THE \((a, d)\)-ASCENDING SUBGRAPH DECOMPOSITION

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Abstract. Let \(G\) be a graph of size \(q\) and \(a, n, d\) be positive integers for which \(\frac{q}{2} (2a + (n-1)d) \leq q < \left(\frac{n+1}{2}\right) (2a + nd)\). Then \(G\) is said to have \((a, d)\)-ascending subgraph decomposition into \(n\) parts \(((a, d) - ASD)\) if the edge set of \(G\) can be partitioned into \(n\)-non-empty sets generating subgraphs \(G_1, G_2, G_3, \ldots, G_n\) without isolated vertices such that each \(G_i\) is isomorphic to a proper subgraph of \(G_{i+1}\) for \(1 \leq i \leq n-1\) and \(|E(G_i)| = a + (i-1)d\). In this paper, we find \((a, d) - ASD\) into \(n\) parts for \(W_m\).

1. Introduction

By a graph we mean a finite undirected graph without loops or multiple edges. A wheel on \(p\) vertices is denoted by \(W_p\). A path of length \(t\) is denoted by \(P_{t+1}\). Terms not defined here are used in the sense of Harary [4]. Throughout this paper \(G \subseteq H\) means \(G\) is a subgraph of \(H\). Let \(G = (V, E)\) be a simple graph of order \(p\) and size \(q\). If \(G_1, G_2, \ldots, G_n\) are edge disjoint subgraphs of \(G\) such that \(E(G) = E(G_1) \cup E(G_2) \cup \cdots \cup E(G_n)\), then \(\{G_1, G_2, \ldots, G_n\}\) is said to be a decomposition of \(G\).

The concept of ASD was introduced by Alavi et al. [1]. The graph \(G\) of size \(q\) where \(\binom{n+1}{2} \leq q < \binom{n+2}{2}\), is said to have an ascending subgraph decomposition (ASD) if \(G\) can be decomposed into \(n\)-subgraphs \(G_1, G_2, \ldots, G_n\) without isolated vertices such that each \(G_i\) is isomorphic to a proper subgraph of \(G_{i+1}\) for \(1 \leq i \leq n-1\) and \(|E(G_i)| = i\).

We generalize this concept into \((a, d) - ASD\) as follows:

\(G\) is a simple graph of size \(q\) and \(a, n, d\) are positive integers for which \(\frac{q}{2} (2a + (n-1)d) \leq q < \left(\frac{n+1}{2}\right) (2a + nd)\). Then \((a, d)\)-ascending subgraph decomposition \(((a, d) - ASD)\) of \(G\) is the edge disjoint decomposition of \(G\) into subgraphs \(G_1, G_2, \ldots, G_n\) without isolated vertices such that each \(G_i\) is isomorphic to a proper subgraph of \(G_{i+1}\) for \(1 \leq i \leq n-1\) and \(|E(G_i)| = a + (i-1)d\).

2. Main Results

Definition 2.1. Let \(G\) be a graph of size \(q\) and \(a, n, d\) be positive integers for which \(\frac{q}{2} (2a + (n-1)d) \leq q < \left(\frac{n+1}{2}\right) (2a + nd)\). Then \(G\) is said to have \((a, d)\)-ascending subgraph decomposition into \(n\) parts \(((a, d) - ASD)\) if the edge set of \(G\) can be partitioned into \(n\) non-empty sets generating subgraphs \(G_1, G_2, \ldots, G_n\) without isolated vertices.

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such that each $G_i$ is isomorphic to a proper subgraph of $G_{i+1}$ for $1 \leq i \leq n - 1$ and $|E(G_i)| = a + (i-1)d$.

**Remark 2.2.** From the above definition, the usual ASD of $G$ coincides with $(1, 1) - ASD$ of $G$.

**Example 2.3.** Consider the Graph $G$.

![Graph G](image)

Clearly $\{G_1, G_2, G_3\}$ is a $(1, 2) - ASD$ of $G$.

**Theorem 2.4.** Let $G$ be a graph of size $q$, where $\frac{n}{2}(2a + (n-1)d) \leq q < \left(\frac{n+1}{2}\right)(2a + nd)$ for some positive integer $n$, such that $G$ has $(a, d) - ASD$ into $n$ parts, then $G$ has an $(a, d) - ASD$ into $n$ parts $G_1, G_2, \ldots, G_n$ such that each $G_i$ has size $a + (i-1)d$ for $1 \leq i \leq n - 1$ and $G_n$ has size $q - \left(\frac{n-1}{2}\right)(2a + (n-2)d)$.

