ON δ-PERFECT FUNCTIONS

C. K. BASU

Abstract. δ-continuous [6] and δ-perfect [5] functions are both introduced by T. Noiri in the similar fashion as continuous and perfect functions. The purpose of the present paper is to investigate several properties of δ-perfect functions and also to determine some topological properties which are preserved by δ-continuous δ-perfect functions.

1. Introduction

T. Noiri initiated the concepts of δ-perfect [5] and δ-continuous [6] functions. The purpose of this paper is to investigate certain properties of δ-perfect functions specially, in addition, when the function is also δ-continuous. We start this discussion with a new characterization of δ-perfect functions. A new class of functions under the terminology N-compact function are defined and investigated w.r.t. their relationship with δ-perfect functions; further, we have established that for a δ-continuous function, the concepts of δ-perfectness and N-compactness are identical when the range space is locally nearly compact and Hausdorff. Preservation of certain topological properties by δ-perfect δ-continuous functions are also investigated.

Throughout this paper, by X or Y we shall mean topological spaces. A set A is called regular open if $A = \text{int} (\text{cl} A)$ and regular closed if $A = \text{cl} (\text{int} A)$. The collection of all regular open sets containing the point $x$ of $X$ is denoted by $RO(x)$. A point $x$ is said to be in the δ-closure [12] of a subset $A$ of $X$, denoted by $δ-cl A$, if for every $U \in RO(x)$, $U \cap A \neq \emptyset$. A is δ-closed if $A = δ - \text{cl} A$. The complement of δ-closed set is called δ-open. A subset $A$ of $X$ is said to be an NC-set [1] if every regular open cover of $A$ has a finite subcover. If $A = X$ and $A$ is an NC-set, then $X$ is called a nearly compact space [10]. A space $X$ is said to be locally nearly compact [1] if for each point $x$ of $X$, there exists a neighbourhood $U$ of $x$ such that cl$U$ is an NC-set in $X$. A function $f : X \rightarrow Y$ is said to be δ-continuous [6] if for each $x \in X$ and each $V \in RO(f(x))$, there exist a $U \in RO(x)$ such that $f(U) \subset V$. A function $f : X \rightarrow Y$ is said to be δ-perfect [5] if for every filter base $\mathcal{F}$ in $f(X)$ δ-converging to $y \in Y$, $f^{-1}(3)$ is δ-directed towards $f^{-1}(y)$. Equivalently $f$ is δ-perfect iff point inverses are NC-sets in $X$ and $f$ is δ-closed i.e. images of every δ-closed sets in $X$ is δ-closed in $Y$ [5]. A space $X$ is said to be almost regular compact, nearly compact, NC-sets, locally nearly compact, nearly paracompact, N-compact.

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2. δ-Continuous δ-Perfect Functions

**Theorem 2.1.** For a function \( F : X \to Y \), where \( Y \) is Hausdorff, the following are equivalent:

(i) \( f \) is δ-perfect,

(ii) for each \( y \in Y \), \( f^{-1}(y) \) is a δ-closed subset of \( X \), and if \( U \) is a δ-open cover of \( X \) that is closed under finite unions, then \( \{ Y - f[X - U] : U \in \mathcal{U} \} \) is a δ-open cover of \( Y \).

**Proof.** The proof is similar to the proof of Theorem 1.8 (c) [7] and is thus omitted.

**Lemma 2.2.** [5] If \( f : X \to Y \) is a δ-perfect function, then \( f^{-1}(K) \) is an NC-set in \( X \) for every NC-set \( K \) of \( Y \).

**Theorem 2.3.** A composition of δ-perfect functions is δ-perfect.

**Proof.** Since the composition of δ-closed function is δ-closed, the proof follows from the Lemma 2.2.

**Definition 2.4.** Let \( f : X \to Y \) and \( g : X \to Z \) be two functions. The function \( F : X \to Y \times Z \) defined by \( F(x) = (f(x), g(x)) \) for each \( x \in X \), is called the Diagonal product of \( f \) and \( g \).

