ON ALMOST KENMOTSU MANIFOLDS WITH NULLITY DISTRIBUTIONS

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Abstract. The object of this paper is to characterize the curvature conditions $R \cdot P = 0$ and $P \cdot S = 0$ with its characteristic vector field $\xi$ belonging to the $(k, \mu)'\text{-nullity distribution}$ and $(k, \mu)\text{-nullity distribution}$ respectively, where $P$ is the Weyl projective curvature tensor. As a consequence of the main results we obtain several corollaries.

1. Introduction

In the present time the study of nullity distributions has become very interesting topic in Differential Geometry. Gray [7] and Tanno [12] introduced the notion of $k$-nullity distribution ($k \in \mathbb{R}$) in the study of Riemannian manifolds $(M, g)$, which is defined for any $p \in M$ and $k \in \mathbb{R}$ as follows:

$$N_p(k) = \{Z \in T_p M : R(X, Y)Z = k[g(Y, Z)X - g(X, Z)Y]\}, \quad (1.1)$$

for any $X, Y \in T_p M$, where $T_p M$ denotes the tangent vector space of $M$ at any point $p \in M$ and $R$ denotes the Riemannian curvature tensor of type $(1,3)$.

Next Blair, Koufogiorgos and Papantoniou [3] introduced the $(k, \mu)$-nullity distribution which is a generalized notion of the $k$-nullity distribution on a contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ and defined for any $p \in M^{2n+1}$ and $k, \mu \in \mathbb{R}$ as follows:

$$N_p(k, \mu) = \{Z \in T_p M^{2n+1} : R(X, Y)Z = k[g(Y, Z)X - g(X, Z)Y]$$
$$+ \mu[g(Y, Z)hX - g(X, Z)hY]\}, \quad (1.2)$$

where $h = \frac{1}{2} \mathcal{L}_\xi \phi$ and $\mathcal{L}$ denotes the Lie differentiation.

In [5], Dileo and Pastore introduced the notion of $(k, \mu)'\text{-nullity distribution}$, another generalized notion of the $k$-nullity distribution, on an almost Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$, which is defined for any $p \in M^{2n+1}$ and $k, \mu \in \mathbb{R}$ as follows:

$$N_p(k, \mu)' = \{Z \in T_p M^{2n+1} : R(X, Y)Z = k[g(Y, Z)X - g(X, Z)Y]$$

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\[ h' = h \circ \phi. \]

Also, Kenmotsu [9] introduced a new type of almost contact metric manifolds named Kenmotsu manifolds nowadays. A differentiable \((2n + 1)\)-dimensional manifold \(M\) is said to have a \((\phi, \xi, \eta)\)-structure or an almost contact structure, if it admits a \((1, 1)\) tensor field \(\phi\), a characteristic vector field \(\xi\) and a 1-form \(\eta\) satisfying

\[ \phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad (1.4) \]

where \(I\) denote the identity endomorphism. Here we include also \(\phi \xi = 0\) and \(\eta \circ \phi = 0\); both can be derived from \((1.4)\).

If a manifold \(M\) with a \((\phi, \xi, \eta)\)-structure admits a Riemannian metric \(g\) such that

\[ g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \]

for any vector fields \(X\) and \(Y\) of \(T_p M^{2n+1}\), then \(M\) is said to have an almost contact metric structure \((\phi, \xi, \eta, g)\). The fundamental 2-form \(\Phi\) is defined by \(\Phi(X, Y) = g(\phi X, Y)\) for any vector fields \(X, Y\) of \(T_p M^{2n+1}\). The condition for an almost contact metric manifold being normal is equivalent to vanishing of the \((1, 2)\)-type torsion tensor \(N_\phi\), defined by \(N_\phi = [\phi, \phi] + 2d\eta \otimes \xi\), where \([\phi, \phi]\) is the Nijenhuis torsion of \(\phi\) [1]. A normal almost Kenmotsu manifold is a Kenmotsu manifold such that \(d\eta = 0\) and \(d\Phi = 2\eta \wedge \Phi\). Also Kenmotsu manifolds can be characterized by \((\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X\) for any vector fields \(X, Y\). It is well known [9] that a Kenmotsu manifold \(M^{2n+1}\) is locally a warped product \(I \times f N^{2n}\), where \(N^{2n}\) is a Kähler manifold, \(I\) is an open interval with coordinate \(t\) and the warping function \(f\), defined by \(f = ce^t\) for some positive constant \(c\). Let us denote the distribution orthogonal to \(\xi\) by \(\mathcal{D}\) and defined by \(\mathcal{D} = Ker(\eta) = Im(\phi)\). In an almost Kenmotsu manifold, since \(\eta\) is closed, \(\mathcal{D}\) is an integrable distribution.

