FUGLEDE-PUTNAM THEOREM AND QUASI-NILPOTENT PART OF $n$-POWER NORMAL OPERATORS

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Abstract. In this article we show that the following properties hold for $n$-power normal operators $T$:

(i) $T$ has the Bishop's property($\bar{\beta}$).
(ii) $T$ is isoloid.
(iii) $T$ is invariant under tensor product.
(iv) $T$ satisfies the Fuglede-Putnam theorem.
(v) $T$ is of finite ascent and descent.
(vi) The Quasi-nilpotent part of $T$ reduces $T$.

1. Introduction

In this introductory section, we indicate the main trend of the ideas to be developed in this paper. Let $H$ and $K$ be complex Hilbert spaces and $T$ a bounded linear operator on $H$, whose domain, range and null space lie in $H$. Let $L(H)$ denote the algebra of all bounded linear operators acting on $H$. An operator $T$ is said to be $n$-power normal if $T^* T^n = T^n T^*$ where $n \in \mathbb{N}$. The class of $n$-power normal operators is denoted by $[nN]$. The class $[nN]$ was introduced by A. S. Jibril [15] and he characterized several properties of class $[nN]$. One of the properties frequently used in this paper is that $T \in [nN]$ if and only if $T^n$ is normal. The normality of $T^n$ enable us to study several properties of class $[nN]$. For example, in section 2 we give matrix representation for $T$ and prove property($\bar{\beta}$).

Definition 1.1. An operator $T \in B(H)$ is said to have the property($\bar{\beta}$) at $\lambda \in \mathbb{C}$ if the following assertion holds:-

If $D \subset \mathbb{C}$ is an open neighbourhood of $\lambda$ and if $f_n : D \to H(n = 1, 2, \ldots)$ are vector valued analytic functions such that $(T - \mu)f_n(\mu) \to 0$ uniformly on every compact subset of $D$, then $f_n(\mu) \to 0$, again uniformly on every compact subset of $D$, for all $\mu \in D$.
Property($\beta$) has been proved for several operators such as hyponormal operators [21], $(p,k)$-quasi-hyponormal operators [26], class A operators [5], class $A(k)$ operators [19], paranormal operators [27], $*$-paranormal operators [7] and $k$-quasi-$M$-hyponormal operators [24].

**Definition 1.2.** An operator $T \in L(H)$ is said to be isoloid if every isolated point of $\sigma(T)$ is an eigen value of $T$.

Throughout this paper, the range, null space and the closure of the range of a bounded linear operator $T$, are denoted by ran $T$, ker $T$ and $\overline{\text{ran}T}$ respectively. For convenience we write $(T - \lambda)$ in the place of $(T - \lambda I)$.

Two important subspaces in local spectral theory are $\chi_T(F)$, the glocal spectral subspace and $\chi_T(C - \{\lambda\})$.

**Definition 1.3.** For $T \in B(H)$ and a closed subset $F$ of $C$ the glocal spectral subspace $\chi_T(F)$ is defined as the set of all $x \in H$ such that there is an analytic $H$-valued function $f : C\setminus F \to H$ for which $(T - \lambda)f(\lambda) = x$ for all $\lambda \in C\setminus F$.

The quasinilpotent part of $(T - \lambda)$ is denoted by $H_0(T - \lambda)$ and defined as follows:

**Definition 1.4.**

$$H_0(T - \lambda) = \left\{ x \in H : \lim_{n \to \infty} \| (T - \lambda)^n x \|^{\frac{1}{n}} = 0 \right\}.$$  

Note that the subspace $\chi_T(\{\lambda\})$ coincides with the quasinilpotent part of $(T - \lambda)$ while $\chi_T(C - \{0\})$ coincides with the analytic core $K(T)$ defined as the set $K(T - \lambda)$ of all $x \in H$ such that there exists $c > 0$ and a sequence $\{x_n\} \in H$ for which $(T - \lambda)x_1 = x, (T - \lambda)x_{n+1} = x_n$ and $\|x_n\| \leq c^n \|x\|$ for all $n \in \mathbb{N}$.

Vrbova’ [28] introduced the subspace $K(T)$ which is the analytic counterpart of the algebraic core $C(T)$. Saphar [25] introduced the subspace $C(T)$ in purely algebraic terms.

**Definition 1.5.** Let $T$ be a linear operator on $H$. The algebraic core $C(T)$ is defined to be the greatest subspace $M$ of $H$ for which $T(M) = M$.

We note that $T^n(M) = M$ for all $n \in \mathbb{N}$.

The class of all upper semi-Fredholm operators is denoted by $\Phi_+(H)$ and is defined as,

$$\Phi_+(H) = \{ T \in L(H) : \alpha(T) < \infty \text{ and } T(H) \text{ is closed} \}$$

and the class of all lower semi-Fredholm operators is denoted by $\Phi_-(H)$ and is defined as,

$$\Phi_-(H) = \{ T \in L(H) : \beta(T) < \infty \}$$
where \( \alpha(T) \) and \( \beta(T) \) denote the dimension of the kernel of \( T \) and the codimension of the range of \( T \). The class of all semi-Fredholm operators is denoted by \( \Phi_{\pm}(H) \) and is defined as \( \Phi_{\pm}(H) = \Phi_{+}(H) \cup \Phi_{-}(H) \) and the class of Fredholm operators is denoted by \( \Phi(H) \) and is defined as \( \Phi(H) = \Phi_{+}(H) \cap \Phi_{-}(H) \).

