AN EXPLICIT VISCOSITY ITERATIVE ALGORITHM FOR FINDING THE SOLUTIONS OF A GENERAL EQUILIBRIUM PROBLEM SYSTEMS

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Abstract. We suggest an explicit viscosity iterative algorithm for finding a common element of the set of solutions for a general equilibrium problem system (GEPS) involving a bifunction defined on a closed, convex subset and the set of fixed points of a nonexpansive semigroup on another one in Hilbert's spaces. Furthermore, we present some numerical examples (by using MATLAB software) to guarantee the main result of this paper.

1. Preliminaries

Let $H$ be a real Hilbert space with norm $\| \cdot \|$ and inner product $\langle \cdot , \cdot \rangle$. Let $C$ be a nonempty closed convex subset of $H$. Then, for any $x \in H$, there exists a unique nearest point in $C$, denoted by $P_C x$, such that

$$\| x - P_C x \| \leq \| x - y \|, \text{ for all } y \in C.$$  

Such $P_C$ is called the metric projection of $H$ onto $C$. We know that $P_C$ is nonexpansive. The strongly (weak) convergent of $\{x_n\}$ to $x$ is written by $x_n \rightharpoonup x$ ($x_n \to x$) as $n \to \infty$. Moreover, $H$ satisfies the Opial's condition [15], if for any sequence $\{x_n\}$ with $x_n \to x$, the inequality

$$\liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\|,$$

holds for every $y \in H$ with $x \neq y$.

Recall that a mapping $T : C \to C$ is said to be nonexpansive, if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. $F(T)$ denotes the set of fixed points of $T$. Let $\{T(s) : s \in [0, \infty)\}$ be a nonexpansive semigroup on a closed convex subset $C$, that is,

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(i) \( T(0)x = x \), for all \( x \in C \);

(ii) \( T(s + t) = T(s)T(t) \), for all \( s, t \geq 0 \);

(iii) \( \| T(s)x - T(s)y \| \leq \| x - y \| \), for all \( x, y \in C \) and \( s \geq 0 \);

(iv) \( s \mapsto T(s)x \) is continuous for all \( x \in C \).

Denote by \( F(S) = \bigcap_{s \geq 0} F(T(s)) \). It is well known that \( F(S) \) is closed and convex subset in \( H \) and \( F(S) \neq \emptyset \) if \( C \) is bounded [1]. Recall that a self mapping \( f : H \to H \) is a contraction if there exists \( \rho \in (0, 1) \) such that \( \| f(x) - f(y) \| \leq \rho \| x - y \| \) for each \( x, y \in H \).

A mapping \( B : C \to H \) is called \( \alpha \)–inverse strongly monotone [14, 20] if there exists a positive real number \( \alpha > 0 \) such that for all \( x, y \in C \)

\[
\langle Bx - By, x - y \rangle \geq \alpha \| Bx - By \|^2.
\]

Shimizu and Takahashi [18] studied the strongly convergent of the sequence \( \{x_n\} \) which is defined by:

\[
x_{n+1} = \alpha_n x + (1 - \alpha_n) \frac{1}{t_n} \int_{0}^{t_n} T(s)x_n ds, \quad x \in C,
\]

in a real Hilbert space, where \( \{T(s) : s \in [0, \infty)\} \) is a strongly continuous semigroup of non-expansive mappings on a closed convex subset \( C \) of a Hilbert space and \( \lim_{n \to \infty} t_n = \infty \). Later, Plubtieng and Punpaeng [17] introduced the following iterative method:

\[
x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n) \frac{1}{t_n} \int_{0}^{t_n} T(s)x_n ds,
\]

where \( \{\alpha_n\}, \{\beta_n\} \) are the sequences in \((0, 1)\), \( \{s_n\} \) is a positive real divergent sequence and \( f : C \to C \) is a contraction. Under the conditions \( \sum_{n=1}^{\infty} \alpha_n = \infty \), \( \alpha_n + \beta_n < 1 \), \( \lim_{n \to \infty} \alpha_n = 0 \) and \( \lim_{n \to \infty} \beta_n = 0 \), they proved the strong convergence of the sequence.

Also, Plubtieng and Punpaeng [16] introduced the following iterative scheme:

Let \( S : C \to H \) be a nonexpansive mapping, defined sequences \( \{x_n\} \) and \( \{u_n\} \) by

\[
\begin{align*}
F(u_n, y) + \frac{1}{t_n} \langle y - u_n, u_n - x_n \rangle & \geq 0; \\
x_{n+1} = \alpha_n y f(x_n) + (I - \alpha_n A)S u_n, \quad \forall y \in H.
\end{align*}
\]

They proved, under the certain appropriate conditions, the sequence \( \{x_n\} \) converges strongly to the unique solution of the variational inequality

\[
\langle (A - y f)z, x - z \rangle \geq 0, \quad \forall x \in F(S) \cap EP(F),
\]

which is the optimality condition for the minimization problem

\[
\min_{x \in F(S) \cap EP(F)} \frac{1}{2} \langle Ax, x \rangle - h(x),
\]
where $h$ is a potential function for $\gamma f$.

A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mappings on a real Hilbert space $H$:

$$
\min \frac{1}{2} \langle Ax, x \rangle - h(x),
$$

where $A$ is strongly positive linear bounded operator and $h$ is a potential function for $\gamma f$, i.e., $h'(x) = \gamma f$, for all $x \in H$.

Let $A : H \rightarrow H$ be an inverse strongly monotone mapping and $F : C \times C \rightarrow \mathbb{R}$ be a bifunction.

Then we consider the following GEPS

$$\text{Find } \bar{x} \in C \text{ such that } F(\bar{x}, y) + \langle Ax, y - x \rangle \geq 0, \text{ for all } y \in C. \quad (1.1)$$

The set of such $x \in C$ is denoted by $\text{GEPS}(F, A)$, i.e.,

$$\text{GEPS}(F, A) = \{x \in C : F(x, y) + \langle Ax, y - x \rangle \geq 0, \forall y \in C\}.$$

To study the generalized equilibrium problem (1.1), let $F$ satisfies the following conditions:

(A1) $F(x, x) = 0$, for all $x \in C$;

(A2) $F$ is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$, $\limsup_{t \to 0} F(tz + (1 - t)x, y) \leq F(x, y)$;

(A4) for each $x \in C y \mapsto F(x, y)$ is convex and weakly lower semi-continuous.

Recently, Kamraska and Wangkeeree [8] introduced a new iterative by viscosity approximation methods in a Hilbert space. To be more precisely, they proved the following result:

**Theorem 1.1.** Let $S = (T(s))_{s \geq 0}$ be a nonexpansive semigroup on a real Hilbert space $H$. Let $f : H \rightarrow H$ be an $\alpha$-contraction, $A : H \rightarrow H$ a strongly positive linear bounded self-adjoint operator with coefficient $\gamma$. Let $\gamma$ be a real number such that $0 < \gamma < \frac{\gamma}{\alpha}$. Let $G : H \times H \rightarrow \mathbb{R}$ be a mapping satisfying hypotheses (A1)-(A4) and $\Psi : H \rightarrow H$ an inverse-strongly monotone mapping with coefficients $\delta > 0$ such that $F(S) \cap \text{GEP}(G, \Psi) \neq \emptyset$. Let the sequences $\{x_n\}, \{u_n\}$ and $\{y_n\}$ be generated by

$$\begin{cases}
    x_1 \in H \text{ chosen arbitrary}, \\
    G(u_n, y) + \langle \Psi x_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle, \\
    y_n = \beta_n x_n + (1 - \beta_n) \frac{1}{s_n} \int_0^s T(s) u_n ds, x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)y_n, \forall n \geq 1.
\end{cases}$$
Under the certain appropriate conditions, they proved that the sequences \{x_n\}, \{u_n\} and \{y_n\} is strongly convergent to \(z\), which is a unique solution in \(F(S) \cap \text{GEP}(G, \Psi)\) of the variational inequality
\[
\langle (\gamma f - A)z, p - z \rangle \leq 0, \forall p \in F(S) \cap \text{GEP}(G, \Psi).
\]
The problem studied in this paper is formulated as follows (By intuition from [6], [7], [8]): Let \(C_1\) and \(C_2\) be closed convex subsets in \(H\). Suppose that \(F(x, y)\) be a bifunction satisfy conditions (\(A_1\)) – (\(A_4\)) with \(C\) replaced by \(C_1\) and let \(\{T(S) : s \in [0, \infty)\}\) be a nonexpansive semigroup on \(C_2\). Find an element
\[
x^* \in \bigcap_{i=1}^{k} \text{GEPS}(F_i, \Psi_i) \cap F(S),
\]
where \(\text{GEPS}(F_i, \Psi_i)\) and \(F(S)\) the set of solutions of an general equilibrium problem system(\(\text{GEPS}\)) involving by a bifunction \(F_k(u_i^{(i)}, y)\) on \(C_1 \times C_1\) and the fixed point set of a nonexpansive semigroup \(\{T(S) : s \in [0, \infty)\}\) on a closed convex subset \(C_2\), respectively.