**Theorem 2.5.** Let \( f : X \to Y \) and \( g : X \to Z \) (where \( Z \) is Hausdorff) be δ-perfect and δ-continuous functions respectively and also let both be surjective. Then the set \( \{(f(x), g(x)) : x \in X \} \) is δ-closed in \( Y \times Z \).

**Proof.** Let \((y, z) \notin \{(f(x), g(x)) : x \in X \} = F(X) \) (say) where \( y \in Y \) and \( z \in Z \) i.e. \( f^{-1}(y) \cap g^{-1}(z) = \phi \). This implies that \( z \notin gf^{-1}(y) \). Since \( gf^{-1}(y) \) is an NC-set in the Hausdorff space \( Z \), there exist disjoint regular open sets \( U \) and \( V \) in \( Z \) such that \( z \in U \) and \( gf^{-1}(y) \subset V \). Since \( f \) is δ-closed and \( f^{-1}(y) \subset g^{-1}(V) \), there exists a regular open set \( V_y \) in \( Y \) containing \( y \) such that \( f^{-1}(V_y) \subset g^{-1}(V) \). Therefore \( U \cap gf^{-1}(V_y) = \phi \) i.e. \( g^{-1}(U) \cap f^{-1}(V_y) = \phi \). Now \( V_y \times U \) is a regular open set in \( Y \times Z \) containing the point \((y, z)\) disjoint from \( F(X) \).

**Theorem 2.6.** If \( f : X \to Y \) is δ-perfect and \( g : X \to Z \) is δ-continuous, where \( X \) and \( Z \) are Hausdorff spaces, then the diagonal product of \( f \) and \( g \) is δ-perfect.

**Proof.** Let \((y, z) \in F(X)\), where \( F : X \to Y \times Z \) is the diagonal product of \( f \) and \( g \). We have \( F^{-1}(y, z) = f^{-1}(y) \cap g^{-1}(z) \). Since \( Z \) is Hausdorff, it is clear that every
one pointic set is $\delta$-closed and since $g$ is $\delta$-continuous, $g^{-1}(z)$ is $\delta$-closed in $X$. As $f$ is $\delta$-perfect, $f^{-1}(y)$ is an NC-set in the Hausdorff space $X$ and hence it is $\delta$-closed in $X$. Therefore $F^{-1}(y,z) = f^{-1}(y) \cap g^{-1}(z)$ is $\delta$-closed in $X$ and is contained in the NC-set $F^{-1}(y)$. So $F^{-1}(y,z)$ is an NC-set. Next we shall show that $F : X \to F(X)$ is a $\delta$-closed function. Let $A$ be any $\delta$-closed subset of $X$. To show $F(A)$ is $\delta$-closed it is sufficient to show that for any point $x^* \notin A$, either (1) $F(x^*) \notin F(A)$ or (2) there is a regular open set in $Y \times Z$ of the point $F(x^*)$ which does not meet $F(A)$. Let $y^* = f(x^*)$ and $z^* = g(x^*)$ and $D = f^{-1}(y^*)$, $E = D \cap A$ and $G = g(E)$. If $g(x^*) \notin G$, then $g(x^*) = g(x_1)$ for some $x_1 \in E$. Then $F(x_1) = (f(x_1), g(x_1)) = (f(x_1), g(x_1)) = F(x_1) \in F(A)$. Thus (1) is valid. Now suppose that $g(x^*) \notin G$. Since $g$ is $\delta$-continuous and $E$ is an NC-set, by Lemma 5.7 of T. Noiri [6], $g(E) = G$ is an NC-set in the Hausdorff space $Z$. There exist disjoint regular open sets $V^*$ and $U^*$ in $Z$ such that $g(x^*) \in V^*$ and $G \subset U^*$. The set $U = g^{-1}(U^*) \cup (X - A)$ is $\delta$-open in $X$ and $f^{-1}(y^*) \subset U$. Since $f$ is $\delta$-closed function, there exists a regular open set $V_y^*$ in $Y$ containing $y^*$ such that $f^{-1}(V_y^*) \subset U$. The set $V_y^* \times V^*$ is a regular open set in $Y \times Z$ containing $F(x^*) = (f(x^*), g(x^*)) = (y^*, g(x^*))$. We claim that $(V_y^* \times V^*) \cap F(A) = \phi$. In fact, if for some $x \in A$, $F(x) \cap (V_y^* \times V^*) \neq \phi$ then $f(x) \in V_y^*$ and $g(x) \in V^*$. $f(x) \in V_y^*$ implies $x \in U$ and $g(x) \in V^*$ implies $x \notin U$ — a contradiction. Hence the proof.

