TAMKANG JOURNAL OF MATHEMATICS
Volume 35, Number 3, Autumn 2004

ON A TRAPEZOIDAL TYPE RULE FOR WEIGHTED INTEGRALS

ZHENG LIU

Abstract. An error runs through a paper by Cerone and Dragomir [1] is corrected. Thus enable us to get a right form of a trapezoidal type rule for weighted integrals and its applications in numerical integration.

1. Preliminaries

Some definitions are required to simplify the subsequent work.

Definition 1. Let $\omega(x)$ be a positive integrable function on $[a,b]$. Let μ and ν be its zeroth and first moments about zero so that

$$\mu=\int_a^b \omega(x)dx<\infty \tag{1.1}$$

and

$$\nu=\int_a^b x\omega(x)dx<\infty \tag{1.2}$$

Definition 2. P and Q will be used to denote the zeroth and first moments of $\omega(x)$ over a subinterval $[a,b]$. In particular, for $\lambda>0$ the subscript a or b will be used to indicate the intervals $[a,a+\lambda]$ and $[b-\lambda,b]$ respectively. Thus, for example,

$$P_a=\int_a^{a+\lambda} \omega(x)dx$$

and

$$Q_b=\int_{b-\lambda}^{b} x\omega(x)dx.$$

The following theorem is due to Hayashi [2, pp.331-312].

Theorem 1. *Let* $h:[a,b]\to\mathbb{R}$ *be a nonincreasing mapping on* $[a,b]$ *and* $g:[a,b]\to\mathbb{R}$ *an integrable mapping on* $[a,b]$ *with*

$$0\le g(x)\le A,\quad \textit{for all } x\in[a,b].$$

Received January 6, 2003; revised March 12, 2003.
2000 *Mathematics Subject Classification.* 26D15, 41A55.
Key words and phrases. Hayashi inequality, Iyengar inequality, weighted quadrature rule.

Then

$$A\int_{b-\lambda}^{b} h(x)dx \le \int_a^b h(x)g(x)dx \le A\int_a^{a+\lambda} h(x)dx \tag{1.3}$$

where

$$\lambda = \frac{1}{A}\int_a^b g(x)dx.$$

Hayashi's inequality (1.3) will now be used to obtain inequalities for weighted integrals to give trapezoidal type quadrature rules.

2. Trapezoidal Inequality for Weighted Integrals

Lemma 1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $\mathring{I}$ (the interior of I) and $[a,b] \subset \mathring{I}$ with $M = \sup_{x\in[a,b]} f'(x) < \infty$, $m = \inf_{x\in[a,b]} f'(x) > -\infty$ and $M > m$. Let $\omega(x) \ge 0$ for all $x \in [a,b]$ and $\mu = \int_a^b \omega(x)dx < \infty$, $\nu = \int_a^b x\omega(x)dx < \infty$ be the zeroth and first moments of $\omega(\cdot)$ on $[a,b]$. If f' is integrable on $[a,b]$ then the following inequality holds:*

$$\begin{aligned}(M-m)[Q_b-(b-\lambda)P_b] &\le \int_a^b \omega(x)f(x)dx - \mu(f(a)-ma) - m\nu \\ &\le (M-m)[Q_a-(\lambda+a)P_a+\lambda\mu]\end{aligned} \tag{2.1}$$

where P, Q are as describe in Definition 2 and $\lambda = \frac{b-a}{M-m}(S-m)$, $S = \frac{f(b)-f(a)}{b-a}$.

Proof. Let $h_b(x) = \int_x^b \omega(u)du$ and $g(x) = f'(x)-m$. Then from Hayashi's inequality (1.3)

$$L_b \le I_b \le U_b \tag{2.2}$$

where

$$I_b = \int_a^b h_b(x)(f'(x)-m)dx,$$
$$\lambda = \frac{1}{M-m}\int_a^b (f'(x)-m)dx,$$

and

$$L_b = (M-m)\int_{b-\lambda}^b h_b(x)dx,$$
$$U_b = (M-m)\int_a^{a+\lambda} h_b(x)dx.$$

Now, an integration by parts gives

$$I_b = -\mu(f(a)-ma) - m\nu + \int_a^b \omega(x)f(x)dx. \tag{2.3}$$

Also,

$$\lambda = \frac{b-a}{M-m}(S-m) \tag{2.4}$$

where

$$S = \frac{f(b)-f(a)}{b-a},$$

the slope of the secant over $[a,b]$.