Recall that the ascent \( p(T) \) of an operator \( T \) is the smallest non-negative integer \( p \) such that \( \ker T^p = \ker T^{p+1} \) and if such an integer does not exist then we put \( p(T) = \infty \). Analogously, descent \( q(T) \) of the operator \( T \) is the smallest non-negative integer \( q \) such that \( \operatorname{ran} T^q = \operatorname{ran} T^{q+1} \) and if such an integer does not exist then we put \( q(T) = \infty \). If \( p(T) \) and \( q(T) \) are finite then \( p(T) = q(T) \) \cite{12, Proposition 38.3}.

The class of all Weyl operators denoted by \( W(H) \) is defined by,

\[
W(H) = \{ T \in \Phi(H) : \operatorname{ind} T = 0 \text{ where } \operatorname{ind} T = \alpha(T) - \beta(T) \}.
\]

2. Main results

We begin with the matrix representation for \( T \in [nN] \).

**Lemma 2.1** \cite{15}. \( T \in [nN] \) if and only if \( T^n \) is normal.

**Lemma 2.2.** Suppose \( T \in [nN] \) then \( \operatorname{ran} T^n \) reduces \( T \).

**Proof.** Since \( T \in [nN], T^n T^* = T^* T^n \). \( \operatorname{ran} T^n \) is invariant under \( T \) is obvious. We shall show that \( \operatorname{ran} T^n \) is invariant under \( T^* \). Let \( x \in \operatorname{ran} T^n \). Then \( x = T^n y \) for some \( y \in H \) and \( T^* x = T^* T^n y = T^n T^* y \in \operatorname{ran} T^n \).

Suppose \( z \) is a limit point of \( \operatorname{ran} T^n \), then there is a sequence \( \{ z_n \} \) in \( \operatorname{ran}(T^n) \) such that \( z_n \to z \). Since \( \{ z_n \} \) is a sequence in \( \operatorname{ran} T^n, z_n = T^n x_n, n = 1, 2, \ldots, n \in \mathbb{N}, x_n \in H \). \( T^* z_n = T^* T^n x_n = T^n T^* x_n \in \operatorname{ran} T^n \).

So \( \{ T^* z_n \} \) is a sequence in \( \operatorname{ran} T^n \). By the continuity of \( T^* \), the sequence \( \{ T^* z_n \} \to T^* z \in [\operatorname{ran} T^n] \). Thus \( [\operatorname{ran} T^n] \) is invariant under \( T^* \) and \( [\operatorname{ran} T^n] \) reduces \( T \).

**Theorem 2.3.** If \( T \) is \( n \)-power normal then \( T \) has the following matrix representation, \( T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} \) on \( H = [\operatorname{ran} T^n] \oplus \ker T^* \) where \( T_1 = T|_{\operatorname{ran} T^n} \) is also an \( n \)-power normal operator and \( T_2 \) is a nilpotent operator with nilpotency \( n \). Furthermore \( \sigma(T) = \sigma(T_1) \cup \{0\} \).

**Proof.** By Lemma 2.2, \( [\operatorname{ran} T^n] \) reduces \( T \). Hence \( T \) has the matrix representation, \( T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} \) on \( H = [\operatorname{ran} T^n] \oplus \ker T^* \). Let \( P \) be the orthogonal projection onto \([\operatorname{ran} T^n]\). Then
\[
\begin{pmatrix}
T_1 & 0 \\
0 & 0
\end{pmatrix} = TP = PT = PTP.
\]

\[
P(T^n T^*) P = \begin{pmatrix}
T_1^n T_1^* & 0 \\
0 & 0
\end{pmatrix}.
\]

Also \(P(T^* T^n) P = \begin{pmatrix}
T_1^* T_1^n & 0 \\
0 & 0
\end{pmatrix}\).

Since \(T \in [nN]\), \(P(T^n T^*) P = P(T^* T^n) P\), implying \(T_1^n T_1^* = T_1^* T_1^n\). Hence \(T_1 \in [nN]\).

For any \(z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in H\),
\[
\langle T_2^n z_2, z_2 \rangle = \langle T^n (I - P) z, (I - P) z \rangle = \langle (I - P) z, T^{*n}(I - P) z \rangle = 0.
\]

Therefore \(T_2^n = 0\). Since \([ran T^n]\) reduces \(T\), \(\sigma(T) = \sigma(T_1) \cup \sigma(T_2) = \sigma(T_1) \cup \{0\}\). \(\square\)

**Lemma 2.4.** If \(T\) is an \(n\)-power normal operator and \(M\) is a reducing subspace of \(T\) then \(T|_M\) is also an \(n\)-power normal operator.