A Riemannian manifold \((M^{2n+1}, g)\) is called locally symmetric if its curvature tensor \(R\) is parallel, that is, \(\nabla R = 0\), where \(\nabla\) is the Levi-Civita connection. The notion of semisymmetric manifold, a proper generalization of locally symmetric manifold, is defined by \(R(X, Y) \cdot R = 0\), where \(R(X, Y)\) is considered as a field of linear operators, acting on \(R\). A complete intrinsic classification of these manifolds was given by Szabó in [11]. In a recent paper [8] Jun, De and Pathak studied Weyl semisymmetric Kenmotsu manifolds.

Let \(M\) be a \((2n + 1)\)-dimensional Riemannian manifold. If there exists a one-to-one correspondence between each coordinate neighborhood of \(M\) and a domain in Euclidean space such that any geodesic of the Riemannian manifold corresponds to a straight line in the Euclidean space, then \(M\) is said to be locally projectively flat. For \(n \geq 1\), \(M\) is locally projectively
flat if and only if the well-known Weyl projective curvature tensor $P$ vanishes. Here $P$ is defined by [10]

$$P(X, Y)Z = R(X, Y)Z - \frac{1}{2n}[S(Y, Z)X - S(X, Z)Y],$$

(1.5)

for all $X, Y, Z \in T_pM$, where $R$ is the curvature tensor and $S$ is the Ricci tensor of type (0, 2) of $M$. In fact, $M$ is Weyl projectively flat if and only if the manifold is of constant curvature [17]. Thus the Weyl projective curvature tensor is the measure of the failure of a Riemannian manifold to be of constant curvature. A Riemannian manifold is said to be Weyl projective semisymmetric if the curvature tensor $P$ satisfies $R(X, Y) \cdot P = 0$.

In [4], Dileo and Pastore studied locally symmetric almost Kenmotsu manifolds. We refer the reader to ([4],[5],[6]) for more related results on $(k, \mu)'$-nullity distribution and $(k, \mu)$-nullity distribution on almost Kenmotsu manifolds. In recent papers ([13],[14],[15],[16]) Wang and Liu studied almost Kenmotsu manifolds with nullity distributions. In [14], Wang and Liu studied $\xi$-Riemannian semisymmetric almost Kenmotsu manifolds with $\xi$ belonging to the $(k, \mu)'$-nullity distribution and $(k, \mu)$-nullity distribution.

Motivated by the above studies we study Weyl projective semisymmetric $(R \cdot P = 0)$ and the curvature condition $P \cdot S = 0$ in an almost Kenmotsu manifolds with nullity distributions.

The paper is organized as follows:
Section 2 focuses on almost Kenmotsu manifolds with $\xi$ belonging to the $(k, \mu)'$-nullity distribution and $\xi$ belonging to the $(k, \mu)$-nullity distribution. In sections 3 and 4 we study Weyl projective semisymmetric almost Kenmotsu manifolds and almost Kenmotsu manifolds satisfying the curvature condition $P \cdot S = 0$ with characteristic vector field $\xi$ belonging to the $(k, \mu)'$-nullity distribution and $(k, \mu)$-nullity distribution respectively. As a consequence of the main results we obtain several corollaries.

2. Almost Kenmotsu manifolds

Let $M^{2n+1}$ be an almost Kenmotsu manifold. We denote by $h = \frac{1}{2} L_\xi \phi$ and $l = R(\cdot, \xi)\xi$ on $M^{2n+1}$. The tensor fields $l$ and $h$ are symmetric operators and satisfy the following relations [4]

$$h\xi = 0, \ l\xi = 0, \ tr(h) = 0, \ tr(h\phi) = 0, \ h\phi + \phi h = 0.$$  

(2.1)

Moreover, we have the following results [4, 5]

$$\nabla_X\xi = -\phi^2 X - \phi hX (\Rightarrow \nabla_\xi\xi = 0),$$

(2.2)

$$\phi l\phi - l = 2(h^2 - \phi^2),$$

(2.3)

$$R(X, Y)\xi = \eta(X)(Y - \phi hY) - \eta(Y)(X - \phi hX) + (\nabla_Y\phi h)X - (\nabla_X\phi h)Y,$$

(2.4)
for any vector fields $X, Y$. The $(1, 1)$-type symmetric tensor field $h' = h \circ \phi$ is anticommuting with $\phi$ and $h' \xi = 0$. Also it is clear that
\[
h = 0 \iff h' = 0, \quad h'^2 = (k + 1)\phi^2 (\Leftrightarrow h^2 = (k + 1)\phi^2), \tag{2.5}
\]
which holds on $(k, \mu)'$-almost Kenmotsu manifold.