It should be noted that $0 < \lambda \le b-a$ since $S \le M$.

For the lower bound L_b a change of order of integration gives

$$\begin{aligned}\frac{L_b}{M-m} &= \int_{b-\lambda}^{b} \omega(u) \int_{b-\lambda}^{u} dxdu \\ &= (\lambda - b)P_b + Q_b \end{aligned} \tag{2.5}$$

where P_b and Q_b are as describe in Definition 2.

Similarly, the upper bound U_b may be obtained through a change of order of integration to give

$$\begin{aligned}\frac{U_b}{M-m} &= \int_{a}^{a+\lambda} \omega(u) \int_{a}^{u} dxdu + \int_{a+\lambda}^{b} \omega(u) \int_{a}^{a+\lambda} dxdu \\ &= \int_{a}^{a+\lambda} (u-a)\omega(u)du + \lambda \int_{a+\lambda}^{b} \omega(u)du \\ &= Q_a - (\lambda + a)P_a + \lambda\mu \end{aligned} \tag{2.6}$$

where P_a and Q_a are as describe in Definition 2 and μ is the zeroth moment of $\omega(x)$ on $[a,b]$.

Using (2.2)-(2.6) the lemma is thus proved.

Lemma 2. *Let the conditions be as in Lemma* 1 *then the following inequality holds:*

$$\begin{aligned}(M-m)[Q_b - (\lambda - b)P_b - \lambda\mu] &\le \int_a^b \omega(x)f(x)dx - \mu(f(b) - mb) - m\nu \\ &\le (M-m)[Q_a - (\lambda + a)P_a]. \end{aligned} \tag{2.7}$$

Proof. The proof follows along similar lines to that of Lemma 1.

Let $h_a(x) = -\int_a^x \omega(u)du$ and $g(x) = f'(x) - m$. Then using Hayashi's inequality (1.2) gives:

$$L_a \le I_a \le U_a \tag{2.8}$$

where

$$I_a = \int_a^b h_a(x)(f'(x) - m)dx$$

and

$$L_a = (M-m)\int_{b-\lambda}^{b} h_a(x)dx,$$
$$U_a = (M-m)\int_{a}^{a+\lambda} h_a(x)dx.$$

Now, a straight forward integration by parts yields

$$I_a = -\mu(f(b)-mb) - m\nu + \int_a^b \omega(x)f(x)dx. \tag{2.9}$$

Further, an interchange of the order of integration and simplification of results yields

$$\frac{L_a}{M-m} = Q_b + (\lambda - b)P_b - \lambda_\mu \tag{2.10}$$

and

$$\frac{U_a}{M-m} = Q_a - (\lambda + a)P_a. \tag{2.11}$$

Hence, using (2.8)-(2.11) the lemma is proved.

Theorem 2. *Let the conditions of Lemmas* 1 *and* 2 *be maintained. Then the following inequality holds:*

$$(M-m)[Q_b - (b-\lambda)P_b - \frac{\lambda}{2}\mu] \le \int_a^b \omega(x)f(x)dx - \frac{\mu}{2}[f(a)+f(b)-m(a+b)] - m\nu$$
$$\le (M-m)[Q_a - (\lambda+a)P_a + \frac{\lambda}{2}\mu] \tag{2.12}$$

where the P's and Q's are as defined in Definition 2.

Proof. Addition of (2.1) and (2.7) produces (2.12) upon division by 2.

Corollary 1. *Let the conditions be as in the previous Lemmas and Theorem* 2*. Then,*

$$\left|\int_a^b \omega(x)f(x)dx - \frac{\mu}{2}[f(a)+f(b)-m(a+b)] - m\nu\right| \le \frac{\mu}{2}(b-a)(S-m)$$
$$\le \frac{M-m}{2}\mu(b-a) \tag{2.13}$$

where S is the slope of the secant on $[a,b]$.