**Definition 2.7.** A function $f : X \to Y$ is said to be N-compact if $f^{-1}(K)$ is an NC-set in $X$ whenever $K$ is an NC-set in $Y$.

**Remark 2.8.** Clearly by Lemma 2.2, every $\delta$-perfect function is N-compact but that the converse is not true follows from the following example.

**Example 2.9.** Consider the identity function $i : (N, T_1) \to (N, T_2)$, where $N$ is the set of naturals, $T_1$ is the discrete topology and $T_2$ is the topology generated by the collection $\{\{1, 2\}, \{3, 4\}, \ldots\}$. Only finite sets are NC-sets in $(N, T_2)$ and as such $i$ is N-compact but $\{1\}$ is $\delta$-closed in $(N, T_1)$ but is not so in $(N, T_2)$.

It is therefore natural under what conditions an N-compact function would be a $\delta$-perfect function. The following theorem establishes one such condition.

**Theorem 2.10.** If $f : X \to Y$ is $\delta$-continuous N-compact function from a Hausdorff space $X$ into a locally nearly compact Hausdorff space $Y$ then $f$ is $\delta$-perfect.

**Proof.** Since $f$ is N-compact function, the point inverses are NC-sets. Let $A$ be a $\delta$-closed subset of $X$ and let $y \notin f(A)$ be in the $\delta$-closure of $f(A)$. Since $Y$ is locally nearly compact, there exists a regular open set $U$ in $Y$ such that $y \in U$ and $\text{cl } U$ is an NC-set in $Y$. Now $f(A) \cap \text{cl } U$ cannot be an NC-set. In fact, if it is an NC-set, there exist disjoint regular open sets $V_1$ and $V_2$ in $Y$ such that $y \in V_1$ and $f(A) \cap \text{cl } U \subset V_2$ (since $Y$ is Hausdorff). Then $U \cap V_1 \cap f(A) \subset V_1 \cap \text{cl } U \cap f(A) = \phi$ — which contradicts the fact that $y$ is in the $\delta$-closure of $f(A)$. As $\text{cl } U$ is an NC-set and $f$ is N-compact, $f^{-1}(\text{cl } U)$ is an NC-set in $X$. Therefore $A \cap f^{-1}(\text{cl } U)$ is an NC-set in the Hausdorff space $X$ and so $f[A \cap f^{-1}(\text{cl } U)] = f(A) \cap \text{cl } U$ is an NC-set — a contradiction. So $y \in f(A)$. 

Corollary 2.11. A $\delta$-continuous function $f : X \to Y$, where $X$ is Hausdorff and $Y$ is locally nearly compact Hausdorff is $N$-compact iff it is $\delta$-perfect.

In the above discussion $\delta$-continuity plays a very crucial role. It is of interest under what conditions on the domain and co-domain spaces, the other restrictions on $f$ may imply that $f$ is $\delta$-continuous.

Theorem 2.12. If $f : X \to Y$ is a surjective function from a almost regular space $X$ onto a nearly compact space $Y$ with the property that $f$ is $\delta$-closed and point inverses are $\delta$-closed sets, then $f$ is $\delta$-continuous.

