0}^{p-1} q_i(t) \omega_i(t^{p-i}) \omega_i(\Gamma) + \sum_{i=p}^{n-1} q_i(t) \omega_i(P_i(t)) + \sup_{(t,x) \in [0,1] \times [0,P]} g_i(t, x_i) \right] dt. \]

By (H_1), we know that \( \phi, h_j \in L_1(J), q_i \in L_\infty(J), \int_0^1 \omega_i(t^{p-i}) dt < \infty, i = 0, \ldots, p - 1, \int_0^1 \omega_i(P_j(s)) ds < \infty, j = p, \ldots, n - 1. \) Consequently, for each \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for any at most countable set \( \{(a_j, b_j)\} \in J \) of mutually disjoint intervals \( (a_j, b_j) \subset J \) with \( \sum_{j \in J} (b_j - a_j) < \delta. \) So (2.17) holds.

3. Existence results

**Theorem 3.1.** Suppose that the assumption (H_1) is satisfied and \( A \in R_+. \) Then for each \( m \in N \) there exists a solution \( x_m \) of BVP (2.5), (1.6), (1.7) with \( \mu = \mu_m \) in (2.5), and
\[ \|x_m^{(j)}\| \leq P \quad \text{for } m \in N, j = 0, \ldots, n - 1 \] (3.1)
and
\[ 0 < \mu_m \leq \mu*. \] (3.2)

**Proof.** Fix \( m \in N. \) Set
\[ \Omega = \{(x, \mu) : (x, \mu) \in C^n(J) \times R, \|x^{(j)}\| < P + 1 \text{ for } j = 0, \ldots, n - 1, |\mu| < \mu* + 1\}. \]

Then \( \Omega \) is a bounded, open and symmetric with respect to \((0,0)\) subset of the Banach space \( C^n(J) \times R \) endowed with the norm \( \|(x, \mu)\| = \sum_{i=0}^{n-1} \|x^{(i)}\| + |\mu|. \) Define the operator \( F_1 : \Omega \rightarrow C^n(R) \times R \) by
\[ F_1(x, \mu) = \left( (-1)^{n-p-1} \int_0^1 G(t, s) \Phi_m' \left( \mu \int_s^1 f_m(r, x(r), \ldots, x^{(n-1)}(r)) dr \right) ds, \right. \]
\[ \left. \max\{x(t) : t \in J\} + \min\{x(t) : t \in J\} + \mu, \right) \]
where \( G \) is defined in Lemma 2.1. We first show that
\[ D(I - F_1, \Omega, 0) \neq 0, \] (3.3)
where $D$ stands for the Leray-Schauder degree and $I$ is the identity operator on $C^n(J) \times R$. To prove (3.3) we define the operator $H : [0, 1] \times \overline{\Omega} \rightarrow C^{n-1}(J) \times R$,

$$H(\lambda, x, \mu) = \left( -1 \right)^{n-p-1} \int_0^1 G(t, s) \Phi_{m'} \left[ \mu(1-\lambda) + \mu \lambda \int_s^1 f_m(r, x(r), \ldots, x^{(n-1)}(r))dr \right] ds,$$

$$\lambda \left[ \max \{x(t) : t \in J\} + \min \{x(t) : t \in J\} \right] + (1 - \lambda)x(1/2) + \mu.$$

Then

$$H(0, -x, -\mu) = \left( -1 \right)^{n-p-1} \int_0^1 G(t, s) \Phi_{m'}(\mu)ds, -x(1/2) - \mu = -H(0, x, \mu)$$

for $(x, \mu) \in \overline{\Omega}$ and so $H$ is an odd operator. Due to the fact that $f_m \in Car(J \times R^{n-1})$, $H$ is a compact operator. Assume that $H(\lambda_0, x_0, \mu_0) = (x_0, \mu_0)$ for some $\lambda_0 \in [0, 1]$ and $(x_0, \mu_0) \in \partial\Omega$. Then

$$x_0(t) = (-1)^{n-p-1} \int_0^1 G(t, s) \Phi_{m'} \left[ \mu(1-\lambda_0) + \mu \lambda_0 \int_s^1 f_m(r, x_0(r), \ldots, x_0^{(n-1)}(r))dr \right] ds$$

for $t \in J$ and

$$\lambda_0 \left[ \max \{x_0(t) : t \in J\} + \min \{x_0(t) : t \in J\} \right] + (1 - \lambda_0)x_0(1/2) = 0. \quad (3.5)$$