Proof. The corollary follows readily from (2.12) on noting that

$$Q_b = \int_{b-\lambda}^{b} x\omega(x)dx \ge (b-\lambda)\int_{b-\lambda}^{b}\omega(x)dx,$$
$$Q_a = \int_a^{a+\lambda} x\omega(x)dx \le (\lambda+a)\int_a^{a+\lambda}\omega(x)dx$$

and substituting $(M-m)\lambda = (b-a)(S-m)$.

Remark 1. Allowing $\omega(x) \equiv 1$ in (2.12) gives from Definitions 1 and 2

$$\mu = b-a,\quad \nu = \frac{b^2-a^2}{2},\quad P_a = P_b = \lambda,\quad Q_a = \frac{\lambda}{2}(\lambda+2a) \text{ and } Q_b = \frac{\lambda}{2}(2b-\lambda).$$

This reveals the lower bound to be negative the upper bound and we have the result of Cerone and Dragomir [3] as

$$\left|\int_a^b f(x) - \frac{b-a}{2}[f(a)+f(b)]\right| \le \frac{(b-a)^2}{2(M-m)}(S-m)(M-S) \tag{2.14}$$

$$\le \frac{M-m}{2}\left(\frac{b-a}{2}\right)^2 \tag{2.15}$$

where $S = \frac{f(b)-f(a)}{b-a}$. It should be mentioned that (2.14) is first proved by Agarwal and Dragomir [4] which is a generalization of the well known Iyengar inequality [5].

Remark 2. The bounds in (2.12) are not symmetric in general since for this to be so they must sum to zero. Let L_1 be the lower bound and U_1 be the upper bound. Then

$$U_1 + L_1 = (M-m)[(Q_b - (b-\lambda)P_b) - ((\lambda+a)P_a - Q_b)].$$

We know from the proof of Corollary 1 that $Q_b \ge (b-\lambda)P_b$ and $Q_a \le (\lambda+a)P_a$, so $U_1 + L_1 = 0$ when $Q_b - (b-\lambda)P_b = (\lambda+a)P_a - Q_a$.

Lemma 3. *Let the Conditions of Theorem* 2 *and Lemmas* 1 *and* 2 *hold. Then, for* $\omega(x)$ *symmetric about the mid-point* $\frac{a+b}{2}$, *the bounds in* (2.12) *are symmetric. Hence*

$$\left|\int_a^b \omega(x)f(x)dx - \frac{\mu}{2}[f(a)+f(b)-m(a+b)] - m\nu\right|$$
$$\le (M-m)\left[\frac{\lambda}{2}\mu - \int_0^\lambda u\omega(\lambda+a-u)du\right].$$

Proof. From Remark 2 and Definition 2, the sum of the upper and lower bounds in (2.12), U_1 and L_1 respectively is:

$$U_1 + L_1 = (M-m)\left[\int_{b-\lambda}^b [x-(b-\lambda)]\omega(x)dx - \int_a^{a+\lambda}(\lambda+a-x)\omega(x)dx\right]$$
$$= (M-m)\left[\int_0^\lambda u\omega(b-\lambda+u)du - \int_0^\lambda u\omega(\lambda+a-u)du\right].$$

Now,

$$U_1 + L_1 = (M-m)\int_0^\lambda u\left[\omega\left(\frac{a+b}{2}+z\right) - \omega\left(\frac{a+b}{2}-z\right)\right]du$$

where $z = \frac{b-a}{2} - \lambda + u$.

Thus

$$U_1 + L_1 = (M-m) \int_{\frac{b-a}{2}-\lambda}^{\frac{b-a}{2}} \left(z + \lambda - \frac{b-a}{2} \right) \left[\omega \left(\frac{a+b}{2} + z \right) - \omega \left(\frac{a+b}{2} - z \right) \right] dz = 0$$

for $\omega(\cdot)$ symmetric about $\frac{a+b}{2}$. Hence the bounds in (2.12) are symmetric.