**Proof.** Since \(M\) is a reducing subspace of \(T\), it has the matrix representation, \(T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}\) on \(H = M \oplus M^\perp\). Let \(P\) be the orthogonal projection onto \(M\). Then \(\begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} = TP = PT = PTP\).

\[
P(T^n T^*) P = \begin{pmatrix}
T_1^n T_1^* & 0 \\
0 & 0
\end{pmatrix}, \quad P(T^* T^n) P = \begin{pmatrix}
T_1^* T_1^n & 0 \\
0 & 0
\end{pmatrix}.
\]

Since \(T \in [nN]\), \(T_1^n T_1^* = T_1^* T_1^n\). Therefore \(T_1 \in [nN]\). Hence \(T|_M\) is \(n\)-power normal. \(\square\)

**Theorem 2.5.** If \(T \in [nN]\) then \(T\) has the property(\(\beta\)).

**Proof.** Consider an open neighbourhood \(D \subset \mathbb{C}\) of \(\lambda \in \mathbb{C}\) and \(f_m(m = 1, 2, \ldots)\), the vector valued analytic functions on \(D\) such that \((T - \mu)f_m(\mu) \to 0\) uniformly on every compact subset of \(D\).

Decompose \(H\) as \(H = [ran T^n] \oplus ker T^{*n}\), by Theorem 2.3, \(T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}\) where \(T_1 \in [nN]\) and \(T_2\) is a nilpotent operator with nilpotency \(n\).

\((T - \mu)f_m(\mu) \to 0\) implies,
\[
\begin{pmatrix}
T_1 - \mu & 0 \\
0 & T_2 - \mu
\end{pmatrix} \begin{pmatrix} f_{m_1}(\mu) \\ f_{m_2}(\mu) \end{pmatrix} = \begin{pmatrix} (T_1 - \mu)f_{m_1}(\mu) \\ (T_2 - \mu)f_{m_2}(\mu) \end{pmatrix} \to 0.
\]

Since \(T_2\) is nilpotent, it has property(\(\beta\)) and therefore \(f_{m_2}(\mu) \to 0\).
Also since $T_1^n$ is normal, it has property($\beta$) and therefore by Theorem 3.39 [17], $T$ has property($\beta$).

**Corollary 2.6.** If $T \in [nN]$ then $T$ has the single-valued extension property.

The following two Examples show that for a $n$-power normal operator $T$, the corresponding eigenspaces need not be reducing subspaces of $T$.

**Example 2.7.** $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Clearly $T$ is a 2-power normal operator and the eigenspace of $T$ is \( \begin{pmatrix} x \\ 0 \end{pmatrix} \) but it is not a reducing subspace of $T$.

**Example 2.8.** $T = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$. Here $T$ is a 2-power normal operator and the corresponding eigenspaces of $T$ are \( \begin{pmatrix} x \\ 0 \end{pmatrix} \) and \( \begin{pmatrix} x \\ -x \end{pmatrix} \) but these are not reducing subspaces of $T$.

Also the $n$-power normal operators are not semiregular. For example consider the multiplication operator $T$ defined by \((Tf)(t) = tf(t)\) for $f \in L^2[0,1]$ and $t \in [0,1]$. Then $T$ is normal, injective and has dense range. Since the range of $T$ is not closed, $T$ is not semiregular.

**Lemma 2.9.** Let $T \in [nN]$ and $\lambda \in \sigma(T)$ be an isolated point. Then $\lambda^n$ is an isolated point of $\sigma(T^n)$.

**Proof.** Since $\lambda \in \sigma(T)$ is an isolated point there is a neighbourhood $V$ of $\lambda$ with radius $\delta$ which contains no point of $\sigma(T)$ other than $\lambda$. By Spectral mapping Theorem, $\lambda^n \in \sigma(T^n)$. Suppose $\lambda^n \in \sigma(T^n)$ is not an isolated point of $\sigma(T^n)$, then every neighbourhood of $\lambda^n$ contains atleast one point of $\sigma(T^n)$ other than $\lambda^n$. Consequently, let $\mu_n$ in $\sigma(T^n)$ be a point in a neighbourhood $V_n$ of $\lambda^n$ with radius $\frac{\delta}{2} \rho$ where $\rho = |\sum_{k=0}^{n-1} \lambda^{n-k-1} \mu^k|$. It follows from Spectral mapping Theorem that $\mu \in \sigma(T)$. Then

$$|\lambda^n - \mu^n| \leq \frac{\delta}{2} \rho \tag{2.1}$$

$$|\lambda^n - \mu^n| = |\lambda - \mu| \left| \sum_{k=0}^{n-1} \lambda^{n-k-1} \mu^k \right|$$

$$= |\lambda - \mu| \rho$$

Consequently, $|\lambda - \mu| \leq \frac{\delta}{2}$ by (2.1).

This shows that $\mu \neq \lambda$ is a point in $V$, contradicting the hypothesis that $\lambda$ is an isolated point of $\sigma(T)$. Thus, $\lambda^n$ is an isolated point of $\sigma(T^n)$. \qed
Lemma 2.10. Let \( T \in \{nN\} \) then \( T \) is isoloid.