**Proof.** If $q = \left(\frac{n}{2}\right)(2a + (n-1)d)$, then there is nothing to prove.

Now, suppose $\left(\frac{n}{2}\right)(2a + (n-1)d) < q < \left(\frac{n+1}{2}\right)(2a + nd)$. Suppose $G$ has $H_1, H_2, \ldots, H_n$ as $(a, d) - ASD$. If the size of $H_{n-1}$ is $a + (n-2)d$, then this decomposition has the desired properties. Therefore assume that the size of $H_{n-1}$ exceeds $a + (n-1)d$. The size of $H_1$ must exceed $a$. Select the edges $e_{11}, e_{12}, \ldots, e_{1a}$ from $H_1$, inorder to
define \( G_1 \), a subgraph of \( G \) induced by the set of edges \( \{e_{11}, e_{12}, \ldots, e_{1a}\} \). Now let \( G_2 \) be a graph induced by the edges \( e_{21}, e_{22}, \ldots, e_{2(a+d)} \) from \( H_2 \) so that \( G_1 \subset G_2 \). Since \( H_2 \) is isomorphic to a subgraph \( H_3' \) of \( H_3 \), we can choose edges \( e_{31}, e_{32}, \ldots, e_{3d} \) from \( E(H_3) - E(H_3') \) so as to define \( G_3 \), a subgraph of \( G \), induced by edges of \( E(H_3') \) and the edges \( e_{31}, e_{32}, \ldots, e_{3d} \). Then it is clear that \( |E(G_3)| = a + 2d \) and \( G_2 \subset G_3 \).

Proceeding as before, we may define the graphs \( G_1, G_2, \ldots, G_k \) \((3 \leq k \leq n - 2)\) such that \( |E(G_k)| = a + (k-1)d \) and \( G_{k-1} \subset G_k \). From the above construction, we observe that each \( G_k \) \((1 \leq k \leq n - 2)\) is a subgraph of \( H_k \). Now we construct \( G_{k+1} \) as follows:

Since \( G_k \) is isomorphic to a subgraph \( H_k' \) of \( H_k \), we choose the edges \( e_{k1}, e_{k2}, \ldots, e_{kd} \) from \( E(H_k) - E(H_k') \) such that the subgraph \( G_{k+1} \) is induced by the edges of \( E(H_k') \) and \( \{e_{k1}, e_{k2}, \ldots, e_{kd}\} \). Also note that \( |E(G_{k+1})| = a + (k-1)d + d = a + kd \). Therefore there exist graphs \( G_1, G_2, \ldots, G_{n-1} \) such that \( |E(G_i)| = a + (i-1)d \) for \( 1 \leq i \leq n - 1 \) and \( G_i \subset G_{i+1} \) for \( 1 \leq i \leq n - 2 \). Now define \( G_n \), the subgraph of \( G \) induced by the edges of \( E(G) - \bigcup_{i=1}^{n-1} E(G_i) \). Hence \( G \) has the required \((a, d) - ASD\) into \( n \) parts namely \( G_1, G_2, \ldots, G_n \). Clearly every graph does not posses \((a, d) - ASD\) into \( n \) parts. Now we wish to identify those graphs which admit \((a, d) - ASD\) into \( n \) parts.

The following number theoretical result will be useful for proving further results.

**Lemma 2.5.** Given that the numbers \( a, a + d, a + 2d, \ldots, a + (n-1)d \) are in A.P \((a, d \in \mathbb{Z})\). Then their sum is

i) \( S_n = (a - d)n + d\binom{n+1}{2} \) if \( d \leq a \) and

ii) \( S_n = a\binom{n+1}{2} + (d - a)\binom{n}{2} \) if \( d \geq a \).

**Theorem 2.6.** \( G \) admits \((a, d) - ASD\) into \( n \) parts. Then \( a = q - k, 2 \leq k \leq q - 1 \) if and only if \( d = \frac{2(nk - (n-1)q)}{n(n-1)} \).