Now, from the upper bound in (2.12), U_1 is such that

$$\begin{aligned} \frac{U_1}{M-m} &= \frac{\lambda}{2}\mu - [(\lambda + a)P_a - Q_a] \\ &= \frac{\lambda}{2}\mu - \int_a^{a+\lambda} (\lambda + a - x)\omega(x)dx \\ &= \frac{\lambda}{2}\mu - \int_0^{\lambda} u\omega(\lambda + a - u)du. \end{aligned}$$

Thus, the lemma is proved.

It should be noted that the expression for U_1 obtained above may be written as

$$\begin{aligned} \frac{U_1}{M-m} &= \frac{\lambda}{2}\mu - \int_0^{\lambda} u\omega \left(\frac{a+b}{2} - z \right) dz \\ &= \frac{\lambda}{2}\mu - \int_0^{\lambda} u\omega \left(z - \frac{a+b}{2} \right) dz \end{aligned}$$

where $z = u + \frac{b-a}{2} - \lambda$. Here, we are using the fact that the weight function $\omega(\cdot)$ is symmetric about the mid-point.

Corollary 2. *Let the conditions be as in the previous lemmas and Theorem* 2. *Then*

$$\begin{aligned} &(M-m)[Q_b - (b-\lambda)P_b] - \mu \left[\left(\frac{b-a}{2} \right) S + am \right] + m\nu \\ &\le \int_a^b \omega(x)f(x)dx - \frac{\mu}{2}[f(a) + f(b)] \\ &\le (M-m)[Q_a - (\lambda + a)P_a] + \mu \left[\left(\frac{b-a}{2} \right) S - bm \right] + m\nu. \end{aligned}$$

Proof. A simple rearrangement of the terms in (2.12), collecting the coefficient of μ and using the fact that $(M-m)\lambda = (b-a)(S-m)$ produces the result.

Remark 3. Using similar approximation as those in Corollary 1, simpler bounds may be obtained viz.,

$$\begin{aligned} &m\nu - \mu \left[\left(\frac{b-a}{2} \right) S + am \right] \\ &\le \int_a^b \omega(x)f(x)dx - \frac{\mu}{2}[f(a) + f(b)] \le m\nu + \mu \left[\left(\frac{b-a}{2} \right) S - bm \right]. \end{aligned}$$

3. Application in Numerical Integration

In this section we will demonstrate how the results obtained in Section 2 may be utilized to obtain quadrature rules for weighted functions.

Theorem 3. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $M = \sup_{x\in[a,b]} f'(x) < \infty$, $m = \inf_{x\in[a,b]} f'(x) > -\infty$, and $M > m$. Let I_n be a partition of $[a,b]$ such that $I_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$. Further, let $\omega(x) \ge 0$ for all $x \in [a,b]$ and $\mu = \int_a^b \omega(x)dx < \infty$, $\nu = \int_a^b x\omega(x)dx < \infty$ be the zeroth and first moments of $\omega(\cdot)$ on $[a,b]$. Then, the following weighted quadrature rule holds*

$$\int_a^b \omega(x)f(x)dx = A(\omega, f, I_n) + R(\omega, f, I_n)$$

where $A(\omega, f, I_n)$ is an approximation to the weighted integral. Namely,

$$\begin{aligned} A(\omega, f, I_n) &= \frac{1}{2}\sum_{i=0}^{n-1} \mu_i[f(x_i) + f(x_{i+1}) - m(x_i + x_{i+1})] + m\nu \\ &= \frac{1}{2}\left[\mu_0 g_0 + \mu_{n-1}g_n + \sum_{i=1}^{n-1}(u_{i-1} + \mu_i)g_i\right] + m\nu \end{aligned}$$

with $g_i = f(x_i) - mx_i$, $u_i = \int_{x_i}^{x_{i+1}} \omega(x)dx$, $\nu_i = \int_{x_i}^{x_{i+1}} x\omega(x)dx$, $i = 0, 1, \ldots, n-1$. In addition, the remainder term $R(\omega, f, I_n)$ satisfies