2. Preliminaries and Lemmas

The following lemmas will be useful for proving the main results of this article. Let \(A\) be a strongly positive linear bounded operator on \(H\): that is, there exists a constant \(\tilde{\gamma} > 0\) such that
\[
\langle Ax, x \rangle \geq \tilde{\gamma} \|x\|^2, \text{ for all } x \in H.
\]

Lemma 2.1 ([13]). Assume \(A\) is a strongly positive linear bounded operator on a Hilbert space \(H\) with coefficient \(\tilde{\gamma} > 0\) and \(0 < \rho < \|A\|^{-1}\). Then \(\|I - \rho A\| \leq I - \rho \tilde{\gamma}\).

Lemma 2.2 ([2]). Let \(C\) be a nonempty closed convex subset of \(H\) and \(F : C \times C \to \mathbb{R}\) be a bifunction satisfying (\(A_1\))–(\(A_4\)). Then, for any \(r > 0\) and \(x \in H\) there exists \(z \in C\) such that
\[
F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \forall y \in C.
\]
Further, define
\[
T_r x = \{z \in C : F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \forall y \in C\}
\]
for all \(r > 0\) and \(x \in H\). Then
(a) \(T_r\) is single-valued;
(b) \(T_r\) is firmly nonexpansive, i.e., for any \(x, y \in H\)
\[
\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle;
\]
(c) \(F(T_r) = \text{GEP}(F)\);
(d) \( \| T_s x - T_r x \| \leq \frac{s-r}{s} \| T_s x - x \| \);

(e) \( \text{GEP}(F) \) is closed and convex.

**Remark 2.3.** It is clear that for any \( x \in H \) and \( r > 0 \), by Lemma 2.2(a), there exists \( z \in H \) such that

\[
F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \text{ for all } y \in H.
\]  

Replacing \( x \) with \( x - r\Psi x \) in (2.1), we obtain

\[
F(z, y) + \langle \Psi x, y - z \rangle + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \text{ for all } y \in H.
\]

**Lemma 2.4 (**[19]**) Let \( \{x_n\} \) and \( \{y_n\} \) be bounded sequences in a Banach space \( E \) and \( \{\beta_n\} \) be a sequence in \([0,1]\) with \( 0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1 \). Suppose \( x_{n+1} = (1 - \beta_n) y_n + \beta_n x_n \) for all integers \( n \geq 1 \) and \( \limsup_{n \to \infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0 \). Then \( \lim_{n \to \infty} \|y_n - x_n\| = 0 \).**

**Lemma 2.5 (**[18]**) Let \( C \) be a nonempty bounded closed convex subset of \( H \) and let \( S = \{T(s) : s \in [0,\infty)\} \) be a nonexpansive semigroup on \( C \). Then for any \( h \in [0,\infty) \),

\[
\limsup_{t \to \infty} \frac{1}{t} \int_0^t T(s)xds - T(h)\left(\frac{1}{t} \int_0^t T(s)xds\right) = 0,
\]

for \( x \in C \) and \( t > 0 \).

**Lemma 2.6 (**[23]**) Assume \( \{\alpha_n\} \) is a sequence of nonnegative numbers such that

\[
a_{n+1} \leq (1 - \alpha_n) a_n + \delta_n,
\]

where \( \{\alpha_n\} \) is a sequence in \((0,1)\) and \( \{\delta_n\} \) is a sequence in real number such that

(i) \( \lim_{n \to \infty} \alpha_n = 0 \) and \( \sum_{n=1}^{\infty} \alpha_n = \infty \);

(ii) \( \limsup_{n \to \infty} \frac{\delta_n}{\alpha_n} \leq 0 \) or \( \sum_{n=1}^{\infty} |\delta_n| < \infty \);

Then \( \lim_{n \to \infty} a_n = 0 \).

**Lemma 2.7 (**[4]**) If \( C \) is a closed convex subset of \( H \), \( T \) is a nonexpansive mapping on \( C \), \( \{x_n\} \) is a sequence in \( C \) such that \( x_n \rightharpoonup x \in C \) and \( x_n - Tx_n \rightharpoonup 0 \), then \( x - Tx = 0 \).**
3. Explicit viscosity iterative algorithm

The viscosity method has been successfully applied to various problems coming from calculus of variations, minimal surface problems, plasticity theory and phase transition. It plays a central role too in the study of degenerated elliptic and parabolic second order equations [9], [11], [12]. First abstract formulation of the properties of the viscosity approximation have been given by Tykhonov [22] in 1963 when studying ill-posed problems (see [3] for details). The concept of viscosity solution for Hamilton-Jacobi equations, which plays a crucial role in control theory, game theory and partial differential equations has been introduced by Crandall and Lions [5]. In this section, we introduce a explicit viscosity iterative algorithm for finding a common element of the set of solution for an equilibrium problem involving a bifunction defined on a closed convex subset and the set of fixed points for a nonexpansive semigroup.

In this section, we introduce a new iterative for finding a common element of the set of solution for an equilibrium problem involving a bifunction defined on a closed convex subset and the set fixed points for a nonexpansive semigroup.

**Theorem 3.1.** Let $H$ be a real Hilbert space. Assume that

- $C_1, C_2$ are two nonempty convex closed subsets $H$,
- $F_1, F_2, \ldots, F_k$ be bifunctions from $C_1 \times C_1$ to $\mathbb{R}$ satisfying (A1) – (A4),
- $\Psi_1, \Psi_2, \ldots, \Psi_k$ is $\mu_i$–inverse strongly monotone mapping on $H$,
- $f : H \rightarrow H$ is a $\frac{1}{2}$– contraction,
- $A$ is a strongly positive linear bounded operator on $H$ with coefficient $\lambda$ and $0 < \gamma < \frac{1}{\rho}$,
- $F(S) = \{ T(s) : s \in [0, \infty) \}$ is a nonexpansive semigroup on $C_2$ such that $\bigcap_{i=1}^{k} F(S) \cap \text{GEPS}(F_i, \Psi_i) \neq \emptyset$,
- $\{x_n\}$ is a sequence generated in the following manner:

$$
\begin{aligned}
& x_1 \in H, \text{ and } u_n^{(i)} \in C_1, \\
& F_1(u_n^{(1)}, y) + \langle \Psi_1 x_n, y - u_n^{(1)} \rangle + \frac{1}{r_n} \langle y - u_n^{(1)}, u_n^{(1)} - x_n \rangle \geq 0, \text{ for all } y \in C_1, \\
& F_2(u_n^{(2)}, y) + \langle \Psi_2 x_n, y - u_n^{(2)} \rangle + \frac{1}{r_n} \langle y - u_n^{(2)}, u_n^{(2)} - x_n \rangle \geq 0, \text{ for all } y \in C_1, \\
& \vdots \\
& F_k(u_n^{(k)}, y) + \langle \Psi_k x_n, y - u_n^{(k)} \rangle + \frac{1}{r_n} \langle y - u_n^{(k)}, u_n^{(k)} - x_n \rangle \geq 0, \text{ for all } y \in C_1, \\
& \omega_n = \frac{1}{k} \sum_{i=1}^{k} u_n^{(i)}, \\
& x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + ((1 - \beta_n) I - \alpha_n A) \frac{1}{r_n} \int_{0}^{t_n} T(s) P_{C_2} \omega_n \, ds,
\end{aligned}
$$

where $\{\alpha_n\}, \{\beta_n\}$ are the sequences in $[0, 1]$ and $\{r_n\} \subset (0, \infty)$ is a real sequence which satisfy the following conditions:
(C1) \( \lim_{n \to \infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty; \)

(C2) \( 0 < \lim \inf \beta_n \leq \lim \sup \beta_n < 1; \)

(C3) \( \lim_{n \to \infty} |r_{n+1} - r_n| = 0 \) and \( 0 < b < r_n < a < 2\mu_i \) for \( i \in \{1, 2, \ldots, k\}; \)

(C4) \( \lim_{n \to \infty} t_n = \infty, \) and \( \sup |t_{n+1} - t_n| \) is bounded.