**Proof.** Suppose \( a = q - k, 2 \leq k \leq q - 1 \).

As \( G \) admits \((a, d) - ASD\) into \( n \)-parts, we have

\[
a + (a + d) + (a + 2d) + \cdots + a + (n-1)d = q
\]

\[
n(a + d)\binom{n}{2} = q
\]

\[
n(n-1)d = 2(q - na)
\]

\[
n(n-1)d = 2(q - n(q - k)) \quad \text{as} \quad a = (q - k)
\]

\[
n(n-1)d = 2(nk - (n-1)q).
\]

Hence \( d = \frac{2(nk - (n-1)q)}{n(n-1)} \).

Conversely, suppose \( d = \frac{2(nk - (n-1)q)}{n(n-1)} > (1) \).
As $G$ admits $(a, d) - ASD$ into $n$ parts, we have
\[
a + (a + d) + (a + 2d) + \cdots + a + (n - 1)d = q
\]
\[
na + d \binom{n}{2} = q
\]
\[
a + [nk - (n - 1)q] = q \text{ by (1)}
\]
\[
n(q - a) = nk.
\]
Hence $a = q - k$.

**Corollary 2.7.** If $G$ admits $(a, d) - ASD$ into $n$ even number of parts and let $a = q - k$, $2 \leq k \leq q - 1$, then $k \equiv 0 \pmod{n - 1}$.

**Proof.** Given $a = q - k$, $2 \leq k \leq q - 1$.

By 2.6, \( n(n - 1)d = 2nk - 2(n - 1)q \)
\[
(n - 1)[nd + 2q] = 2nk
\]
\[
dl + 2q = \frac{2nk}{(n - 1)} \quad (n > 3).
\]
As \((n - 1, n) = 1\) and $n$ is even, $n - 1$ divides $k$. Therefore, $k \equiv 0 \pmod{n - 1}$.

**Observation 2.8.** If $G$ admits $(a, d) - ASD$ into $n$ parts, then $1 \leq a \leq \frac{q - \binom{n}{2}}{n}$ and $1 \leq d \leq \frac{q - n}{(n - 2)}$.

**Proof.** Suppose $G$ admits $(a, d) - ASD$ into $n$ parts. Then we have,
\[
a + (a + d) + (a + 2d) + \cdots + a + (n - 1)d = q
\]
\[
na + d \binom{n}{2} = q \quad \text{ (1)}
\]
\[
a + \binom{n}{2} \leq q \text{ as } d \geq 1, \text{ therefore } a \leq \frac{q - \binom{n}{2}}{n}.
\]
Alos from (1) and since $a \geq 1$, $n + \binom{n}{2}d \leq q$, $d \leq \frac{q - n}{(n - 2)}$.
Hence we have $1 \leq a \leq \frac{q - \binom{n}{2}}{n}$ and $1 \leq d \leq \frac{q - n}{(n - 2)}$.

**Corollary 2.9.** If $G$ admits $(a, d) - ASD$ into two parts, then $1 \leq a \leq \frac{q - 1}{2}$ and $1 \leq d \leq q - 2$.

**Corollary 2.10.** If $G$ admits $(a, d) - ASD$ into two parts and if $a = \frac{q - 1}{2}$, then $d = 1$.

**Corollary 2.11.** If $G$ admits $(a, d) - ASD$ into two parts and if $d = q - 2$, then $a = 1$. 
Corollary 2.12. If $G$ admits $(a, d) - ASD$ into two parts and let $d = q - k$ where $2 \leq k \leq q - 1$, then $k$ is even.

Proof. Since $G$ admits $(a, d) - ASD$ into 2 parts
\[
a + (a + d) = q
\]
\[
2a + d = q
\]
\[
2a + q - k = q, \text{ as } d = q - k.
\]
Therefore, $k = 2a$.

Corollary 2.13. If $G$ admits $(a, d) - ASD$ into three parts, then $1 \leq a \leq \frac{2m - 3}{3}$ and $1 \leq d \leq \frac{2m - 3}{3}$.

3. $(a, d) - ASD$ on Wheel

In this section for proving $W_m = K_1 + C_{m-1}$ $(m \geq 4)$ admits $(a, d) - ASD$ into $n$ parts, we need the following results.