Proof. Let $f$ be not $\delta$-continuous. Then by Theorem 2.2 of T. Noiri [6], there exist a point $x \in X$ and a $V \in RO(f(x))$ such that for every $U \in RO(x)$, $f(U) \cap (Y - V) \neq \emptyset$. Since $f$ is $\delta$-closed $f(clU) \cap (Y - V)$ is a $\delta$-closed set in $Y$. The collection $\{f(clU) \cap (Y - V) : U \in RO(x)\}$ has the finite inter-section property. If not i.e. if there exist $U_1, U_2, \ldots, U_n \in RO(x)$ such that $\cap_{i=1}^n [f(clU_i) \cap (Y - V)] = \emptyset$, then it can be easily shown that $f(\cap_{i=1}^n U_i) \cap (Y - V) = \emptyset$, which shows that $f$ is $\delta$-continuous — a contradiction. As $Y$ is nearly compact, $\cap_{U \in RO(x)} [f(clU) \cap (Y - V)] \neq \emptyset$. Let $y^*$ belong to the intersection, then clearly $f(x) \neq y^*$. So $x \notin f^{-1}(y^*)$. By the almost regularity of $X$, there exist disjoint regular open sets $U_1^*$ and $U_2^*$ in $X$ such that $x \in U_1^*$ and $f^{-1}(y^*) \subset U_2^*$. So $y^* \notin f(clU_1^*)$. But $y^* \in f(clU_1^*)$ — a contradiction. So $f$ is $\delta$-continuous.

Next we shall show that the product of two $\delta$-perfect functions is $\delta$-perfect.

Lemma 2.13. Let $X_1$ and $X_2$ be two topological spaces and let $K_i$ be NC-sets in $X_i$ for $i = 1, 2$. If $Y$ be a regular open set of $X_1 \times X_2$ containing $K_1 \times K_2$, there exist $\delta$-open sets $U_i$ of $X_i$ containing $K_i$ such that $K_1 \times K_2 \subseteq U_1 \times U_2 \subseteq Y$.

Proof. We fix $x \in K_1$ and then for each $y \in K_2$, $(x, y) \in Y$. So there exist open sets $W_1$ of $X_1$ such that $x \in W_1$ and $y \in W_2$ and $(x, y) \in W_1 \times W_2 \subset intcl W_1 \times intcl W_2 \subset intcl V = Y$. Then the collection $\{intcl W_2 : y \in K_2\}$ covers $K_2$. Since $K_1$ is an NC-set, there exist $y_1, \ldots, y_n \in K_2$ such that $K_2 \subseteq \bigcup_{i=1}^n intcl W_2^{y_i} = W_z$ (say). Let $U_z = \bigcap_{i=1}^n intcl W_2^{y_i}$. Clearly $\{x\} \times K_2 \subseteq U_z \times W_z \subseteq V$. Since $K_1$ is an NC-set and the collection $\{U_z : x \in K_1\}$ is a regular open cover of $K_1$, then there exist $x_1, \ldots, x_m \in K_1$ such that $K_1 \subseteq \bigcup_{i=1}^m U_{z_{x_i}} = U_1$ (say) and $U_2 = \bigcap_{i=1}^m W_{z_{x_i}}$. Then $K_1 \times K_2 \subseteq U_1 \times U_2 \subseteq Y$.

Theorem 2.14. Let $f_i : X_i \to Y_i$ $(i = 1, 2)$ be two $\delta$-perfect functions, then the function $f = f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ defined by $(f_1 \times f_2)(x_1, x_2) = (f_1(x_1), f_2(x_2))$ is $\delta$-perfect.

Proof. Let $y = (y_1, y_2) \in Y_1 \times Y_2$. Then $f^{-1}(y) = (f_1 \times f_2)^{-1}(y_1, y_2) = f_1^{-1}(y_1) \times f_2^{-1}(y_2)$, which is an NC-set in $X_1 \times X_2$. Let $P$ be any $\delta$-closed set in $X_1 \times X_2$. Let $(y_1, y_2) \notin f(P)$. Then $(f_1 \times f_2)^{-1}(y_1, y_2) = f_1^{-1}(y_1) \times f_2^{-1}(y_2) \subset X_1 \times X_2 - P$. By Lemma 2.13, there exist $\delta$-open sets $U_i$ containing $y_i$ such that $(f_1 \times f_2)^{-1}(y_1, y_2) \subseteq
of regular closed sets of $X$. Following:

Perfect continuous functions preserve, in both directions, certain topological properties. Here, we shall investigate certain topological properties which are preserved by $\delta$-perfect $\delta$-continuous functions.