Then

(i) the sequence \( \{x_n\} \) is bounded;

(ii) \( \lim_{n \to \infty} \|\Psi_i x_n - \Psi_i x^*\| = 0, \) for \( i \in \{1, 2, \ldots, k\}, x^* \in \bigcap_{i=1}^{k} \text{GEPS}(F_i, \Psi_i) \cap F(S); \)

(iii) \( \lim_{n \to \infty} \left\| x_n - \frac{1}{t_n} \int_0^{t_n} T(s)P_{C_2} \omega_n ds \right\| = 0 \) and \( \lim_{n \to \infty} \left\| \omega_n - \frac{1}{t_n} \int_0^{t_n} T(s)P_{C_2} \omega_n ds \right\| = 0. \)

**Proof.** (i) By the same argument in [7, 10],

\[ \|(1 - \beta_n)I - \alpha_n A\| \leq 1 - \beta_n - \alpha_n \lambda. \]

Let \( q \in \bigcap_{i=1}^{k} F(S) \cap \text{GEPS}(F_i, \Psi_i). \) Observe that \( I - r_n \Psi_i \) for any \( i = 1, 2, \ldots, k \) is a nonexpansive mapping. Indeed, for any \( x, y \in H, \)

\[ \|(I - r_n \Psi_i)x - (I - r_n \Psi_i)y\|^2 = \|(x - y) - r_n(\Psi_i x - \Psi_i y)\|^2 \]
\[ = \|x - y\|^2 - 2r_n\langle x - y, \Psi_i x - \Psi_i y \rangle + r_n^2 \|\Psi_i x - \Psi_i y\|^2 \]
\[ \leq \|x - y\|^2 - r_n(2\mu_i - r_n)\|\Psi_i x - \Psi_i y\|^2 \]
\[ \leq \|x - y\|^2. \]

So

\[ \|u_n^{(i)} - q\| \leq \|x_n - q\|, \] \( (3.1) \)

and hence

\[ \|\omega_n - q\| \leq \|x_n - q\|. \] \( (3.2) \)

Thus

\[ \|x_{n+1} - q\| = \|\alpha_n yf(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n A)\| \frac{1}{t_n} \int_0^{t_n} T(s)P_{C_2} \omega_n ds - q\| \]
\[ \leq \alpha_n \|yf(x_n) - Aq\| + \beta_n \|x_n - q\| \]
\[ + \|(1 - \beta_n)I - \alpha_n A\| \| \frac{1}{t_n} \int_0^{t_n} T(s)P_{C_2} \omega_n ds - q\|. \]
\[ \leq \alpha_n \|\gamma f(x_n) - \gamma f(q)\| + \|\gamma f(q) - Aq\| + \beta_n \|x_n - q\| \]
\[ + (1 - \beta_n - \alpha_n \lambda) \frac{1}{t_n} \int_{0}^{t_n} \| T(s) P C_\omega w_n - P C_\omega q \| ds \]
\[ \leq \alpha_n \rho \gamma \|x_n - q\| + \alpha_n \|\gamma f(q) - Aq\| + \beta_n \|x_n - q\| \]
\[ + (1 - \beta_n - \alpha_n \lambda) \|w_n - q\| \]
\[ \leq (1 - \alpha_n (\lambda - \gamma \rho)) \|x_n - q\| + \alpha_n \|\gamma f(q) - Aq\| \]
\[ \leq \max(\|x_n - q\|, \frac{\|\gamma f(q) - Aq\|}{\lambda - \gamma \rho}). \]

By induction
\[ \|x_n - q\| \leq \max(\|x_1 - q\|, \frac{\|\gamma f(q) - Aq\|}{\lambda - \gamma \rho}). \]

Therefore, the sequence \( \{x_n\} \) is bounded and also \( \{f(x_n)\}, \{w_n\} \) and \( \{\frac{1}{t_n} \int_{0}^{t_n} T(s) P C_\omega w_n ds\} \) are bounded.

(ii) Note that \( u^{(i)}_n \) can be written as \( u^{(i)}_n = T^{(i)}_{t_n}(x_n - r_n \Psi_i x_n) \). By Lemma 2.2, for any \( i = 1, 2, \ldots, k \),
\[ \|u^{(i)}_{n+1} - u^{(i)}_n\| \leq \|T^{(i)}_{t_n}(I - r_{n+1} \Psi_i)x_{n+1} - T^{(i)}_{t_n}(I - r_n \Psi_i)x_n\| \]
\[ + \|T^{(i)}_{t_n}(I - r_n \Psi_i)x_n - T^{(i)}_{t_n}(I - r_n \Psi_i)x_n\| \]
\[ \leq \|(I - r_{n+1} \Psi_i)x_{n+1} - (I - r_n \Psi_i)x_n\| \]
\[ + \|T^{(i)}_{t_n}(I - r_n \Psi_i)x_n - T^{(i)}_{t_n}(I - r_n \Psi_i)x_n\| \]
\[ \leq \|x_{n+1} - x_n\| + |r_{n+1} - r_n| \|\Psi_i x_n\| \]
\[ + \frac{r_{n+1} - r_n}{r_{n+1}} \left\| T^{(i)}_{t_n}(I - r_n \Psi_i)x_n - T^{(i)}_{t_n}(I - r_n \Psi_i)x_n \right\|. \]

Then
\[ \|u^{(i)}_{n+1} - u^{(i)}_n\| \leq \|x_{n+1} - x_n\| + 2M_i |r_{n+1} - r_n|, \] \hspace{1cm} (3.3)

where
\[ M_i = \max(\sup(\frac{\|x_{n+1} - x_n\|}{r_{n+1}}), \sup(\|\Psi_i x_n\|)). \]

Also
\[ \left\| \frac{1}{t_{n+1}} \int_{0}^{t_{n+1}} T(s) P C_\omega w_{n+1} ds - \frac{1}{t_n} \int_{0}^{t_n} T(s) P C_\omega w_n ds \right\| \]
\[ = \left\| \frac{1}{t_{n+1}} \int_{0}^{t_{n+1}} [T(s)w_{n+1} - T(s)w_n] ds + \left( \frac{1}{t_{n+1}} - \frac{1}{t_n} \right) \int_{0}^{t_n} [T(s)w_n - T(s)q] ds \right\| \]
\[ + \frac{1}{t_{n+1}} \int_{t_n}^{t_{n+1}} [T(s)w_n - T(s)q] ds \]
\[
\leq \|\omega_{n+1} - \omega_n\| + \frac{2|\ell_{n+1} - t_n|}{t_{n+1}} \|\omega_n - q\|.
\]

Let \( M = \frac{1}{\bar{\beta}} \sum_{i=1}^{k} 2M_i < \infty \), since

\[
\|\omega_{n+1} - \omega_n\| \leq \frac{1}{\bar{\beta}} \sum_{i=1}^{k} \|u_{n+1}^{(i)} - u_n^{(i)}\| \leq \|x_{n+1} - x_n\| + M|r_{n+1} - r_n|,
\]

hence

\[
\left\| \frac{1}{t_{n+1}} \int_{t_n}^{t_{n+1}} T(s)P_{C_2}\omega_{n+1}ds - \frac{1}{t_n} \int_{0}^{t_n} T(s)P_{C_2}\omega_n ds \right\| \\
\leq \|x_{n+1} - x_n\| + M|r_{n+1} - r_n| + \frac{2|t_{n+1} - t_n|}{t_{n+1}} \|\omega_n - q\|. \tag{3.4}
\]

Suppose \( z_n = \frac{\alpha_n\gamma f(x_n) + ((1 - \beta_n)I - \alpha_nA)\Lambda_n}{1 - \beta_n} \), where \( \Lambda_n := \frac{1}{t_n} \int_{0}^{t_n} T(s)P_{C_2}\omega_n ds \). It follows from (3.3), (3.4)

\[
\|z_{n+1} - z_n\| = \left\| \frac{\alpha_n+1\gamma f(x_{n+1}) + ((1 - \beta_{n+1})I - \alpha_{n+1}A)\Lambda_{n+1}}{1 - \beta_{n+1}} \right\|
\]

\[
- \frac{\alpha_n\gamma f(x_n) + ((1 - \beta_n)I - \alpha_nA)\Lambda_n}{1 - \beta_n}
\]

\[
= \|\frac{\alpha_n+1\gamma f(x_{n+1}) + (\frac{(1 - \beta_{n+1})\Lambda_{n+1}}{1 - \beta_{n+1}}) - \frac{\alpha_{n+1}A\Lambda_{n+1}}{1 - \beta_{n+1}}}{1 - \beta_{n+1}} - \frac{\alpha_n\gamma f(x_n) + (\frac{(1 - \beta_n)\Lambda_n}{1 - \beta_n}) + \frac{\alpha_nA\Lambda_n}{1 - \beta_n}}{1 - \beta_n}\|
\]

\[
= \|\frac{\alpha_n+1}{1 - \beta_{n+1}}(\gamma f(x_{n+1}) - A\Lambda_{n+1}) + \frac{\alpha_n}{1 - \beta_n}(A\Lambda_n - \gamma f(x_n)) + (\Lambda_{n+1} - \Lambda_n)\|
\]

\[
\leq \frac{\alpha_n+1}{1 - \beta_{n+1}}\|\gamma f(x_{n+1}) - A\Lambda_{n+1}\| + \frac{\alpha_n}{1 - \beta_n}\|A\Lambda_n - \gamma f(x_n)\| + \|\Lambda_{n+1} - \Lambda_n\|
\]

\[
\leq \frac{\alpha_n+1}{1 - \beta_{n+1}}\|\gamma f(x_{n+1}) - A\Lambda_{n+1}\| + \frac{\alpha_n}{1 - \beta_n}\|A\Lambda_n - \gamma f(x_n)\| + \|x_{n+1} - x_n\|
\]

\[
+ M|r_{n+1} - r_n| + \frac{2|t_{n+1} - t_n|}{t_{n+1}} \|\omega_n - q\|.
\]