Theorem 3.1. If $W_m$ admits $(a, d) - ASD$ into $n$-parts, then

a) For $n \equiv 0 \pmod{4}$,
   i) either $a \geq 1$ and $d \equiv 1 \pmod{2}$ or $a \geq 1$ and $d \equiv 0 \pmod{2}$
   ii) $m \equiv \frac{a}{2} + 1 \pmod{\frac{n^2}{2}}$ when $a \geq 1$ and $d \equiv 1 \pmod{2}$ and
   iii) $m \equiv 1 \pmod{\frac{a}{2}}$ when $a \geq 1$ and $d \equiv 0 \pmod{2}$.

b) For $n \equiv 1 \pmod{4}$,
   i) $m \equiv 1 \pmod{a}$ and ii) $a \equiv 0 \pmod{2}$.

c) For $n \equiv 2 \pmod{4}$,
   i) $m \equiv 1 \pmod{\frac{a}{2}}$ and ii) $d \equiv 0 \pmod{2}$.

d) For $n \equiv 3 \pmod{4}$,
   i) $m \equiv 1 \pmod{a}$ and ii) $a$ is even (odd) if and only if $d$ is even (odd).

Proof. Suppose $W_m$ admits $(a, d) - ASD$ into $n$-parts. Then we have,
\[
a + (a + d) + (a + 2d) + \cdots + a + (n - 1)d = q
\]
\[
\frac{n}{2}(2a + (n - 1)d) = 2(m - 1) \quad \text{as } q = 2(m - 1)
\]
\[
n(2a + (n - 1)d) = 4(m - 1) \quad > (1)
\]

Case (a): Suppose $n \equiv 0 \pmod{4}$.
Let $n = 4k$, $(k \in \mathbb{Z}^+)$.
Sub case (a)(i): Suppose $k$ is odd, then by (i) $(m - 1)$ is either odd or even.
Suppose $(m - 1)$ is odd, then $a \geq 1$ and $d \equiv 1 \pmod{2}$.
Suppose $(m - 1)$ is even, then $a \geq 1$ and $d \equiv 0 \pmod{2}$.
Sub case (a)(i)(a): Suppose \( k \) is even. 
Then \( (m - 1) \) must be even. Therefore \( d \equiv 0 \pmod{2} \) or \( d \equiv 1 \pmod{2} \).
Hence either \( a \geq 1 \) and \( d \equiv 1 \pmod{2} \) or \( a \geq 1 \) and \( d \equiv 0 \pmod{2} \).

Sub case a(ii): Suppose \( a \geq 1 \) and \( d \equiv 1 \pmod{2} \).
Let \( d = 2r + 1 \) \((r \in \mathbb{Z}^+ \cup \{0\})\). By using (1) we have,
\[
\begin{align*}
n[2a + (n - 1)d] &= 4(m - 1) \\
k[2a + (4k - 1)(2r + 1)] &= (m - 1) \text{ since } n = 4k \\
k[2a + (8kr - 2r + 4k - 2) + 1] &= m - 1 \\
2k[a + (4kr - r + 2k - 1)] &= m - (k + 1).
\end{align*}
\]
Therefore \( m \equiv k + 1 \pmod{2k} \).
Hence \( m \equiv \frac{k}{2} + 1 \pmod{2} \).

Sub case a(iii): Suppose \( a > 1 \) and \( d \equiv 0 \pmod{2} \).
Let \( d = 2r \) \((r \in \mathbb{Z}^+)\). By using (1) we have,
\[
\begin{align*}
n[2a + (n - 1)d] &= 4(m - 1) \\
k[2a + (n - 1)d] &= 4(m - 1) \text{ since } n = 4k \\
2k[a + (n - 1)r] &= m - 1.
\end{align*}
\]
Therefore \( m \equiv 1 \pmod{2k} \).
Hence \( m \equiv 1 \pmod{\frac{n}{2}} \).

Case (b): Suppose \( n \equiv 1 \pmod{4} \).
Let \( n = 4k + 1 \) \((k \in \mathbb{Z}^+)\). By using (1) we have,
\[
\begin{align*}
n[2a + (n - 1)d] &= 4(m - 1) \\
n[2a + 4kd] &= 4(m - 1) \\
n[a + 2kd] &= 2(m - 1).
\end{align*}
\]
As \( a, d \) are integers and \( n \) is odd, (b)(i) follows clearly.
As \( n \) is odd, (b)(ii) follows clearly.