Proof. Let \( \lambda \in \sigma(T) \) be an isolated point. By Lemma 2.9, \( \lambda^n \) is an isolated point of \( \sigma(T^n) \). Since \( T^n \) is normal, it is isoloid. Therefore \( \lambda^n \) is in the point spectrum of \( T^n \). This implies that \( (\lambda^n - T^n)^{-1} \) does not exist.

\[
(\lambda^n - T^n)^{-1} = \frac{1}{\lambda^n} \left(I - \frac{T^n}{\lambda^n}\right)^{-1} = \mu^n (I - \mu^n T^n)^{-1} \quad \text{where} \quad \mu = \frac{1}{\lambda}.
\]

Recall the identity,

\[
(I - \mu^n T^n)^{-1} = \frac{1}{n} [(I - \mu T)^{-1} + (I - \mu w T)^{-1} + (I - \mu w^2 T)^{-1} + \cdots + (I - \mu w^{n-1} T)^{-1}]
\]

\[
(I - \mu^n T^n)^{-1} = \frac{1}{n} [(I - \mu T)^{-1} + \sum_{k=1}^{n-1} (I - \mu w^k T)^{-1}] \tag{2.2}
\]

where \( w \) is the primitive root of unity. Since \( (I - \mu^n T^n)^{-1} \) does not exist, at least one term of the expression on the right-hand side of (2.2) does not exist. Hence there exist two cases.

Case(i):
Suppose \( (I - \mu T)^{-1} \) does not exist. Since \( \mu = \frac{1}{\lambda} \), \( (\lambda I - T)^{-1} \) does not exist, which implies \( \lambda \in P_\sigma(T) \).

Case(ii):
Suppose \( (I - \mu w^k T)^{-1} \) does not exist for some \( k = 1, 2, 3, \ldots, n - 1 \). From (2.2),

\[
n(I - \mu^n T^n)^{-1} - (I - \mu w^k T)^{-1} = (I - \mu T)^{-1} + \cdots + (I - \mu w^{k-1} T)^{-1} + (I - \mu w^{k+1} T)^{-1} + \cdots + (I - \mu w^{n-1} T)^{-1}.
\]

Since \( n(I - \mu^n T^n)^{-1} - (I - \mu w^k T)^{-1} \) does not exist the expression on the other side also does not exist. In that expression, at least one term does not exist. On repeating a similar argument as above, we arrive at a stage where \( (I - \mu T)^{-1} \) does not exist. That is \( \frac{1}{\lambda} (\lambda I - T)^{-1} \) does not exist, hence \( \lambda \in P_\sigma(T) \). It follows that \( T \) is isoloid. \( \square \)

Tensor product of class[nN] operators

For \( A, B \in L(H) \), a number of authors have considered variously, the tensor product \( A \otimes B \), on the product space \( H \otimes H \). The operation of taking tensor products \( A \otimes B \) preserves many a property of \( A, B \in L(H) \), but by no means all of them. For instance the normaloid property is invariant under tensor products, whereas the spectroloid property is not [23, pp.623 and 631]. H. Jinchuan [16] proved that \( A \otimes B \) is normal if and only if \( A \) and \( B \) are so, where \( A \) and \( B \) are non-zero operators. Similar results were proved for subnormal operators [18], hyponormal operators [13], \( p \)-hyponormal operators [6], class \( A \) operators [14] and \( p \)-quasihyponormal operators [9]. But there exists paranormal operators \( A \) and \( B \) such that \( A \otimes B \) is not paranormal [3]. We show that if \( A \) and \( B \) are of class n-power normal then \( A \otimes B \) is also of the class n-power normal.
Lemma 2.11 ([13]). If $A \in L(H)$ and $B \in L(K)$ are non-zero operators, then $A \otimes B$ is normal if and only if so are $A$ and $B$.

Theorem 2.12. $T_1 \otimes T_2$ is an $n$-power normal operator if and only if $T_1$ and $T_2$ are so.

Proof. First we begin with the observations that $(T_1 \otimes T_2)^*(T_1 \otimes T_2) = T_1^* T_1 \otimes T_2^* T_2$ and $(T_1 \otimes T_2)^n = T_1^n \otimes T_2^n$. Suppose $T_1$ and $T_2$ are $n$-power normal operators, then

$$(T_1 \otimes T_2)^n(T_1 \otimes T_2)^* = T_1^n T_1^* \otimes T_2^n T_2^*$$
$$= T_1^* T_1^n \otimes T_2^* T_2^n$$
$$= (T_1 \otimes T_2)^*(T_1 \otimes T_2)^n.$$ Therefore $T_1 \otimes T_2$ is an $n$-power normal operator. Conversely, suppose $T_1 \otimes T_2$ is an $n$-power normal operator, then $(T_1 \otimes T_2)^n$ is normal. By Lemma 2.11, we have $T_1^n$ and $T_2^n$ are normal. Then by Lemma 2.1, $T_1$ and $T_2$ are $n$-power normal operators.

Fuglede-Putnam Theorem for $n$-power normal operators

Fuglede-Putnam Theorem is well known in operator theory. It affirms that if $A$ and $B$ are normal operators and $AX = XB$ for some operator $X$ then $A^* X = X B^*$. First, Fuglede [10] proved it in the case when $A = B$ and then Putnam [22] proved it in a general case. There exists many generalizations of this Theorem of which most of them go into unwinding the normality of $A$ and $B$ (see [11, 20] and some of the references cited in these papers).