(C1), (C3) and (C4) implies that

\[
\limsup_{n \to \infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0.
\]

By Lemma 2.4

\[
\lim_{n \to \infty} \|z_n - x_n\| = 0.
\]

Consequently

\[
\lim_{n \to \infty} \|x_{n+1} - x_n\| = \lim_{n \to \infty} (1 - \beta_n)\|z_n - x_n\| = 0. \tag{3.5}
\]

Moreover, for any \( i \in \{1, 2, \ldots, k\}, \)

\[
\|u_n^{(i)} - q\|^2 \leq \|(x_n - q) - r_n(\Psi_i x_n - \Psi_i q)\|^2
\]
\[
\|x_n - q\|^2 - 2r_n \langle x_n - q, \Psi_i x_n - \Psi_i q \rangle + r_n^2 \|\Psi_i x_n - \Psi_i q\|^2 \\
\leq \|x_n - q\|^2 - r_n(2\mu_i - r_n) \|\Psi_i x_n - \Psi_i q\|^2 ,
\]
and then
\[
\|\omega_n - q\|^2 = \|\sum_{i=1}^k \frac{1}{k} (u^{(i)}_n - q)\|^2 \leq \frac{1}{k} \|u^{(i)}_n - q\|^2 \leq \frac{1}{k} \|x_n - q\|^2 - r_n(2\mu_i - r_n) \|\Psi_i x_n - \Psi_i q\|^2 .
\] (3.6)

By (3.6), we have
\[
\|x_{n+1} - q\|^2 = \|\alpha_n (\gamma f(x_n) - Aq) + \beta_n (x_n - q) + (1 - \beta_n) I - \alpha_n A)(\Lambda_n - q)\|^2 \\
\leq \alpha_n \|\gamma f(x_n) - Aq\|^2 + \beta_n \|x_n - q\|^2 + (1 - \beta_n - \alpha_n \lambda) \|\Lambda_n - q\|^2 \\
\leq \alpha_n \|\gamma f(x_n) - Aq\|^2 + \beta_n \|x_n - q\|^2 + (1 - \beta_n - \alpha_n \lambda) \|\omega_n - q\|^2 \\
\leq \alpha_n \|\gamma f(x_n) - Aq\|^2 + \beta_n \|x_n - q\|^2 + (1 - \beta_n - \alpha_n \lambda) \|x_n - q\|^2 \\
- \frac{1}{k} \sum_{i=1}^k r_n(2\mu_i - r_n) \|\Psi_i x_n - \Psi_i q\|^2 \\
\leq \alpha_n \|\gamma f(x_n) - Aq\|^2 + \|x_n - q\|^2 \\
- (1 - \beta_n - \alpha_n \lambda) \frac{1}{k} \sum_{i=1}^k r_n(2\mu_i - r_n) \|\Psi_i x_n - \Psi_i q\|^2 ,
\]
and hence
\[
(1 - \beta_n - \alpha_n \lambda) \frac{1}{k} \sum_{i=1}^k b(2\mu_i - a) \|\Psi_i x_n - \Psi_i q\|^2 \\
\leq \alpha_n \|\gamma f(x_n) - Aq\|^2 + \|x_n - q\|^2 - \|x_{n+1} - q\|^2 \\
\leq \alpha_n \|\gamma f(x_n) - Aq\|^2 + \|x_{n+1} - x_n\| \|x_{n+1} - q\| - \|x_n - q\|).
\]

Since \(\alpha_n \to 0\) and \(\|x_{n+1} - x_n\| \to 0\), it follows that
\[
\lim_{n \to \infty} \|\Psi_i x_n - \Psi_i q\| = 0, \forall i = 1, 2, \ldots, k.
\] (3.7)

(iii) By Lemma 2.2, for any \(i = 1, 2, \ldots, k\),
\[
\|u^{(i)}_n - q\|^2 \leq \langle (I - r_n \Psi_i)x_n - (I - r_n \Psi_i)q, u^{(i)}_n - q \rangle \\
= \frac{1}{2} \| (I - r_n \Psi_i)x_n - (I - r_n \Psi_i)q \|^2 + \|u^{(i)}_n - q\|^2 \\
- \| (I - r_n \Psi_i)x_n - (I - r_n \Psi_i)q - (u^{(i)}_n - q) \|^2 \\
\leq \frac{1}{2} \| (I - r_n \Psi_i)x_n - (I - r_n \Psi_i)q - (u^{(i)}_n - q) \|^2 \\
\leq \frac{1}{2} \| x_n - q \|^2 + \|u^{(i)}_n - q\|^2 - \|x_n - u^{(i)}_n - r_n(\Psi_i x_n - \Psi_i q)\|^2 \\
= \frac{1}{2} \| x_n - q \|^2 + \|u^{(i)}_n - q\|^2 - (\|x_n - u^{(i)}_n\|^2 \\
- 2r_n \langle x_n - u^{(i)}_n, \Psi_i x_n - \Psi_i q \rangle + r_n^2 \|\Psi_i x_n - \Psi_i q\|^2 ) .
\]
This implies
\[
\| u_n^{(i)} - q \|^2 \leq \| x_n - q \|^2 - \| x_n - u_n^{(i)} \|^2 + 2r_n \| x_n - u_n^{(i)} \| \| \Psi_i x_n - \Psi_i q \|, \tag{3.8}
\]

and hence
\[
\| \omega_n - q \|^2 = \| \frac{1}{k} \sum_{i=1}^{k} (u_n^{(i)} - q) \|^2 
\leq \frac{1}{k} \sum_{i=1}^{k} \| u_n^{(i)} - q \|^2 
\leq \| x_n - q \|^2 - \frac{1}{k} \sum_{i=1}^{k} \| u_n^{(i)} - x_n \|^2 + \frac{2}{k} \sum_{i=1}^{k} 2r_n \| x_n - u_n^{(i)} \| \| \Psi_i x_n - \Psi_i q \|. \tag{3.9}
\]

Observe that
\[
\| x_{n+1} - q \|^2 \leq \alpha_n \| \gamma f(x_n) - Aq \|^2 + \beta_n \| x_n - q \|^2 + (1 - \beta_n - \alpha_n \lambda) \| \omega_n - q \|^2 
\leq \alpha_n \| \gamma f(x_n) - Aq \|^2 + \beta_n \| x_n - q \|^2 + (1 - \beta_n - \alpha_n \lambda) \{ \| x_n - q \|^2 
\leq \frac{1}{k} \sum_{i=1}^{k} \| u_n^{(i)} - x_n \|^2 + \frac{2}{k} \sum_{i=1}^{k} 2r_n \| x_n - u_n^{(i)} \| \| \Psi_i x_n - \Psi_i q \|.
\]

It follows that
\[
(1 - \beta_n - \alpha_n \lambda) \frac{1}{k} \sum_{i=1}^{k} \| u_n^{(i)} - x_n \|^2 \leq \alpha_n \| \gamma f(x_n) - Aq \|^2 + \| x_n - q \|^2 - \| x_{n+1} - q \|^2 
\leq \alpha_n \| \gamma f(x_n) - Aq \|^2 + \| x_{n+1} - x_n \| \| x_n - q \| - \| x_{n+1} - q \| 
\leq \alpha_n \| \gamma f(x_n) - Aq \|^2 + \| x_{n+1} - x_n \| \| x_n - q \| - \| x_{n+1} - q \| 
\leq \alpha_n \| \gamma f(x_n) - Aq \|^2 + \| x_{n+1} - x_n \| \| x_n - q \| - \| x_{n+1} - q \|.
\]

Since \( \alpha_n \to 0 \) and \( \| x_{n+1} - x_n \| \to 0 \), we have
\[
\lim_{n \to \infty} \| u_n^{(i)} - x_n \| = 0. \tag{3.10}
\]

It is easy to prove
\[
\lim_{n \to \infty} \| \omega_n - x_n \| = 0. \tag{3.11}
\]

The definition of \( \{ x_n \} \) shows
\[
\| \Lambda_n - x_n \| \leq \| x_{n+1} - x_n \| + \| x_{n+1} - \Lambda_n \| 
\leq \| x_{n+1} - x_n \| + \| \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n A) \Lambda_n - \Lambda_n \| 
\leq \| x_{n+1} - x_n \| + \alpha_n \| \gamma f(x_n) - A \Lambda_n \| + \beta_n \| x_n - \Lambda_n \|.
\]
That is,
\[ \|\Lambda_n - x_n\| \leq \frac{1}{1 - \beta_n} \|x_{n+1} - x_n\| + \frac{\alpha_n}{1 - \beta_n} \|\gamma f(x_n) - A\Lambda_n\|. \]