Case (c): Suppose \( n \equiv 2 \pmod{4} \).
Let \( n = 4k + 2 \) \((k \in \mathbb{Z}^+)\). By using (1) we have,
\[
\begin{align*}
n[2a + (n - 1)d] &= 4(m - 1) \\
(4k + 2)[2a + (n - 1)d] &= 4(m - 1) \\
2(m - 1) &= (2k + 1)\ell \text{ where } \ell = 2a + (n - 1)d.
\end{align*}
\]
The above equation is true only when \( \ell \) is even. Then (c)(i) follows. Further, since \( \ell \) is even and \( n \) is even, then (c)(ii) follows.
Case (d): Suppose \( n \equiv 3 \pmod{4} \).

Let \( n = 4k + 3 \) \((k \in \mathbb{Z}^+ \cup \{0\})\), By using (1) we have,

\[
\begin{align*}
    n(2a + (n - 1)d) &= 4(m - 1) \\
    n(2a + (4k + 2)d) &= 4(m - 1) \\
    n(a + (2k + 1)d) &= 2(m - 1).
\end{align*}
\]

As \( a, d \) are integers and \( n \) is odd, then (d)(i) follows clearly.

As \( n \) is odd, (d)(ii) follows clearly.

**Theorem 3.2.** If \( W_m \) admits \((a, d)\)-ASD into \( n\)-parts, then \( 1 \leq a \leq \frac{q - n}{2} \) and \( 1 \leq d \leq \frac{q - n}{n(n-2)}. \)

**Proof.** Suppose \( W_m \) admits \((a, d)\)-ASD into \( n\)-parts. Then by 2.8, we have \( 1 \leq a \leq \frac{q - n}{2} \) and \( 1 \leq d \leq \frac{q - n}{n(n-2)}. \)

**Theorem 3.3.** \( W_m \) admits \((a, d)\)-ASD into \( n\)-parts if and only if

a) For \( n \equiv 0 \pmod{4} \),
   i) either \( a \geq 1 \) and \( d \equiv 1 \pmod{2} \) or \( a \geq 1 \) and \( d \equiv 0 \pmod{2} \).
   ii)\( a \equiv 0 \pmod{2} \) and \( b) m \geq \frac{n(n+1)}{4} + 1 \) when \( a \geq 1 \) and \( d \equiv 1 \pmod{2} \).
   iii)\( a \equiv 1 \pmod{2} \) and \( b) m \geq \frac{n^2}{2} + 1 \) when \( a \geq 1 \) and \( d \equiv 0 \pmod{2} \).

b) For \( n \equiv 1 \pmod{4} \),
   i) \( m \equiv 1 \pmod{4} \), ii) \( a \equiv 0 \pmod{2} \) and iii) \( m \geq \frac{n(n+1)}{4} + 1 \).

c) For \( n \equiv 2 \pmod{4} \),
   i) \( m \equiv 1 \pmod{4} \), ii) \( d \equiv 0 \pmod{2} \) and iii) \( m \geq \frac{n^2}{2} + 1 \).

d) For \( n \equiv 3 \pmod{4} \),
   i) \( m \equiv 1 \pmod{4} \), ii) \( a \text{ and } d \text{ are both even or both odd} \) and iii) \( m \geq \frac{n(n+1)}{4} + 1 \).

**Proof.** The proof of the necessary part follows from 3.1. Conversely,

Let \( V(W_m) = \{v_1, v_2, \ldots, v_m\} \) and \( E(W_m) = \{(v_i, v_{i+1}) | 1 \leq i \leq m - 1\} \cup \{(v_m, v_1) | 1 \leq i \leq m - 1\}. \)

Define \( L_i = (v_i, v_{i+1}) \cup (v_m, v_i), \quad 1 \leq i \leq m - 1. \)

Case (a): Let \( n \equiv 0 \pmod{4} \).

Subcase (a)(i): Suppose \( a \) and \( d \) are even.
\[ a = 4, \; d = 2 \]

\[
\begin{array}{c}
\text{Figure 3.1.} \\
\end{array}
\]

Define \( G_1 = a^2 \bigcup_{i=1}^{L_i} \) and for \( 2 \leq j \leq n \), \( G_j = \left\{ \begin{array}{c}
\frac{j-1}{2} \sum_{k=0}^{j-1} (a + kd) \\
\frac{j-2}{2} \sum_{k=0}^{j-2} (a + kd) + 1 \\
\end{array} \right\} \cup \bigcup_{i=1}^{L_i} L_i \}

Clearly \( G_j \subset G_{j+1} \) for \( 1 \leq j \leq n - 1 \).
Therefore \( G_1, G_2, \ldots, G_n \) is an \((a, d) - ASD\) into \( n \)-parts of \( W_m \).