Berbarian [4] unwinds the hypothesis on $A$ and $B$ by assuming $A$ and $B^*$ are hyponormal operators and $X$ to be a Hilbert-Schmidt class. The operators in $H$ which are of Hilbert-Schmidt class form an ideal $\mathcal{H}$ in the algebra $L(H)$ of all operators in $H$. $\mathcal{H}$ itself is a Hilbert space for the inner product

$$\langle X, Y \rangle = \sum \langle X e_i, Y e_i \rangle = Tr(Y^* X) = Tr(X Y^*),$$

where $\{e_i\}$ is any orthonormal basis of $H$. For each pair of operators $A, B \in L(H)$, there is an operator $\Gamma$ defined on $L(\mathcal{H})$ via the formula $\Gamma(X) = AXB$ as in [4]. Obviously, $\|\Gamma\| \leq \|A\| \|B\|$. The adjoint of $\Gamma$ is given by the formula $\Gamma^*(X) = A^* X B^*$. Also if $A \geq 0, B \geq 0$ then $\Gamma \geq 0$ [4].

Lemma 2.13. If $A$ and $B^*$ are of class $[nN]$ then the operator $\Gamma$ is of class $[nN]$.

Proof. By hypothesis, $A^* A^n = A^n A^*, B B^* = B^* B$. Since, $\Gamma(X) = AXB$ and $\Gamma^*(X) = A^* X B^*$ for any pair $A, B \in L(H)$,

$$\Gamma^* \Gamma - \Gamma \Gamma^* = \Gamma^* \Gamma^n X - \Gamma^n \Gamma^* X$$
$$= \Gamma^*(A^n X B^*) - \Gamma^n(A^* X B)$$
\[ \begin{align*}
A^* A^n X B^{*n} B &= A^* A^n X B B^{*n} B \\
A^* A^n X B^{*n} B - A^* A^n X B B^{*n} B &= 0.
\end{align*} \]

The above equality shows that \( \Gamma \in [nN] \).

\[ \Box \]

**Lemma 2.14.** If \( A \in [nN] \) and \( A \) is invertible, then \( A^{-1} \in [nN] \).

**Proof.** By hypothesis \( A^* A^n = A^n A^* \), we need to prove that \( A^{-1} A^{-1} = A^{-1} A^{-1} \).

\[ A^{-1} A^{-1} = (A^n A^*)^{-1} = (A^* A^n)^{-1} = A^{-1} A^{-1}. \]

Hence \( A^{-1} \in \text{class}[nN] \).

\[ \Box \]

**Lemma 2.15 (15).** Let \( T \in L(H) \) such that \( T \in [2N] \cap [3N] \), then \( T \in [nN] \) for all positive integers \( n \geq 4 \).

**Theorem 2.16.** Let \( A \) and \( B^* \) be in class \([2N \cap 3N]\) such that \( B^* \) is invertible and \( X \) be a Hilbert-Schmidt operator. Suppose that \( AX = XB \) then \( A^* X = XB^* \).

**Proof.** Let \( \Gamma \) be the Hilbert-Schmidt operator defined by, \( \Gamma Y = AY B^{-1} \), where \( Y \in L(H) \). By hypothesis \( A \) and \( B^* \) are of class \([nN]\), by Lemma 2.14 \( (B^*)^{-1} \) is of class \([nN]\). Since \( (B^*)^{-1} = (B^{-1})^* \), it follows by Lemma 2.13 that \( \Gamma \) is of class \([nN]\). The hypothesis \( AX = XB \) implies that \( \Gamma X = X \) and also by Lemma 2.15, \( T \in [nN] \) for all \( n \geq 2 \), it follows that,

\[ \| \Gamma^* X \|^2 = \langle \Gamma^* X, \Gamma^* X \rangle = \langle \Gamma^* X, \Gamma^* X \rangle = \langle \Gamma^* X, \Gamma^* X \rangle = \langle \Gamma^* X, \Gamma^* X \rangle = \| X \|^2. \]

The above equality gives,

\[ \| \Gamma^* X - X \|^2 = \langle \Gamma^* X - X, \Gamma^* X - X \rangle = \langle \Gamma^* X - X, \Gamma^* X - X \rangle = \| \Gamma^* X \|^2 - \langle X, \Gamma X \rangle - \langle \Gamma X, X \rangle + \| X \|^2 \]

\[ = \| X \|^2 - \langle X, X \rangle - \langle X, X \rangle + \| X \|^2 \]
= 0.

Therefore $\Gamma^* X = X$ and hence $A^* X = X B^*$.

\section*{Ascent and Descent}

The non-negative integers $p(T)$ and $q(T)$ known as the ascent and descent of $T$ respectively play a vital role to generate several classes of Browder operators and related spectrum. So we may anticipate if an n-power normal operator $T$ have finite ascent(descent) or not. Infact, $T^n$ has finite ascent since it is normal. Indeed, the following Lemma shows that the ascent and descent of $T \in [n\mathbb{N}]$ are finite.

\begin{lemma}
For any operator $T \in L(H)$ with $T^n$ normal, the following assertions hold:

(i) $p(T) = q(T) \leq n$.

(ii) $N^\infty(T) = \ker T^n$ and $T^\infty(H) = \mathrm{ran} T^n$, where $N^\infty(T) = \bigcup_{k \in \mathbb{N}} \ker T^k$ and $T^\infty(H) = \bigcap_{k \in \mathbb{N}} T^k(H)$ are the hyper kernel and hyper range respectively.
\end{lemma}

\begin{proof}
(i) It is well known that, for any normal operator $A$, $\ker A^2 = \ker A$ and $[\mathrm{ran} A^2] = [\mathrm{ran} A]$.