The condition (C1) together (3.5) implies that
\[ \lim_{n \to \infty} \|\Lambda_n - x_n\| = 0. \] (3.12)

Moreover, \( \|\omega_n - \Lambda_n\| \leq \|\omega_n - x_n\| + \|x_n - \Lambda_n\| \), we get
\[ \lim_{n \to \infty} \|\Lambda_n - \omega_n\| = 0. \] (3.13)

Then,
\[ \lim_{n \to \infty} \left\| x_n - \frac{1}{t_n} \int_0^{t_n} T(s)PC_2\omega_n ds \right\| = 0, \]
\[ \lim_{n \to \infty} \left\| \omega_n - \frac{1}{t_n} \int_0^{t_n} T(s)PC_2\omega_n ds \right\| = 0. \]

**Theorem 3.2.** Suppose all assumptions of Theorem 3.1 are hold. Then the sequence \( \{x_n\} \) is strongly convergent to a point \( \tilde{x} \), where \( \tilde{x} \in \bigcap_{i=1}^k F(S) \cap \text{GEP}(F_i, \Psi_i) \), which solves the variational inequality
\[ \langle (A - \gamma f)\tilde{x}, \tilde{x} - x \rangle \leq 0. \]
Equivalently, \( \tilde{x} = P_{\bigcap_{i=1}^k F(S) \cap \text{GEP}(F_i, \Psi_i)}(I - A + \gamma f)(\tilde{x}) \).

**Proof.** For all \( x, y \in H \), we have
\[
\|P_{\bigcap_{i=1}^k F(S) \cap \text{GEP}(F_i, \Psi_i)}(I - A + \gamma f)(x) - P_{\bigcap_{i=1}^k F(S) \cap \text{GEP}(F_i, \Psi_i)}(I - A + \gamma f)(y)\|
\leq \|(I - A + \gamma f)(x) - (I - A + \gamma f)(y)\|
\leq \|I - A\| \|x - y\| + \gamma \|f(x) - f(y)\|
\leq (1 - \lambda) \|x - y\| + \gamma \rho \|x - y\|
= (1 - (\lambda - \gamma \rho)) \|x - y\|.
\]

This implies that \( P_{\bigcap_{i=1}^k F(S) \cap \text{GEP}(F_i, \Psi_i)}(I - A + \gamma f) \) is a contraction of \( H \) into itself. Since \( H \) is complete, then there exists a unique element \( \tilde{x} \in H \) such that
\[ \tilde{x} = P_{\bigcap_{i=1}^k F(S) \cap \text{GEP}(F_i, \Psi_i)}(I - A + \gamma f)(\tilde{x}). \]

Next, we prove
\[ \limsup_{n \to \infty} \langle (A - \gamma f)\tilde{x}, \tilde{x} - \frac{1}{t_n} \int_0^{t_n} T(s)PC_2\omega_n ds \rangle \leq 0. \]
Let $\tilde{x} = P_{\cap_{i=1}^{k} F(S) \cap \text{GEPS}(F_i, \psi_i)} x_1$, set

$$
\Sigma = \{ \bar{y} \in C_2 : \| \bar{y} - \tilde{x} \| \leq \| x_1 - \tilde{x} \| + \frac{\| \gamma f(\tilde{x}) - A \tilde{x} \|}{\lambda - \gamma \rho} \}.
$$

It is clear, $\Sigma$ is nonempty closed bounded convex subset of $C_2$ and $S = \{ T(s) : s \in [0, \infty) \}$ is a nonexpansive semigroup on $\Sigma$.

Let $\Lambda_n = \frac{1}{t_n} \int_0^{t_n} T(s) P_{C_2} \omega_n ds$, since $\{ \Lambda_n \} \subset \Sigma$ is bounded, there is a subsequence $\{ \Lambda_{n_j} \}$ of $\{ \Lambda_n \}$ such that

$$
\limsup_{n \to \infty} \langle (A - \gamma f) \bar{x}, \bar{x} - \Lambda_n \rangle = \lim_{j \to \infty} \langle (A - \gamma f) \bar{x}, \bar{x} - \Lambda_{n_j} \rangle. \tag{3.14}
$$

As $\{ \omega_n \}$ is also bounded, there exists a subsequence $\{ \omega_{n_{j_l}} \}$ of $\{ \omega_{n_j} \}$ such that $\omega_{n_{j_l}} \to \xi$. Without loss of generality, let $\omega_{n_j} \to \xi$. From $(iii)$ in Theorem 3.1, we have $\Lambda_{n_j} \to \xi$.

Since $\{ \omega_n \} \subset C_1$ and $\{ \Lambda_n \} \subset C_2$ and $C_1, C_2$ are two closed convex subsets in $H$, we obtain that $\xi \in C_1 \cap C_2$.

Now, we prove the following items:

(i) $\xi \in F(S) = \bigcap_{s \geq 1} F(T(s))$.

Since $\{ \Lambda_n \} \subset C_2$, we have

$$
\| \Lambda_n - P_{C_2} \omega_n \| = \| P_{C_2} \Lambda_n - P_{C_2} \omega_n \|
\leq \| \Lambda_n - \omega_n \|.
$$

By $(iii)$ in Theorem 3.1, we have

$$
\lim_{n \to \infty} \| \Lambda_n - P_{C_2} \omega_n \| = 0. \tag{3.15}
$$

By using $(iii)$ in Theorem 3.1 and (3.15), we obtain

$$
\lim_{n \to \infty} \| \omega_n - P_{C_2} \omega_n \| = 0. \tag{3.16}
$$

This shows that the sequence $P_{C_2} \omega_{n_j} \to \xi$ as $j \to \infty$.

For each $h > 0$, we have

$$
\| T(h) P_{C_2} \omega_n - T(h) \Lambda_n \|
\leq \| T(h) P_{C_2} \omega_n - T(h) \Lambda_n \|
+ \| T(h) \Lambda_n - \Lambda_n \|
+ \| \Lambda_n - P_{C_2} \omega_n \|
\leq 2 \| \Lambda_n - P_{C_2} \omega_n \| + \| T(h) \Lambda_n - \Lambda_n \|.
$$

The lemma 2.5 implies that

$$
\lim_{n \to \infty} \| T(h) \Lambda_n - \Lambda_n \| = 0, \tag{3.17}
$$
the equalities (3.15, 3.16) and (3.17) implies that

$$\lim_{n \to \infty} \|T(h)P_{C_2} \omega_n - \omega_n\| = 0.$$  

Note that $F(TP_C) = F(T)$ for any mapping $T : C \to C$. The Lemma 2.7 implies that $\xi \in F(T(h)P_{C_2}) = F(T(h))$ for all $h > 0$. This shows that $\xi \in F(S)$.

(ii) $\xi \in \bigcap_{i=1}^{k} GEPS(F_i, \Psi_i)$.

Since $\{\omega_n\}$ is bounded and as respects (3.13), there exists a subsequence $\{\omega_{n_j}\}$ of $\{\omega_n\}$ such that $\omega_{n_j} \to \xi$. By intuition from [8],

$$F_i(u^{(i)}_n, y) + \langle \Psi_i x_n, y - u^{(i)}_n \rangle + \frac{1}{r_n} \langle y - u^{(i)}_n, u^{(i)}_n - x_n \rangle \geq 0, \text{ for all } y \in C_1.$$  

By (A2), we have

$$\langle \Psi_i x_n, y - u^{(i)}_n \rangle + \frac{1}{r_n} \langle y - u^{(i)}_n, u^{(i)}_n - x_n \rangle \geq F_i(y, u^{(i)}_n).$$  

Substitute $n$ by $n_j$, we get

$$\langle \Psi_i x_{n_j}, y - u^{(i)}_{n_j} \rangle + \frac{1}{r_{n_j}} \langle y - u^{(i)}_{n_j}, u^{(i)}_{n_j} - x_{n_j} \rangle \geq F_i(y, u^{(i)}_{n_j}). \quad (3.18)$$  