Also from (2.3) it follows that

$$x_0(0) = 0, x'_0(t) \geq 0, t \in J.$$

So $x_0(t) > 0$ for $t \in (0, 1)$ and $\min \{x_0(t) : t \in J\} = 0$. Therefore

$$\lambda_0 \left[ \max \{x_0(t) : t \in J\} + \min \{x_0(t) : t \in J\} \right] + (1 - \lambda_0)x_0(1/2)$$

$$= \lambda_0 \max \{x_0(t) : t \in J\} + (1 - \lambda_0)x_0(1/2) > 0,$$

contrary to (3.5). If $\mu_0 < 0$, then $x_0(t) < 0$ for $t \in (0, 1)$. By (2.3) $x_0(0) = 0, x'_0(0) \leq 0, t \in J$, so $\max \{x_0(t) : t \in J\} = 0$. Hence

$$\lambda_0 \left[ \max \{x_0(t) : t \in J\} + \min \{x_0(t) : t \in J\} \right] + (1 - \lambda_0)x_0(1/2)$$

$$= \lambda_0 \min \{x_0(t) : t \in J\} + (1 - \lambda_0)x_0(1/2) < 0,$$

contrary to (3.5). If $\mu_0 = 0$ and then (3.4) gives $x_0 = 0$. Consequently, $(x_0, \mu_0) = (0, 0)$, contrary to $(x_0, \mu_0) \in \partial\Omega$. The Borsuk antipodal theorem and the Leray-Schauder degree theory lead to $D(I - H(0, \cdot, \cdot), \Omega, 0) \neq 0$ and

$$D(I - F_1, \Omega, 0) = D(I - H(1, \cdot, \cdot), \Omega, 0) = D(I - H(0, \cdot, \cdot), \Omega, 0)$$

which implies (3.3).
Finally, let $\mathcal{F} : \overline{\Omega} \to C^n(J) \times R$ be defined by the formula

$$\mathcal{F}(x, \mu) = \left( (-1)^{n-p-1} \int_0^1 G(t, s) \Phi_{m'} \left[ \mu \int_s^1 f_m(r, x(r), \ldots, x^{(n-1)}(r)) \, dr \right] ds, \right.$$

$$\left. \max\{x(t) : t \in J\} + \min\{x(t) : t \in J\} - A + \mu \right).$$

We claim that to prove our theorem it is sufficient to verify:

$$D(I - \mathcal{F}, \Omega, 0) \neq 0.$$  \hfill (3.6)

In fact, if (3.6) is true, then there exists a fixed point $(\hat{x}, \hat{\mu}) \in \Omega$ of the operator $\mathcal{F}$. Hence

$$\hat{x}(t) = (-1)^{n-p-1} \int_0^1 G(t, s) \Phi_{m'} \left[ \hat{\mu} \int_s^1 f_m(r, \hat{x}(r), \ldots, x^{(n-1)}(r)) \, dr \right] ds$$

for $t \in J$ and

$$\max\{\hat{x}(t) : t \in J\} + \min\{\hat{x}(t) : t \in J\} = A.$$  \hfill (3.8)

Moreover, $\hat{\mu} > 0$ since in the case of $\hat{\mu} \leq 0$ (3.7) and Lemma 2.1 gives for $\hat{x}(t) \leq 0$ for $t \in J$, so $\max\{\hat{x}(t) : t \in J\} = 0$, contrary to (3.8). Therefore (see (3.7)) $\hat{x}$ is a solution of BVP (2.4), (1.6) with $\mu = \hat{\mu}$ in (2.4), and for $t \in (0, 1)$. So $\min\{\hat{x}(t) : t \in J\} = 0$. Then, by (3.8), $\max\{\hat{x}(t) : t \in J\} = A$, and we see that $\hat{x}$ is a solution of BVP (2.4), (1.6), (1.7).

In order to prove (3.6) we consider the operator $\mathcal{H} : [0, 1] \times \overline{\Omega} \to C^n(J) \times R$,

$$\mathcal{H}(\lambda, x, \mu) = \left( (-1)^{n-p-1} \int_0^1 G(t, s) \Phi_{m'} \left[ \mu \int_s^1 f_m(r, x(r), \ldots, x^{(n-1)}(r)) \, dr \right] ds, \right.$$

$$\left. \max\{x(t) : t \in J\} + \min\{x(t) : t \in J\} - \lambda A + \mu \right).$$

Then, $\mathcal{H}(1, \cdot, \cdot) = \mathcal{F}$, $\mathcal{H}(0, \cdot, \cdot) = \mathcal{F}_1$ and, by (3.3),

$$D(I - \mathcal{H}(0, \cdot, \cdot), \Omega, 0) \neq 0.$$  \hfill (3.9)