$$\begin{aligned} |R(\omega, f, I_n)| &\le \frac{1}{2}\sum_{i=0}^{n-1} \mu_i[f(x_{i+1}) - f(x_i) - m(x_{i+1} - x_i)] \\ &= \frac{1}{2}\left[\mu_{n-1}g_n - \mu_0 g_0 + \sum_{i=1}^{n-1}(\mu_{i-1} - \mu_i)g_i\right] \\ &\le \frac{M-m}{2}\sum_{i=0}^{n-1}\mu_i h_i, \end{aligned}$$

where $h_i = x_{i+1} - x_i$.

Proof. Applying inequality (2.13) of Corollary 1 on the interval $[x_i, x_{i+1}]$ for $i = 0, 1, \ldots, n-1$ we have

$$\begin{aligned} &\left|\int_{x_i}^{x_{i+1}} \omega(x)f(x)dx - \frac{\mu_i}{2}[f(x_i) + f(x_{i+1}) - m(x_i + x_{i+1})] - m\nu_i\right| \\ &\le \frac{\mu_i}{2}[f(x_{i+1}) - f(x_i) - m(x_{i+1} - x_i)]. \end{aligned}$$

Summing over i for $i = 0, 1, \ldots, n-1$ gives the quadrature rule

$$A(\omega, f, I_n) = \frac{1}{2}\sum_{i=0}^{n-1} \mu_i[f(x_i) + f(x_{i+1}) - m(x_i + x_{i+1})] + m\sum_{i=0}^{n-1}\nu_i$$

$$= \frac{1}{2}\sum_{i=0}^{n-1}\mu_i(g_i+g_{i+1})+m\nu$$

where $g_i = f(x_i) - mx_i$.

Hence

$$A(\omega, f, I_n) = \frac{1}{2}\left[\mu_0 g_0 + \mu_{n-1}g_n + \sum_{i=1}^{n-1}(\mu_{i-1}+\mu_i)g_i\right] + m\nu.$$

The remainder term $R(\omega, f, I_n)$ is such that

$$\begin{aligned}|R(\omega, f, I_n)| &\le \frac{1}{2}\sum_{i=0}^{n-1}\mu_i[f(x_{i+1})-f(x_i)-m(x_{i+1}-x_i)]\\ &= \frac{1}{2}\sum_{i=0}^{n-1}\mu_i[g_{i+1}-g_i]\\ &= \frac{1}{2}\left[\mu_{n-1}g_n - \mu_0 g_0 + \sum_{i=1}^{n-1}(\mu_{i-1}-\mu_i)g_i\right].\end{aligned}$$

Using the second inequality in Corollary 1 gives

$$|R(\omega, f, I_n)| \le \frac{M-m}{2}\sum_{i=0}^{n-1}\mu_i h_i.$$

Hence the theorem is proved.

If a uniform grid is taken so that $x_i = x_0 + ih$, $i = 0, 1, \ldots, n$, then

$$|R(\omega, f, I_n)| \le \frac{M-m}{2}h\mu.$$

References

[1] P. Cerone and S. S. Dragomir, *On a weighted generalization of Iyengar type inequalities involving bounded first derivative*, Mathematical Inequalities & Applications **3**(2000), 35-44.

[2] D. S. Mitrinović, J. E. Pečarić and A. M. Fink, Classical and New Inequalities in Analysis, Kluwer Academic Publishers, 1993.

[3] P. Cerone and S. S. Dragomir, *Lobatto type quadrature rules for functions with bounded derivative*, RGMIA Research Report Collection **2**(1999), 133-146.

[4] R. P. Agarwal and S. S. Dragomir, *An application of Hayashi's inequality for differentiable functions*, Computers Math. Appl. **32**(1996), 95-99.

[5] K. S. K. Iyengar, *Note on an inequality*, Math. Student **6**(1938), 75-76.

Institute of Applied Mathematics, Faculty of Science, Anshan University of Science and Technology, Anshan 114044, Liaoning, People's Republic of China.