For $0 < l \leq 1$ and $y \in C_1$, set $y_l = l\xi + (1 - l)\xi$. We have $y_l \in C_1$ and

$$\langle y_l - u^{(i)}_{n_j}, \Psi_i y_l \rangle \geq \langle y_l - u^{(i)}_{n_j}, \Psi_i y_l \rangle - \langle \Psi_i x_{n_j}, y_l - u^{(i)}_{n_j} \rangle$$  

$$\quad - \langle y_l - u^{(i)}_{n_j}, \frac{u^{(i)}_{n_j} - x_{n_j}}{r_{n_j}} \rangle + F_i(y_l, u^{(i)}_{n_j})$$  

$$\quad = \langle y_l - u^{(i)}_{n_j}, \Psi_i y_l - \Psi_i u^{(i)}_{n_j} \rangle + \langle y_l - u^{(i)}_{n_j}, \Psi_i u^{(i)}_{n_j} - \Psi_i x_{n_j} \rangle$$  

$$\quad - \langle y_l - u^{(i)}_{n_j}, \frac{u^{(i)}_{n_j} - x_{n_j}}{r_{n_j}} \rangle + F_i(y_l, u^{(i)}_{n_j}).$$  

The condition (A4), monotonicity of $\Psi_i$ and (3.10) implies that

$$\langle y_l - u^{(i)}_{n_j}, \Psi_i y_l - \Psi_i u^{(i)}_{n_j} \rangle \geq 0 \text{ and } \|\Psi_i u^{(i)}_{n_j} - \Psi_i x_{n_j}\| \to 0 \text{ as } j \to \infty. \text{ Hence}$$  

$$\langle y_l - \xi, \Psi_i y_l \rangle \geq F_i(y_l, \xi). \quad (3.19)$$  

Now, (A1) and (A4) together (3.19) show

$$0 = F_i(y_l, y_l) \leq lF_i(y_l, y) + (1 - l)F_i(y_l, \xi)$$  

$$\leq lF_i(y_l, y) + (1 - l)\langle y_l - \xi, \Psi_i y_l \rangle$$
\[= lF_i(y_i, y) + (1 - l)\langle y - \xi, \Psi_i y_i \rangle,\]

which yields \(F_i(y_i, y) + (1 - l)\langle y - \xi, \Psi_i y_i \rangle \geq 0.\)

By taking \(l \to 0,\) we have

\[F_i(\xi, y) + \langle y - \xi, \Psi_i \xi \rangle \geq 0.\]

This shows \(\xi \in GEPS(F_i, \Psi_i),\) for all \(i = 1, 2, \ldots, k.\) Then, \(\xi \in \bigcap_{i=1}^k GEPS(F_i, \Psi_i).\)

Now, in view of (3.14), we see

\[\limsup_{n \to \infty} \langle (A - \gamma f)\bar{x}, \bar{x} - \Lambda_n \rangle = \langle (A - \gamma f)\bar{x}, \bar{x} - \xi \rangle \leq 0. \quad (3.20)\]

Finally, we prove \(\{x_n\}\) is strongly convergent to \(\bar{x}.\)

\[\|x_{n+1} - \bar{x}\|^2 = \|\alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n A)\Lambda_n - \bar{x}\|^2\]

\[= \|\alpha_n \gamma f(x_n) - \Lambda\bar{x}\| + \beta_n \|x_n - \bar{x}\|^2 + (1 - \beta_n)I - \alpha_n A) \|\Lambda_n - \bar{x}\|\|^2\]

\[= \beta_n \|x_n - \bar{x}\|^2 + ((1 - \beta_n)I - \alpha_n A) \|\Lambda_n - \bar{x}\|\|^2 + \alpha_n^2 \gamma f(x_n) - A\bar{x}\|^2\]

\[\leq \|\Lambda_n - \bar{x}\| + \beta_n \|x_n - \bar{x}\|^2 + \alpha_n^2 \gamma f(x_n) - A\bar{x}\|^2\]

\[\leq \|\Lambda_n - \bar{x}\| + \beta_n \|x_n - \bar{x}\|^2 + \alpha_n^2 \gamma f(x_n) - A\bar{x}\|^2\]

\[\leq \|\Lambda_n - \bar{x}\| + \beta_n \|x_n - \bar{x}\|^2 + \alpha_n^2 \gamma f(x_n) - A\bar{x}\|^2\]

\[\leq \|\Lambda_n - \bar{x}\| + \beta_n \|x_n - \bar{x}\|^2 + \alpha_n^2 \gamma f(x_n) - A\bar{x}\|^2\]

Consequently

\[\|x_{n+1} - \bar{x}\|^2 \leq \|\alpha_n \lambda\|^2 + 2\rho \alpha_n \beta_n \gamma + 2\rho(1 - \beta_n) \gamma \alpha_n \|x_n - \bar{x}\|^2\]

\[+\alpha_n^2 \gamma f(x_n) - A\bar{x}\|^2 + 2\alpha_n^2 \gamma f(x_n) - A\bar{x}\|^2\]

\[\leq (1 - 2\alpha_n \lambda - \rho \gamma)(x_n - \bar{x}) + 2\alpha_n^2 \gamma f(x_n) - A\bar{x}\|^2\]

\[+\alpha_n^2 \gamma f(x_n) - A\bar{x}\|^2 + 2\alpha_n^2 \gamma f(x_n) - A\bar{x}\|^2\]

\[\leq (1 - 2\alpha_n \lambda - \rho \gamma)(x_n - \bar{x}) + 2\alpha_n^2 \gamma f(x_n) - A\bar{x}\|^2\]

\[+\alpha_n^2 \gamma f(x_n) - A\bar{x}\|^2 + 2\alpha_n^2 \gamma f(x_n) - A\bar{x}\|^2\]

\[\leq (1 - 2\alpha_n \lambda - \rho \gamma)(x_n - \bar{x}) + 2\alpha_n^2 \gamma f(x_n) - A\bar{x}\|^2\]

\[+\alpha_n^2 \gamma f(x_n) - A\bar{x}\|^2 + 2\alpha_n^2 \gamma f(x_n) - A\bar{x}\|^2\]

Since \(\{x_n\}, \{f(x_n)\}\) and \(\{\Lambda_n\}\) are bounded, one can take a constant \(\Gamma > 0\) such that

\[\Gamma \geq \lambda^2 \|x_n - \bar{x}\|^2 + \|\gamma f(x_n) - A\bar{x}\|^2 + 2\|A(\Lambda_n - \bar{x})\| \|\gamma f(x_n) - A\bar{x}\|.\]
Let
\[ \Xi_n = 2\beta_n \langle x_n - \bar{x}, \gamma f(\bar{x}) - A\bar{x} \rangle + 2(1 - \beta_n) \langle \Lambda_n - \bar{x}, \gamma f(\bar{x}) - A\bar{x} \rangle + \Gamma \alpha_n. \]

Hence
\[ \|x_{n+1} - \bar{x}\|^2 \leq (1 - 2\alpha_n(\lambda - \rho \gamma))\|x_n - \bar{x}\|^2 + \alpha_n \Xi_n. \] (3.21)

With respect to (3.20), \( \limsup_{n \to \infty} \Xi_n \leq 0 \) and so all conditions of Lemma 2.6 are satisfied for (3.21). Consequently, the sequence \( \{x_n\} \) is strongly convergent to \( \bar{x} \).

As a result, by intuition from [8], the following mean ergodic theorem for a nonexpansive mapping in Hilbert space is proved.

**Corollary 3.3.** Suppose all assumptions of Theorem 3.1 are holds. Let \( \{T^i\} \) be a family of nonexpansive mappings on \( C_1 \) for all \( i = 1, 2, \ldots, k \) such that \( \bigcap_{i=1}^k F(T^i) \cap GEPS(F_i, \Psi_i) \neq \emptyset \). Let \( \{x_n\} \) and \( \{u^{(i)}_n\} \subset C_1 \) be sequences generated in the following manner:

\[
\begin{align*}
& x_1 \in H \text{ chosen arbitrary}, \\
& F_1(u^{(1)}_n, y) + \langle \Psi_1 x_n, y - u^{(1)}_n \rangle + \frac{1}{r_n} \langle y - u^{(1)}_n, u^{(1)}_n - x_n \rangle \geq 0, \text{ for all } y \in C_1, \\
& F_2(u^{(2)}_n, y) + \langle \Psi_2 x_n, y - u^{(2)}_n \rangle + \frac{1}{r_n} \langle y - u^{(2)}_n, u^{(2)}_n - x_n \rangle \geq 0, \text{ for all } y \in C_1, \\
& \vdots \\
& F_k(u^{(k)}_n, y) + \langle \Psi_k x_n, y - u^{(k)}_n \rangle + \frac{1}{r_n} \langle y - u^{(k)}_n, u^{(k)}_n - x_n \rangle \geq 0, \text{ for all } y \in C_1, \\
& \omega_n = \frac{1}{k} \sum_{i=1}^k u^{(i)}_n, \\
& x_{n+1} = \alpha_n y f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n A) \frac{1}{n+1} \sum_{i=0}^n P_{C_2} T^i \omega_n,
\end{align*}
\]

where \( \{\alpha_n\}, \{\beta_n\} \) are the sequences in \([0,1]\) and \( \{r_n\} \subset (0, \infty) \) is a real sequence. Suppose the following conditions are satisfied:

(C1) \( \lim_{n \to \infty} \alpha_n = 0, \sum_{n=1}^\infty \alpha_n = \infty; \)

(C2) \( 0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1; \)

(C3) \( \lim_{n \to \infty} |r_{n+1} - r_n| = 0 \) and \( 0 < b < r_n < a < 2\mu_i \) for \( i \in \{1, 2, \ldots, k\}. \)

Then the sequence \( \{x_n\} \) is strongly convergent to a point \( \bar{x} \), where
\[
\bar{x} = P_{\bigcap_{i=1}^k F(T^i) \cap GEPS(F_i, \Psi_i)} (I - A + \gamma f)(\bar{x}),
\]
is the unique solution of the variational inequality
\[ \langle (A - \gamma f)\bar{x}, \bar{x} - x \rangle \leq 0, \forall x \in \bigcap_{i=1}^k F(T^i) \cap GEPS(F_i, \Psi_i). \]
4. Application

If $T(s) = T$ for all $s > 0$ and $C_1 = C_2 = C$, then we have the following corollary.