Since $T^n$ is normal, $\ker T^{2n} = \ker T^n$ and $[\mathrm{ran} T^{2n}] = [\mathrm{ran} T^n]$. Consequently, from the chain relations $\ker T \subseteq \ker T^2 \subseteq \cdots \subseteq \ker T^n \subseteq \ker T^{n+1} \subseteq \cdots \subseteq \ker T^{2n} = \ker T^n \subseteq \ker T^{2n+1} \cdots$ and $\cdots [\mathrm{ran} T^n] = [\mathrm{ran} T^{2n}] \subseteq [\mathrm{ran} T^{2n-1}] \subseteq \cdots \subseteq [\mathrm{ran} T^n] \subseteq [\mathrm{ran} T^{n+1}] \subseteq \cdots \subseteq [\mathrm{ran} T]$; we obtain, $\ker T^n = \ker T^{n+1}$ and $\mathrm{ran} T^n = \mathrm{ran} T^{n+1}$. By the definition of $p(T)$ and $q(T)$, we have $p(T) \leq n$ and $q(T) \leq n$. Since both are finite $p(T) = q(T)$ [12].

(ii) Also

$N^\infty(T) = \bigcup_{k \in \mathbb{N}} \ker T^k = \ker T^n$, $T^\infty(H) = \bigcap_{k \in \mathbb{N}} T^k(H) = \mathrm{ran} T^n$.
\end{proof}

\section*{Nullity and Deficiency}

The role of nullity $\alpha(T)$ and deficiency $\beta(T)$ of an operator $T$ are crucial in the class of Fredholm operators and Weyl operators. The following Theorem concerning $\alpha(T)$ and $\beta(T)$ is useful to explore if $T \in [n\mathbb{N}]$ fit into the class of Weyl operators or not. Infact, Aiena [1] proved a Theorem connecting ascent and descent with nullity and deficiency, which is stated below.

\begin{theorem}[1, Theorem 3.4]
If $T$ is a linear operator on a vector space $X$ and if $p(T) = q(T) < \infty$ then $\alpha(T) = \beta(T)$ (possibly infinity).
\end{theorem}

\begin{theorem}
Suppose $T \in [n\mathbb{N}]$ such that $\alpha(T)$ or $\beta(T)$ is finite and $T(H)$ is closed then $T$ is a Weyl operator.
\end{theorem}
**Proof.** We have $p(T) = q(T) \leq n$ by Lemma 2.17. It immediately follows from Theorem 2.18 that $\alpha(T) = \beta(T) < \infty$.

Consequently $T$ is a Fredholm operator with ind $T = 0$ and hence Weyl. □

**Theorem 2.20.** Suppose that $C(T)$ is the algebraic core of $T \in [nN]$ then the following assertions hold:

(i) $C(T)$ is invariant under $T^*n$.

(ii) $T^*(C(T)) \subseteq C(T^n)$.

**Proof.** (i) Since $T \in [nN]$, $T^*T^n = T^nT^*$ or $T^*nT = T^nT^*$. Also by the definition of algebraic core of $T$, $T(C(T)) = C(T)$ or $T^n(C(T)) = C(T)$ for all $n \in \mathbb{N}$. $T^*nT = T^nT^*$ implies $T^*nT(C(T)) = T^nT^*(C(T))$ or $T^*n(C(T)) = T^nT^*(C(T))$.

$C(T)$ being the greatest subspace satisfying $T(C(T)) = C(T)$, we have $T^*n(C(T)) \subseteq C(T)$. Thus $C(T)$ is invariant under $T^*n$.

(ii) $T \in [nN]$ implies $T^*nC(T) = T^nT^*C(T)$ or $T^*(C(T)) = T^nT^*(C(T))$. It follows that $T^*(C(T)) \subseteq C(T^n)$, the algebraic core of $T^n$. □

A. S. Jibril [15] proved that if $T \in [2N] \cap [3N]$, then $T \in [nN]$ for all positive integers $n \geq 4$.

**Theorem 2.21.** If $T \in [2N] \cap [3N]$, then

(i) $H_0(T)$ is a reducing subspace of $T$.

(ii) $x \in H_0(T)$ if and only if $T^*x \in H_0(T)$ where

$$H_0(T) = \left\{ x \in H : \lim_{n \to \infty} \| T^n x \|^{\frac{1}{n}} = 0 \right\}.$$

(iii) $\ker(T - \lambda) \cap H_0(T) = \{0\}$ for every $\lambda \neq 0$.