Subcase(a)(ii): Suppose \( a \) and \( d \) are odd.
Define when \( a = 1, \; d = 1 \), \( G_1 = (v_m, v_1) \) and \( G_2 = (v_1, v_2) \cup (v_m, v_2) \).
Define when \( a = 1, \; d > 1 \), \( G_1 = (v_m, v_1) \)
\[
G_2 = (v_{\ell+1}, v_{\ell+2}) \cup \bigcup_{i=\ell+2}^{p} L_i \cup (v_m, v_{p+1}) \text{ where } \ell = \left\lfloor \frac{a}{2} \right\rfloor \text{ and } p = \left\lfloor \frac{a}{2} \right\rfloor + \left\lfloor \frac{a+d}{2} \right\rfloor.
\]
Define when \( a > 1 \) and \( d > 1 \)
\[
G_1 = \left\{ \bigcup_{i=1}^{\ell} L_i \right\} \cup (v_m, v_{\ell+1}) \]
\[
G_2 = (v_{\ell+1}, v_{\ell+2}) \cup \bigcup_{i=\ell+2}^{p} L_i \cup (v_m, v_{p+1}) \text{ where } \ell = \left\lfloor \frac{a}{2} \right\rfloor \text{ and } p = \left\lfloor \frac{a}{2} \right\rfloor + \left\lfloor \frac{a+d}{2} \right\rfloor.
\]
Let \( m_j = \left\lfloor \frac{a+kd}{2} \right\rfloor + \left\lfloor \frac{1}{2} \right\rfloor \)
When $j \equiv 0 \pmod{4}$, define

$$G_j = \left\{ \sum_{k=0}^{j-1} m_j + 1 \right\} \bigcup_{i=\sum_{k=0}^{j-2} m_j + 1} L_i \right.$$ where $\ell = \sum_{k=0}^{j-2} m_j$.

When $j \equiv 1 \pmod{4}$, define

$$G_j = \left\{ \sum_{k=0}^{j-1} m_j \right\} \bigcup_{i=\sum_{k=0}^{j-2} m_j + 1} L_i \right.$$ where $p = \sum_{k=0}^{j-1} m_j$.

When $j \equiv 1 \pmod{4}, (j > 1)$, define

$$G_j = \left\{ \sum_{k=0}^{j-1} m_j \right\} \bigcup_{i=\sum_{k=0}^{j-2} m_j + 1} L_i \bigcup (v_m, v_{p+1}) \right.$$
When \( j \equiv 2 \pmod{4}, (j > 2) \), define

\[
G_j = (v_{p+1}, v_{p+2}) \cup \left( \bigcup_{k=0}^{j-2} \left( \sum_{m=0}^{j} m_j \right) L_i \right) \cup \bigcup (v_m, v_{\ell+1})
\]

where \( p = \sum_{k=0}^{j-2} m_j \) and \( \ell = \sum_{k=0}^{j-1} m_j \).

In the above construction addition of indices being taken modulo \((m - 1)\) with residues 1, 2, \ldots, \(m - 1\).

Clearly \( G_j \subset G_{j+1} \) for \( 1 \leq j \leq n - 1 \).

Therefore, \( G_1, G_2, \ldots, G_n \) is an \((a, d) - ASD\) of \( W_m \).

Subcase (a)(iii) Suppose \( a \) is even and \( d \) is odd.

\[
G_1 = \bigcup_{i=1}^{\ell} L_i
\]

\[
G_2 = \bigcup_{i=\ell+1}^{p} L_i \cup (v_m, v_{p+1}) \text{ where } \ell = \frac{a}{2} \text{ and } p = \frac{a}{2} + \left\lfloor \frac{a + d}{2} \right\rfloor
\]

\( a = 4, d = 1 \)

Figure 3.3.
Let $m_j = \left\lfloor \frac{a + kd}{2} \right\rfloor + \left\lfloor \frac{j}{4} \right\rfloor$.

When $j \equiv 3 \pmod{4}$, define

$$G_j = (v_{\ell + 1}, v_{\ell + 2}) \cup \left\{ \sum_{k=0}^{j-1} m_j \cup \bigcup_{i=\sum_{k=0}^{j-2} m_j + 2} L_i \right\} \cup (v_m, v_{p+1})$$

where $\ell = \sum_{k=0}^{j-2} m_j$ and $p = \sum_{k=0}^{j-1} m_j$.