**Corollary 4.1.** Let $H$ be a real Hilbert space, $F_1, F_2, \ldots, F_k$ be bifunctions from $C \times C$ to $\mathbb{R}$ satisfying (A1) – (A4), $\Psi_1, \Psi_2, \ldots, \Psi_k$ be $\mu_i$–inverse strongly monotone mapping on $H$, $A$ be a strongly positive linear bounded operator on $H$ with coefficient $\lambda$ and $0 < \gamma < \frac{1}{\rho}$, $f : H \to H$ be a $\rho$–contraction. Suppose that $T$ be a nonexpansive mapping on $C$ such that $\bigcap_{i=1}^{k} F(T) \cap GEPS(F, \Psi_i) \neq \emptyset$. Define the sequence $\{x_n\}$ as follows.

$$\begin{align*}
x_1 \in H, & \text{ and } u_n^{(i)} \in C, \\
F_1(u_n^{(1)}, y) + \langle \Psi_1 x_n, y - u_n^{(1)} \rangle + \frac{1}{r_n} \langle y - u_n^{(1)}, u_n^{(1)} - x_n \rangle \geq 0, & \text{ for all } y \in C, \\
F_2(u_n^{(2)}, y) + \langle \Psi_2 x_n, y - u_n^{(2)} \rangle + \frac{1}{r_n} \langle y - u_n^{(2)}, u_n^{(2)} - x_n \rangle \geq 0, & \text{ for all } y \in C, \\
\vdots & \\
F_k(u_n^{(k)}, y) + \langle \Psi_k x_n, y - u_n^{(k)} \rangle + \frac{1}{r_n} \langle y - u_n^{(k)}, u_n^{(k)} - x_n \rangle \geq 0, & \text{ for all } y \in C, \\
\omega_n = \frac{1}{k} \sum_{i=1}^{k} u_n^{(i)}, \\
x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n A)TP_C \omega_n ds,
\end{align*}$$

where $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$ and $\{r_n\} \subset (0, \infty)$ are the sequences satisfying the conditions (C1) – (C3) in Theorem 3.2. Then the sequence $\{x_n\}$ converges strongly to a point $\bar{x}$, where $\bar{x} \in \bigcap_{i=1}^{k} F(T) \cap GEPS(F, \Psi_i)$ solves the variational inequality

$$\langle (A - \gamma f)\bar{x}, \bar{x} - x \rangle \leq 0.$$ 

We apply Theorem 3.2 for finding a common fixed point of a nonexpansive semigroup mappings and strictly pseudo-contractive mapping and inverse strongly monotone mapping. Recall that, a mapping $T : C \to C$ is called strictly pseudo-contractive if there exists $k$ with $0 \leq k \leq 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \text{ for all } x, y \in C.$$ 

If $k = 0$, then $T$ is nonexpansive. Put $J = I - T$, where $T : C \to C$ is a strictly pseudo-contractive mapping. $J$ is $\frac{1-k}{2}$–inverse strongly monotone and $J^{-1}(0) = F(T)$. Indeed, for all $x, y \in C$ we have

$$\|(I - J)x - (I - J)y\|^2 \leq \|x - y\|^2 + k\|Jx - Jy\|^2.$$ 

Also

$$\|(I - J)x - (I - J)y\|^2 \leq \|x - y\|^2 + \|Jx - Jy\|^2 - 2\langle x - y, Jx - Jy \rangle.$$ 

So, we have

$$\langle x - y, Jx - Jy \rangle \geq \frac{1-k}{2} \|Jx - Jy\|^2.$$
Corollary 4.2. Let $H$ be a real Hilbert space, $F_1, F_2, \ldots, F_k$ be bifunctions from $C \times C$ to $\mathbb{R}$ satisfying (A1) – (A4), $\Psi_1, \Psi_2, \ldots, \Psi_k$ be $\mu_i$–inverse strongly monotone mapping on $H$, $A$ be a strongly positive linear bounded operator on $H$ with coefficient $\lambda$ and $0 < \gamma < \frac{\lambda}{\rho}$, $f : H \to H$ be a $\rho$–contraction. Suppose that $T : C \to H$ be a $k$–strictly pseudo-contractive mapping for some $0 \leq k < 1$ such that $\bigcap_{i=1}^{k} F(T) \cap \text{GEPS}(F_i, \Psi_i) \neq \emptyset$. Define the sequence $\{x_n\}$ as follows.

\[
\begin{align*}
&x_1 \in H, \text{ and } u_n^{(1)} \in C, \\
&F_1(u_n^{(1)}, y) + \langle \Psi_1 x_n, y - u_n^{(1)} \rangle + \frac{1}{r_n} \langle y - u_n^{(1)}, u_n^{(1)} - x_n \rangle \geq 0, \text{ for all } y \in C, \\
&F_2(u_n^{(2)}, y) + \langle \Psi_2 x_n, y - u_n^{(2)} \rangle + \frac{1}{r_n} \langle y - u_n^{(2)}, u_n^{(2)} - x_n \rangle \geq 0, \text{ for all } y \in C, \\
&\vdots \\
&F_k(u_n^{(k)}, y) + \langle \Psi_k x_n, y - u_n^{(k)} \rangle + \frac{1}{r_n} \langle y - u_n^{(k)}, u_n^{(k)} - x_n \rangle \geq 0, \text{ for all } y \in C, \\
&\omega_n = \frac{1}{k} \sum_{i=1}^{k} u_n^{(i)}, \\
&x_{n+1} = \alpha_n y f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n A)P_C J \omega_n ds,
\end{align*}
\]

where $J : C \to H$ is a mapping defined by $J x = k x + (1 - k) T x$ and $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$ and $\{r_n\} \subset (0, \infty)$ are the sequences satisfying the conditions (C1)–(C3) in Theorem 3.1. Then the sequence $\{x_n\}$ is strongly convergent to a point $\bar{x}$, where $\bar{x} \in \bigcap_{i=1}^{k} F(T) \cap \text{GEPS}(F_i, \Psi_i)$ solves the variational inequality

$$
\langle (A - \gamma f)\bar{x}, \bar{x} - x \rangle \leq 0.
$$

Proof. Note that $S : C \to H$ is a nonexpansive mapping and $F(T) = F(S)$. By Lemma 2.3 in [24] and Lemma 2.2 in [21], we have $P_C S : C \to C$ is a nonexpansive mapping and $F(P_C S) = F(S) = F(S)$. Therefore, the result follows from Corollary 4.1.

5. Numerical Examples

In this section, we show numerical examples which guarantee the main theorem. The programming has been provided with Matlab according to the following algorithm.