Definition 2.17. [8] A space $X$ is said to be nearly paracompact if every regular open cover of $X$ has an open locally finite refinement.

Theorem 2.18. If $f : (X, T) \to (Y, \sigma)$ is a surjective $\delta$-perfect $\delta$-continuous function, then the following are true:

i) $(X, T)$ is almost regular iff $(Y, \sigma)$ is almost regular.

ii) $(X, T)$ is Hausdorff iff $(Y, \sigma)$ is Hausdorff.

iii) $(X, T)$ is locally nearly compact iff $(Y, \sigma)$ is nearly compact.

iv) $(X, T)$ is locally nearly compact Hausdorff iff $(Y, \sigma)$ is locally nearly compact Hausdorff.

Proof. $f : (X, T) \to (Y, \sigma)$ is $\delta$-continuous if $f : (X, T_\alpha) \to (Y, \sigma_\alpha)$ is continuous [6] and $f : (X, T) \to (Y, \sigma)$ is $\delta$-perfect if $f : (X, T_\alpha) \to (Y, \sigma_\alpha)$ is perfect [5], where $(X, T_\alpha)$ and $(Y, \sigma_\alpha)$ are semiregularizations of $(X, T)$ and $(Y, \sigma)$ respectively. Also a space $(X, T)$ is almost regular (resp. Hausdorff, nearly compact, locally nearly compact Hausdorff and nearly paracompact iff $(X, T_\alpha)$ is regular (resp. Hausdorff, compact, locally compact Hausdorff [3] and paracompact [4])). Since regularity, Hausdorffness, compactness and local compactness (in presence of Hausdorffness) are preserved in both directions by perfect continuous functions, all the results are immediate.

Definition 2.19. A space $X$ is said to be weakly $T_2$ if every point is the intersection of regular closed sets of $X$.

Theorem 2.20. Let $f : X \to Y$ be a $\delta$-perfect surjective function. Then we have the following:

i) If $X$ is weakly $T_2$ then $Y$ is also weakly $T_2$.

ii) If $f$ is $\delta$-continuous and $Y$ is nearly paracompact, then $X$ is also nearly paracompact.
Proof. i) Let $X$ be weakly $T_2$. Then every point in $X$ is the intersection of regular closed sets of $X$. Therefore every point in $X$ is $\delta$-closed. Let $y \in Y$. Then for every $x \in f^{-1}(y)$, $f(x) = y$. Since $f$ is $\delta$-perfect and hence $\delta$-closed, $\{y\}$ is $\delta$-closed i.e. intersection of regular closed sets of $Y$. Therefore $Y$ is weakly $T_2$.

ii) It is immediate from the argument given in the proof of Theorem 2.18.


Theorem 2.22. Let $X$ be nearly compact and $Y$ be nearly paracompact then $X \times Y$ is nearly paracompact.

Proof. Since $\pi_Y : X \times Y \to Y$ is continuous open and hence $\theta$-continuous almost open, by Lemma 2.21, $\pi_Y$ is $\delta$-continuous. Since $X$ is nearly compact by Lemma 2.15, $\pi_Y : X \times Y \to Y$ is $\delta$-perfect. As $Y$ is nearly paracompact and $\pi_Y$ is $\delta$-perfect $\delta$-continuous surjection, by Theorem 2.20, $X \times Y$ is nearly paracompact.

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References


Department of Mathematics, University of Kalyani, Kalyani, Dist.-Nadia, West Bengal, Pin-741235, India.
E-mail: ckbasu@klyuniv.ernet.in