Assume that $\mathcal{H}(\lambda_1, x_1, \mu_1) = (x_1, \mu_1)$ for some $\lambda_1 \in [0, 1]$ and $(x_1, \mu_1) \in \partial \Omega$. If $\mu_1 = 0$ then from the equality

$$x_1(t) = (-1)^{n-p-1} \int_0^1 G(t, s) \Phi_{m'} \left[ \mu_1 \int_s^1 f_m(r, x_1(r), \ldots, x_1^{(n-1)}(r)) \, dr \right] ds, \quad (3.10)$$

for $t \in J$, we get $x_1 = 0$, contrary to $(x_1, \mu_1) = (0, 0) \in \partial \Omega$. let $\mu_1 < 0$. Then (see (3.10)) $x_1(t) < 0$ on $(0, 1)$, and $\max\{x_1(t) : t \in J\} = 0$, contrary to $\max\{x_1(t) : t \in J\} + \min\{x_1(t) : t \in J\} = \lambda_1 A$. hence $\mu_1 > 0$ and then $x_1$ is a solution of BVP (2.5), (1.6) with $\max\{x(t) : t \in J\} = \lambda A$. Moreover, by Lemma 2.5, $\|x_1^{(j)}\| \leq P$ for $0 \leq j \leq n-1$ and $0 < \mu \leq \mu_*$. Consequently, $(x_1, \mu_1) \notin \partial \Omega$, a contradiction. we have proved $\mathcal{F}(\lambda, x, \mu) \neq (x, \mu)$ for each $\lambda \in [0, 1]$ and $(x, \mu) \in \partial \Omega$, and since $\mathcal{H}$ is a compact homotopy,

$$D(I - \mathcal{F}, \Omega, 0) = D(I - \mathcal{H}(1, \cdot, \cdot), \Omega, 0) = D(I - \mathcal{H}(0, \cdot, \cdot), \Omega, 0).$$
and then (3.9) gives (3.6), which finishes our proof.

**Theorem 3.2.** Suppose the assumptions $(H_1)$ be satisfied and $A \in R^+$. Then there exists a solution of BVP (1.5), (1.6), (1.7) for each $A \in R^+$.

**Proof.** For each $m \in N$, there exists a solution $x_m$ of BVP (2.5), (1.6), (1.7) with a $\mu = \mu_m$ by Theorem 3.1. Consider the sequence $\{x_m\}, \{\mu_m\}$. By Lemma 2.2, Lemma 2.5, $\{x_m^{(i)}\}, \{\mu_m\}$ are bounded for $i = 0, \ldots, n - 1$.

For $t_1, t_2 \in J, t_2 < t_1$,

$$|x_m^{(n-1)}(t_1) - x_m^{(n-1)}(t_2)| = \Phi_m' \left( \left| \int_0^{t_1} \mu_m f_m(t, x_m(t), \ldots, x_m^{(n-1)}(t)) dt \right| \right) - \Phi_m' \left( \left| \int_0^{t_2} \mu_m f_m(t, x_m(t), \ldots, x_m^{(n-1)}(t)) dt \right| \right).$$

We can use Lemma 2.6 and obtain that the sequence $\mu_m f_m(t, x_m(t), \ldots, x_m^{(n-1)}(t))$ is uniformly absolutely continuous on $J$. Moreover by the continuity of $\Phi_m'$ we have $\{x_m^{(n-1)}\}_m$ is equi-continuous on $J$. The Arzalà-Ascoli theorem guarantees the existence of a subsequence, such that $\{x_{m_k}\}, \{\mu_{m_k}\}$ is convergent in $C^0(J)$ and $R$ respectively. Let $\lim_{k \to \infty} x_{m_k} = x, \lim_{k \to \infty} \mu_{m_k} = \hat{\mu}$, then $x \in C^{n-1}(J), x$ satisfies boundary condition (1.6), (1.7) and $0 \leq \hat{\mu} \leq \mu$.