**Proof.** (i) Let $F \subset C$ be a closed set. The glocal spectral subspace $\chi_T(F)$ is defined as, $\chi_T(F) = \{ x \in H : \exists \text{ analytic } f(z) : (T - z)f(z) = x \text{ on } C \setminus F \}$. By Theorem 2.20 [1], we have $H_0(T - \lambda) = \chi_T((\lambda))$. By Theorem 2.5, $T$ has property(β). Also by Proposition 1.2.19 [17], $\chi_T(F)$ is closed and $\sigma(T|_{\chi_T(F)}) \subset F$. Hence $H_0(T - \lambda)$ is closed for $\lambda \in C$, which implies $H_0(T)$ is closed. If $x \in H_0(T)$ then from the inequality $\| T^n T x \| \leq \| T \| \| T^n x \|$, it is easily seen that $T x \in H_0(T)$ and $H_0(T)$ is invariant under $T$.

$$\| T^n T^* x \|^2 = \langle T^n T^* x, T^n T^* x \rangle$$
$$= \langle T^{n+1} T^* x, T^{n+1} x \rangle = \| T^{n+1} x \|^2$$ since $T^{n+1}$ is normal

$$\| T^n T^* x \| = \| T^{n+1} x \|$$ (2.3)
If \( x \in H_0(T) \) then, \( \| T^n T^* x \|^{\frac{1}{n+1}} = \left( \| T^{n+1} x \|^{\frac{1}{n+1}} \right)^{\frac{n+1}{n}} \) by (2.3). It follows that \( T^* x \in H_0(T) \) and \( H_0(T) \) is invariant under \( T^* \).

(ii) \( x \in H_0(T) \) implies \( T^* x \in H_0(T) \) follows by (i). Conversely let \( T^* x \in H_0(T) \). Since by (2.3) \( \| T^{n+1} x \| = \| T^n T^* x \| \),

\[
\lim_{n \to \infty} \| T^{n+1} x \|^{\frac{1}{n+1}} = \lim_{n \to \infty} \left( \| T^n T^* x \|^{\frac{1}{n}} \right)^{\frac{n}{n+1}} = 0.
\]

Thus \( x \in H_0(T) \).

(iii) Suppose \( x \neq 0 \in ker(T - \lambda) \cap H_0(T) \). Then \( x \in ker(T - \lambda) \) implies,

\[
(T - \lambda)x = 0 \Rightarrow Tx = \lambda x \Rightarrow T^n x = \lambda^n x.
\]

By (ii) \( x \in H_0(T) \) if and only if \( T^*(x) \in H_0(T) \) and hence,

\[
0 = \lim_{n \to \infty} \| T^n T^* x \|^{\frac{1}{n}} = \lim_{n \to \infty} \| T^* T^n x \|^{\frac{1}{n}} = \lim_{n \to \infty} \| T^* \lambda^n x \|^{\frac{1}{n}} = \lim_{n \to \infty} |\lambda| \| T^* x \|^{\frac{1}{n}} = |\lambda| \lim_{n \to \infty} \| T^* x \|^{\frac{1}{n}} = |\lambda|.
\]

Which is a contradiction and therefore \( T^* x \notin H_0(T) \Rightarrow x \notin H_0(T) \). Hence \( ker(T - \lambda) \cap H_0(T) = \{0\} \) for every \( \lambda \neq 0 \). \( \square \)

**Remark 2.22.** For \( T \in [nN] \) the restriction \( T^n|_M \) of \( T^n \) to a closed invariant subspace \( M \) is a hyponormal operator, since \( T^n|_M \) is subnormal.

**Theorem 2.23.** Suppose \( T \in [2N] \cap [3N] \), then for every \( m \geq 2, m \in \mathbb{N} \) the following properties hold:

(i) \( H_0(T^m - \lambda) \) is a reducing subspace of \( T \).
(ii) \( H_0(T^m - \lambda) = ker(T^m - \lambda) = ker(T^* m - \lambda) \). In particular \( H_0(T^m) = ker T^m = ker T^* m \).
(iii) If \( M \) is an invariant subspace of \( T \) and \( T_1 = T|_M \) on \( H = M \oplus M^\perp \) then \( H_0(T_1^m - \lambda) = ker(T_1^m - \lambda) \subseteq ker(T^m - \lambda) \).
(iv) \( H_0(T^m - \lambda^m) \supset H_0(T - \lambda) \) and \( H_0(T^m - \lambda^m) = H_0(T - \lambda) \) if \( S = T^{m-1} + \lambda T^{m-2} + \cdots + \lambda^{m-2} T + \lambda^{m-1} \) is invertible.
(v) \( H_0(T - \lambda) \subset ker(T^m - \lambda^m) \) and \( H_0(T - \lambda) = ker(T - \lambda) \) if \( S \) is invertible.
Proof. (i)

\[ H_0(T^m - \lambda) = \left\{ x \in H : \lim_{n \to \infty} \left\| (T^m - \lambda)^n x \right\|^\frac{1}{n} = 0 \right\}. \]

Since \( T \in [2N] \cap [3N] \), \( T \) is n-power normal for all \( n \geq 2 \). Therefore \( (T^m)^n T^* = T^*(T^m)^n \) for all \( m \geq 2, n \geq 1 \). Consequently, \( (T^m - \lambda)^n T^* = T^*(T^m - \lambda)^n \) and hence for \( x \in H_0(T^m - \lambda) \), we have

\[
\lim_{n \to \infty} \left\| (T^m - \lambda)^n x \right\|^\frac{1}{n} = \lim_{n \to \infty} \left\| T^*(T^m - \lambda)^n x \right\|^\frac{1}{n} \leq \lim_{n \to \infty} \left\| T^* \right\|^\frac{1}{n} \lim_{n \to \infty} \left\| (T^m - \lambda)^n x \right\|^\frac{1}{n} = 0.
\]

Thus \( T^* x \in H_0(T^m - \lambda) \). That \( Tx \in H_0(T^m - \lambda) \) is obvious.