Example 5.1. Suppose that $H = \mathbb{R}, C_1 = [-1, 1], C_2 = [0, 1] and$

$$
F_1(x, y) = -3x^2 + xy + 2y^2, F_2(x, y) = -4x^2 + xy + 3y^2, F_3(x, y) = -5x^2 + xy + 4y^2.
$$

Also, we consider $\Psi_1(x) = x, \Psi_2 = 2x$ and $\Psi_3(x) = \frac{x}{10}$. Suppose that $A = \frac{x}{10}, f(x) = \frac{x}{10}$ with coefficient $\gamma = 1$ and $T(s) = e^{-s}$ is a nonexpansive semigroup on $C_2$. It is easy to check that $\Psi_1, \Psi_2, \Psi_3, A, f$ and $T(s)$ satisfy all conditions in Theorem 3.1. For each $y \in C_1$ there exists $z \in C_1$ such that

$$
F_1(z, y) + \langle \Psi_1 x, y - z \rangle + \frac{1}{r} \langle y - z, z - x \rangle \geq 0
$$

Note that $\text{Proof.}$
\[ -3z^2 + zy + 2y^2 + x(y - z) + \frac{1}{r} (y - z) (z - x) \geq 0 \]
\[ 2ry^2 + ((r + 1)z - (r - 1)x) y - 3rz^2 - xzr - z^2 + zx \geq 0. \]

Set \( G(y) = 2ry^2 + ((r + 1)z - (r - 1)x) y - 3rz^2 - xzr - z^2 + zx \). Then \( G(y) \) is a quadratic function of \( y \) with coefficients \( a = 2r, b = (r + 1)z - (r - 1)x \) and \( c = -3rz^2 - xzr - z^2 + zx \). So

\[
\begin{align*}
\Delta &= b^2 - 4ac \\
&= [(r + 1)z - (r - 1)x]^2 - 8r(-3rz^2 - xzr - z^2 + zx) \\
&= x^2(r - 1)^2 + 2zx(r - 1)(5r + 1) + z^2(5r + 1)^2 \\
&= [(x(r - 1) + z(5r + 1))]^2.
\end{align*}
\]

Since \( G(y) \geq 0 \) for all \( y \in C_1 \), if and only if \( \Delta = [(x(r - 1) + z(5r + 1))]^2 \leq 0 \). Therefore, \( z = \frac{1 - r}{5r + 1} x \), which yields
\[
T_{r_n}^{(1)} = u_n^{(1)} = \frac{1 - r}{5r + 1} x_n.
\]
By the same argument, for \( F_2 \) and \( F_3 \), one can conclude
\[
\begin{align*}
T_{r_n}^{(2)} &= u_n^{(2)} = \frac{1 - 2r}{7r + 1} x_n, \\
T_{r_n}^{(3)} &= u_n^{(3)} = \frac{10 - r}{90r + 10} x_n.
\end{align*}
\]
Then
\[
\omega_n = \frac{u_n^{(1)} + u_n^{(2)} + u_n^{(3)}}{3} = \frac{1}{3} \left[ \frac{1 - r}{5r + 1} + \frac{1 - 2r}{7r + 1} + \frac{10 - r}{90r + 10} \right] x_n.
\]

By choosing \( r_n = \frac{n+8}{n} \), \( t_n = n \), and \( \alpha_n = \frac{9}{10n}, \beta_n = \frac{2n-1}{10n-9} \), we have the following algorithm for the sequence \( \{x_n\} \)
\[
x_{n+1} = \frac{200n^2 - 10n - 81}{100n^2 - 90n} x_n + \frac{800n^2 - 890n + 81}{1000n^2 - 900n} \left( \frac{1 - e^{-n}}{n} \right) \omega_n.
\]

Choose \( x_1 = 1000 \). By using MATLAB software, we obtain the following table and figure of the result.

**Example 5.2.** Theorem 3.2 can be illustrated by the following numerical example where the parameters are given as follows:

\[
\begin{align*}
H &= [-10, 10], C_1 = [-1, 1], C_2 = [0, 1], A = I, f(x) = \frac{x}{5} \\
\Psi_1(x) &= x, \Psi_2 = 2x, \Psi_3(x) = \frac{x}{10}, \Psi_4(x) = 3x, \Psi_5(x) = 4x \\
\alpha_n &= \frac{1}{2n}, \beta_n = \frac{n}{2n+1}, r_n = \frac{n+1}{n}
\end{align*}
\]
Moreover, we compute $u_n^{(i)}$ for $i = 1, 2, 3, 4, 5$ as follows:

\[ T_{r_n}^{(1)} = u_n^{(1)} = \frac{1 - r_n}{5r_n + 1} x_n, \]
\[ T_{r_n}^{(2)} = u_n^{(2)} = \frac{1 - 2r_n}{7r_n + 1} x_n, \]
\[ T_{r_n}^{(3)} = u_n^{(3)} = \frac{10 - r_n}{90r_n + 10} x_n, \]
\[ T_{r_n}^{(4)} = u_n^{(4)} = \frac{1 - 3r_n}{11r_n + 1} x_n, \]
\[ T_{r_n}^{(5)} = u_n^{(5)} = \frac{1 - 4r_n}{15r_n + 1} x_n. \]
AN EXPLICIT VISCOSITY ITERATIVE ALGORITHM

<table>
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<th>n</th>
<th>$x_n$</th>
<th>n</th>
<th>$x_n$</th>
<th>n</th>
<th>$x_n$</th>
</tr>
</thead>
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</tr>
<tr>
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<tr>
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</tr>
</tbody>
</table>

Then

$$\omega_n = \frac{u_n^{(1)} + u_n^{(2)} + \ldots + u_n^{(5)}}{5}$$

$$= \frac{1}{5} \left[ 1 - r_n + \frac{1 - 2r_n}{7r_n + 1} + \frac{10 - r_n}{90r_n + 10} + \frac{1 - 3r_n}{11r_n + 1} + \frac{1 - 4r_n}{15r_n + 1} \right] x_n.$$ 

Choose $x_1 = 10$. The detailed results of proposed iterative in Theorem 3.2 are presented in the following table and figure.

**Example 5.3.** Let

$$H = [-10, 10], \ C_1 = [0, 1], \ C_2 = [-1, 1], \ A = \frac{x}{10}, \ f(x) = \frac{x}{10}$$

$$\Psi_1(x) = \Psi_2(x) = 0, \ \Psi_3(x) = x, \ \Psi_4(x) = 2x, \ \Psi_5(x) = \frac{x}{10}$$

$$\Psi_6(x) = 3x, \ \Psi_7(x) = 4x$$

$$\alpha_n = \frac{1}{n}, \ \beta_n = \frac{n}{3n+1}, \ r_n = \frac{n+1}{n}$$
Moreover, 

\[ F_1(x, y) = (1 - x^2)(x - y), \quad F_3(x, y) = -5x^2 + xy + 4y^2; \]
\[ F_2(x, y) = -x^2(x - y)^2, \quad F_6(x, y) = -6x^2 + xy + 5y^2; \]
\[ F_3(x, y) = -3x^2 + xy + 2y^2, \quad F_7(x, y) = -8x^2 + xy + 7y^2; \]
\[ F_4(x, y) = -4x^2 + xy + 3y^2. \]

By the same argument in Example 5.1, we compute \( u_n^{(i)} \) for \( i = 1, 2 \) as follows: For any \( y \in C_1 \) and \( r > 0 \), we have 

\[ F(z, y) + \frac{1}{r}(y - z, z - x) \geq 0 \iff (y - z)(rz^2 + z - r - x) \geq 0. \]

This implies that \( rz^2 + z - r - x = 0 \). Therefore, \( z = \frac{-1 + \sqrt{1 + 4r(r + x)}}{2r} \) which yields 

\[ T_n^{(1)} = \frac{-1 + \sqrt{1 + 4r_n(r_n + x_n)}}{2r_n}. \]

Also, we have 

\[ F(z, y) + \frac{1}{r}(y - z, z - x) \geq 0 \iff -rz^2y^2 + (2rz^3 + z - x)y - rz^4 - z^2 + zx \geq 0. \]

Set \( J(y) = -rz^2y^2 + (2rz^3 + z - x)y - rz^4 - z^2 + zx \). Then \( J(y) \) is a quadratic function of \( y \) with coefficients \( a = -rz^2, b = 2rz^3 + z - x \) and \( c = -rz^4 - z^2 + zx \). So 

\[ \Delta = [2rz^3 + z - x]^2 + 4rz^2(-rz^4 - z^2 + zx) \]
\[ = (z - x)^2. \]

Since \( J(y) \geq 0 \) for all \( y \in H \), if and only if \( \Delta = (z - x)^2 = 0 \). Therefore, \( z = T_n^{(2)} = x \).

Then 

\[ T_n^{(1)} = u_n^{(1)} = \frac{-1 + \sqrt{1 + 4r_n(r_n + x_n)}}{2r_n}, \]
\[ T_n^{(2)} = u_n^{(2)} = x_n, \]
\[ T_n^{(3)} = u_n^{(3)} = \frac{1 - r_n}{5r_n + 1}x_n, \]
\[ T_n^{(4)} = u_n^{(4)} = \frac{1 - 2r_n}{7r_n + 1}x_n, \]
\[ T_n^{(5)} = u_n^{(5)} = \frac{10 - r_n}{90r_n + 10}x_n, \]
\[ T_n^{(6)} = u_n^{(6)} = \frac{1 - 3r_n}{11r_n + 1}x_n, \]
\[ T_n^{(7)} = u_n^{(7)} = \frac{1 - 4r_n}{15r_n + 1}x_n. \]
Then

$$\omega_n = \frac{u_n^{(1)} + u_n^{(2)} + \ldots + u_n^{(7)}}{7}.$$  

We have

$$x_{n+1} = \frac{10n^2+3n+1}{30n^2-10n} x_n + \frac{20n^2+7n-1}{30n^2+10n} \left(1-e^{-n}\right) \omega_n.$$  

Choose $x_1 = 10$. The detailed results of proposed iterative in Theorem 3.2 are presented in the following table and figure.

### References