We now prove $(-1)^{n-p}x^{(n-1)}(t) > 0, t \in [0, 1]$. If not, there exists $t_1 \in (0, 1)$ such that

$$(-1)^{n-p}x^{(n-1)}(t) > 0, t \in [t_1, 1), \quad (-1)^{n-p}x^{(n-1)}(t) = 0, t \in (0, t_1] \quad (3.11)$$

From (2.3) we obtain $x^{(i)}$ has at most one zero $\xi_j$ on $[0, t_1]$ for $i = 0, \ldots, p - 1$. Now from the construction of $f_{m_k} \in Car(J \times R^{n-1})$ it follows that there exists a set $A \in J, \mu(A) = 0$ such that $f_{m_k}(t, \ldots, s)$ are continuous on $R^{n-1}$ for each $t \in J \setminus A$ which implies that

$$\lim_{k \to \infty} \mu_{m_k} f_{m_k}(t, x_{m_k}(t), \ldots, x_{m_k}^{(n-1)}(t)) = \hat{\mu} f(t, x(t), \ldots, x^{(n-1)}(t))$$

for $t \in [0, t_1] \setminus A$. By Lemma 2.6 $\{\mu_{m_k} f_{m_k}(t, x_{m_k}(t), \ldots, x_{m_k}^{(n-1)}(t))\}$ is uniformly absolutely continuous on $[0, t_1]$. Then $\hat{\mu} f(t, x(t), \ldots, x^{(n-1)}(t)) \in L^1[0, t_1]$ and

$$\lim_{k \to \infty} \mu_{m_k} \int_t^{t_1} f_{m_k}(s, x_{m_k}(s), \ldots, x_{m_k}^{(n-1)}(s)) ds = \hat{\mu} \int_t^{t_1} f(s, x(s), \ldots, x^{(n-1)}(s)) ds$$

for $t \in [0, t_1]$ by Vitali’s convergence theorem. Noticing $x_{m_k}^{(n-1)}(t_1)$ is bounded, we assume it is convergent, and let $\lim_{k \to \infty} x_{m_k}^{(n-1)}(t_1) = d$. Taking the limit as $k \to \infty$ in the equality

$$x_{m_k}^{(n-1)}(t) = x_{m_k}^{(n-1)}(t_1) - (-1)^{n-k} \Phi_m' \left( \mu_{m_k} \int_t^{t_1} f_{m_k}(	au, x_{m_k}(	au), \ldots, x_{m_k}^{(n-1)}(\tau)) d\tau \right)$$
we get
\[ x^{(n-1)}(t) = d - (-1)^n k \Phi_m \left( \mu \int_t^{t_1} f(\tau, x(\tau), \ldots, x^{(n-1)}(\tau)) d\tau \right) \]

There are two cases to consider:
Case(i) If \( \hat{\mu} = 0, x^{(n-1)}(t) = 0 \) for \( t \in [0, t_1] \), and the equality \( x^{(n-1)}(t_1) = 0 \) yields \( d = 0 \). Hence \( x^{(n-1)}(t) = 0 \) for \( t \in J \), contrary to (3.10).
Case(ii) If \( \hat{\mu} > 0 \). By (2.4), we have
\[ |x^{(n-1)}(\tau)| \geq \Phi_m \left( \mu_m \int_0^1 f(\theta) d\theta \right), \quad k \in \mathbb{N} \]  
(3.12)

Letting \( k \to \infty \) in (3.11) we have
\[ |x^{(n-1)}(t)| \geq \Phi_m \left( \mu \int_0^1 f(\theta) d\theta \right), \quad t \in J \]

Hence \( |x^{(n-1)}(t)| > 0 \) for \( t \in [0, 1] \), contrary to (3.11). Thus \( |x^{(n-1)}(t)| > 0 \) for \( t \in [0, 1) \). So \( x^{(i)}(t) > 0, 0 \leq i \leq p-1 \) on \( (0, 1) \), \((-1)^{n-p-1}x^{(i)}(t) > 0, p \leq i \leq n-1 \) on \([0, 1)\). Noticing \( \{x^{(n-1)}(0)\} \) is convergent. Let \( \lim_{k \to \infty} x^{(n-1)}(0) = \hat{d} \). Since \( \{\mu_m J_{m_k} (t, x_{m_k}(t), \ldots, x^{(n-1)}_{m_k})\} \) is uniformly absolutely continuous on \( J \) and
\[ \lim_{k \to \infty} \mu_m J_{m_k} (t, x_{m_k}(t), \ldots, x^{(n-1)}_{m_k}) = \mu f(t, x(t), \ldots, x^{(n-1)}(t)). \]