(ii) It is well known that for a totally paranormal operator \( T \), \( H_0(T - \lambda) = ker(T - \lambda) \) for all \( \lambda \in \mathbb{C} \) [2]. The class of totally paranormal operators includes the class of hyponormal operators and hence normal operators. Since \( T^m \) is normal for all \( m \geq 2 \), we have

\[ H_0(T^m - \lambda) = ker(T^m - \lambda) = ker(T^m - \lambda)^*. \]

For \( \lambda = 0 \), \( H_0(T^m) = ker(T^m) = ker(T^{*m}) \).

(iii) By Remark 2.22, \( T_1^m = T^m|_M \) is hyponormal and hence \( H_0(T_1^m - \lambda) = ker(T_1^m - \lambda) \subseteq ker(T^m_1 - \lambda)^* \).

(iv) Let \( x \in H_0(T - \lambda) \) then

\[
\lim_{n \to \infty} \left\| (T - \lambda)^n x \right\|^\frac{1}{n} = 0.
\]

Since \( T^m - \lambda^m = (T - \lambda)(T^{m-1} + \lambda T^{m-2} + \cdots + \lambda m^{-2} T + \lambda m^{-1}) = (T - \lambda)S \), where \( S = (T^{m-1} + \lambda T^{m-2} + \cdots + \lambda m^{-2} T + \lambda m^{-1}) \), we have,

\[
\lim_{n \to \infty} \left\| (T^m - \lambda^m)^n x \right\|^\frac{1}{n} = \lim_{n \to \infty} \left\| (T - \lambda)^n S^n x \right\|^\frac{1}{n} \leq \lim_{n \to \infty} \left\| S^n \right\|^\frac{1}{n} \lim_{n \to \infty} \left\| (T - \lambda)^n x \right\|^\frac{1}{n} \leq \left\| S \right\| \lim_{n \to \infty} \left\| (T - \lambda)^n x \right\|^\frac{1}{n} = 0.
\]

Therefore \( x \in H_0(T^m - \lambda^m) \) and \( H_0(T - \lambda) \subseteq H_0(T^m - \lambda^m) \).
On the other hand if $S$ is invertible then $(T - \lambda) = S^{-1}(T^m - \lambda^m)$. For $x \in H_0(T^m - \lambda^m)$, we have,

$$\lim_{n \to \infty} \| (T - \lambda)^n x \|^{\frac{1}{n}} = \lim_{n \to \infty} \| S^{-n}(T^m - \lambda^m)^n x \|^{\frac{1}{n}}$$

$$\leq \| S^{-1} \| \lim_{n \to \infty} \| (T^m - \lambda^m)^n x \|^{\frac{1}{n}}$$

$$= 0.$$

Consequently, $H_0(T^m - \lambda^m) = H_0(T - \lambda)$ for all $m \geq 2$.

(v) $H_0(T - \lambda) \subset \ker(T^m - \lambda^m)$ follows from (ii) and (iv).

That $S$ is invertible yields $\ker(T^m - \lambda^m) = \ker(T - \lambda)$. Again by (ii) and (iv) $H_0(T - \lambda) = \ker(T - \lambda)$. \hfill \Box

In general $T \in [nN]$ is not translation invariant.

**Example 2.24.** It is easily seen that, for $T = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \in [3N],$

$$(T - i)^3(T - i)^* = \begin{pmatrix} -6i - 8 & -4i + 7 \\ 10i - 1 & -4i - 7 \end{pmatrix}. $$

$$(T - i)^*T(T - i) = \begin{pmatrix} -6i - 8 & 10i + 1 \\ 4i - 7 & -4i - 7 \end{pmatrix}. $$

Therefore $(T - i) \notin [3N]$. Thus $T \in [nN]$ is not translation invariant.

Naturally in view of the above statement, the following question arises: What could be the nature of class $[nN]$ operators satisfying the translation invariant property?

In [8] Eungil Ko proved that if the $k^{th}$ root of a hyponormal operator is translation invariant then it is hyponormal. We use the same technique to prove the following Theorem.

**Theorem 2.25.** Suppose $T \in [nN]$ is translation invariant then $T$ is normal.

**Proof.**

$$(T - \lambda)^n(T - \lambda)^* = (T - \lambda)^*(T - \lambda)^n$$

$$0 = (T - \lambda)^*(T - \lambda)^n - (T - \lambda)^n(T - \lambda)^*$$

$$0 = (T - \lambda)^* \left[ \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} T^{n-k}(-\lambda)^k \right] - \left[ \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} T^{n-k}(-\lambda)^k \right](T - \lambda)^* \quad (2.4)$$

Put $\lambda = \rho e^{i\theta}, \rho > 0, 0 \leq \theta < 2\pi$, in (2.4) and dividing the simplified equation by $\rho^n$ gives,

$$0 = n(T^*T - TT^*)e^{(n-1)i\theta} + \frac{1}{\rho} (\text{the other terms}).$$

Taking limit as $\rho \to \infty$ gives, $TT^* = T^*T$. \hfill \Box
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