When $j \equiv 0 \pmod{4}$, define

$$G_j = (v_{\ell}, v_{\ell + 1}) \cup \left\{ \sum_{k=0}^{j-1} m_j \cup \bigcup_{i=\sum_{k=0}^{j-2} m_j + 1} L_i \right\}$$

where $\ell = \sum_{k=0}^{j-2} m_j$.

When $j \equiv 1 \pmod{4}$, ($j > 1$), define

$$G_j = \left\{ \sum_{k=0}^{j-1} m_j \cup \bigcup_{i=\sum_{k=0}^{j-2} m_j + 1} L_i \right\}$$

When $j \equiv 2 \pmod{4}$, ($j > 2$), define

$$G_j = \left\{ \sum_{k=0}^{j-1} m_j \cup \bigcup_{i=\sum_{k=0}^{j-2} m_j + 1} L_i \right\} \cup (v_m, v_{\ell + 1})$$

where $\ell = \sum_{k=0}^{j-1} m_j$.

In the above construction addition of indices being taken modulo $(m - 1)$ with residues $1, 2, \ldots, m - 1$.

Clearly $G_j \subseteq G_{j+1}$ for $1 \leq j \leq n - 1$.

Therefore, $G_1, G_2, \ldots, G_n$ is an $(a, d) - ASD$ into $n$ parts of $W_m$. 
Subcase (a)(iv): Suppose $a$ is odd and $d$ is even.

Define $G_1 = (v_m, v_1)$ when $a = 1$, $d \geq 2$

$$G_1 = \left\{ \bigcup_{i=1}^{\ell} L_i \right\} \cup (v_m, v_{\ell+1}) \text{ when } a > 1, \; d \geq 2$$

$$G_2 = (v_{\ell+1}, v_{\ell+2}) \cup \bigcup_{i=\ell+2}^{p} L_i \text{ where } \ell = \left\lfloor \frac{a}{2} \right\rfloor \text{ and } p = \left\lfloor \frac{a}{2} \right\rfloor + \left\lfloor \frac{a+d}{2} \right\rfloor + 1.$$  

Let $m_j = \left\lfloor \frac{a + kd}{2} \right\rfloor + \left\lfloor \frac{j}{2} \right\rfloor$.

When $j \equiv 3 \pmod{4}$, define

$$G_j = \left\{ \sum_{k=0}^{j-1} m_j \bigcup_{i=j-2}^{j-1} L_i \right\} \cup (v_m, v_{\ell+1}) \text{ where } \ell = \sum_{k=0}^{j-1} m_j + 1$$

$a = 1$, $d = 2$

Figure 3.4.
When \( j \equiv 0 \pmod{4} \), define

\[
G_j = (v_\ell, v_{\ell+1}) \cup \left\{ \sum_{k=0}^{j-1} m_j L_i \right\} \quad \text{where } \ell = \sum_{k=0}^{j-2} m_j.
\]

When \( j \equiv 1 \pmod{4} \), \((j > 1)\), define

\[
G_j = \left\{ \sum_{k=0}^{j-1} m_j L_i \right\} \cup (v_m, v_{p+1}) \quad \text{where } p = \sum_{k=0}^{j-1} m_j.
\]

When \( j \equiv 2 \pmod{4} \), \((j > 2)\), define

\[
G_j = (v_p, v_{p+1}) \cup \left\{ \sum_{k=0}^{j-1} m_j L_i \right\} \quad \text{where } p = \sum_{k=0}^{j-2} m_j.
\]

In the above construction addition of indices being taken modulo \((m-1)\) with 
residues \(1, 2, \ldots, m-1\).

Clearly \( G_j \subset G_{j+1} \) for \( 1 \leq j \leq n-1 \). Therefore, \( G_1, G_2, \ldots, G_n \) is an
\((a, d)\) - ASD into \(n\) parts of \(W_m\).

Case (b): Let \( n \equiv 1 \pmod{4} \).

The proof of this case is analogous to subcases a(i) and a(iii).

Case (c): Let \( n \equiv 2 \pmod{4} \).

The proof of this case is analogous to subcases a(i) and a(iv).

Case (d): Let \( n \equiv 3 \pmod{4} \).

The proof of this case is analogous to subcases a(i) and a(ii).

References


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