By the Vitali’s Convergence theorem to get \( \hat{\mu} f(t, x(t), \ldots, x^{(n-1)}(t)) \in L_1(J) \) and letting \( k \to \infty \) in the equality
\[ x^{(n-1)}_{m_k}(t) = x^{(n-1)}_{m_k}(0) + (-1)^n k \Phi_m \left( \mu_m \int_0^t f_{m_k}(s, x_{m_k}(s), \ldots, x^{n-1}_{m_k}(s)) ds \right), \quad t \in J \]

we get
\[ x^{(n-1)}(t) = \hat{d} + (-1)^n k \Phi_m \left( \mu \int_0^t f(s, x(s), \ldots, x^{n-1}(s)) ds \right), \quad t \in J. \]  
(3.13)

If \( \hat{\mu} = 0, x^{(n-1)}(t) = \hat{d} \) for \( t \in J \) and condition \( x^{(n-1)}(1) = 0 \) gives \( \hat{d} = 0 \). So \( x^{(n-1)}(t) = 0 \) for \( t \in J \). Without loss of generality, we suppose \( \|x^{(i)}\| = x(\xi) \).
\[ \|x^{(i)}\| > \frac{x(i-1)(\xi_{i-1}) - x^{(i-1)}(0)}{\xi_{i-1}} > \|x^{(i-1)}\|, \quad 0 \leq i \leq p-1 \]

and
\[ \|x^{(i)}\| > \frac{x^{(i-1)}(1) - x^{(i-1)}(\xi_{i-1})}{1 - \xi_{i-1}} > \|x^{(i-1)}\|, \quad p \leq i \leq n-1 \]

Thus \( \|x^{(i)}\| > \|x^{(i-1)}\| \) for \( 0 \leq i \leq n-1 \). The fact \( x^{(n-1)}(t) = 0 \) for \( t \in J \) contradicts \( \|x^{(n-1)}\| > A \).
If $\hat{\mu} > 0$ and from (3.13) we see that $x \in AC^{n-1}(J)$ and $x$ satisfies (1.5) a.e. on $J$. We have proved that $x$ is a solution of BVP (1.5)-(1.7) with $\mu = \hat{\mu}$ in (1.5).

4. Example

Example 4.1. Let us consider the following fourth-order boundary value problem

$$\begin{aligned}
(f(y^{(3)}(t)))' &= \mu \left[ 1 - t + \frac{3}{i=0} q_i(t) y_i^{-\frac{1}{4}} + g_0(t) \sin(\Phi_3(y_0))^\frac{3}{2} + \frac{3}{i=1} g_i(t) \Phi_3(y_i) \right], \\
y(0) &= 0, \quad y'(1) = y''(1) = y^{(3)}(1) = 0,
\end{aligned}$$

with $\max\{y(t) : t \in [0, 1]\} = A, q_i \in L_\infty([0, 1]), g_i \in C[0, 1], m = 3, p = 1$ for $i = 0, 1, 2, 3$.

Corresponding to BVP (1.5)-(1.7) we have

$$f(t, y_0, y_1, y_2, y_3) = 1 - t + \frac{3}{i=0} q_i(t) y_i^{-\frac{1}{4}} + g_0(t) \sin(\Phi_3(y_0))^\frac{3}{2} + \frac{3}{i=1} g_i(t) \Phi_3(y_i)$$

where $\phi(t) = 1 - t, \omega_i(|y_i|) = |y_i|^{-\frac{1}{4}}, i = 0, 1, 2, 3, g_0(t, y_0) = g_0(t) \sin(\Phi_3(y_0))^\frac{1}{2}, g_i(t, y_i) = g_i(t) \Phi_3(y_i), i = 1, 2, 3$.

Then for any $A > 0$, there exists $\mu_A < \mu_* = \Phi_3\left(\frac{A}{T}\right)$ such that BVP (4.1) has a solution $y \in AC^3([0, 1])$.

To see (4.1) has a solution $y \in AC^3([0, 1])$, we apply theorem 3.2, It is easy to verify $(H_1)$

$$\lim_{|y_0| \to \infty} \sup_{t \in [0, 1]} \frac{g_0(t, y_0)}{(\Phi_3(|y_0|)))^{1/2}} = 0,$$

$$\lim_{|y_0| \to \infty} \sup_{t \in [0, 1]} \frac{g_i(t, y_i)}{(\Phi_3(|y_i|)))^{1/2}} = \sup_{t \in [0, 1]} g_i(t) \geq 0,$$

$$\int_0^1 \omega_0(s)ds < \infty,$$

$$\int_0^1 \omega_i(P_i(s))ds < \infty, \quad i = 1, 2, 3$$

hold. So applying Theorem 3.2, for any $A > 0$, there exists $\mu_A$ such that BVP (4.1) has a solution $y \in AC^3([0, 1])$.

References


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