3. $\xi$ belongs to the $(k, \mu)'$-nullity distribution

This section is devoted to study of almost Kenmotsu manifolds with $\xi$ belonging to the $(k, \mu)'$-nullity distribution. Let $X \in \mathcal{D}$ be the eigen vector of $h'$ corresponding to the eigen value $\lambda$. Then from (2.5) it is clear that $\lambda^2 = -(k + 1)$, a constant. Therefore $k \leq -1$ and $\lambda = \pm \sqrt{-k - 1}$. We denote by $[\lambda]'$ and $[-\lambda]'$ the corresponding eigenspaces related to the non-zero eigen value $\lambda$ and $-\lambda$ of $h'$, respectively. Before presenting our main theorems we recall some results:

**Lemma 3.1** (Prop. 4.1 and Prop. 4.3 of [5]). Let $(M_{2n+1}^{2}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold such that $\xi$ belongs to the $(k, \mu)'$-nullity distribution and $h' \neq 0$. Then $k < -1$, $\mu = -2$ and $\text{Spec}(h') = \{0, \lambda, -\lambda\}$, with 0 as simple eigen value and $\lambda = \sqrt{-k - 1}$. The distributions $[\xi] \oplus [\lambda]'$ and $[\xi] \oplus [-\lambda]'$ are integrable with totally geodesic leaves. The distributions $[\lambda]'$ and $[-\lambda]'$ are integrable with totally umbilical leaves. Furthermore, the sectional curvature are given as following:

(a) $K(X, \xi) = k - 2\lambda$ if $X \in [\lambda]'$ and $K(X, \xi) = k + 2\lambda$ if $X \in [-\lambda]'$,

(b) $K(X, Y) = k - 2\lambda$ if $X, Y \in [\lambda]'$;

$c) M^{2n+1}$ has constant negative scalar curvature $r = 2n(k - 2n)$.

**Lemma 3.2** (Lemma 3 of [15]). Let $(M_{2n+1}^{2}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold with $\xi$ belonging to the $(k, \mu)'$-nullity distribution and $h' \neq 0$. If $n > 1$, then the Ricci operator $Q$ of $M^{2n+1}$ is given by
\[
Q = -2nid + 2n(k + 1)\eta \otimes \xi - 2nh'. \tag{3.1}
\]
Moreover, the scalar curvature of $M^{2n+1}$ is $2n(k - 2n)$.

**Lemma 3.3** (Proposition 4.2 of [5]). Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold such that $h' \neq 0$ and $\xi$ belongs to the $(k, -2)'$-nullity distribution. Then for any $X_{\lambda}, Y_{\lambda}, Z_{\lambda} \in [\lambda]'$ and $X_{-\lambda}, Y_{-\lambda}, Z_{-\lambda} \in [-\lambda]'$, the Riemannian curvature tensor satisfies:
\[
R(X_{\lambda}, Y_{\lambda})Z_{-\lambda} = 0,
\]
We obtain from (3.2)
\[ R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)h^'X - \eta(X)h^'Y], \tag{3.2} \]
where \( k, \mu \in \mathbb{R} \). Also we get from (3.2)
\[ R(\xi, X)Y = k[g(Y, X)\xi - \eta(Y)X] + \mu[g(h^'X, Y)\xi - \eta(Y)h^'X]. \tag{3.3} \]
Contracting \( Y \) in (3.2) we have
\[ S(X, \xi) = 2nk\eta(X). \tag{3.4} \]
By applying the above results and Lemma 3.2 we obtain from (1.5)
\[ P(\xi, Y)Z = (k + 1)g(Y, Z)\xi - g(h^'Y, Z)\xi + 2\eta(Z)h^'Y - (k + 1)\eta(Y)\eta(Z)\xi \tag{3.5} \]
for all vector fields \( Y, Z \) on \( M \).

Using the above results we can present our main theorem as follows:

**Theorem 3.1.** Let \((M^{2n+1}, \phi, \xi, \eta, g)(n > 1)\) be an almost Kenmotsu manifold with \( \xi \) belonging to the \((k, \mu)\)'-nullity distribution and \( h^' \neq 0 \). If the manifold \( M^{2n+1} \) is Weyl projective semisymmetric then the manifold is locally isometric to the Riemannian product of an \((n + 1)\)-dimensional manifold of constant sectional curvature \(-4\) and a flat \( n \)-dimensional manifold.

**Proof.** We suppose that the manifold \( M^{2n+1} \) is Weyl projective semisymmetric, that is, \( R \cdot P = 0 \). Then \((R(X, Y) \cdot P)(U, V)W = 0\) for all vector fields \( X, Y, U, V, W \), which implies
\[ R(X, Y)P(U, V)W - P(R(X, Y)U, V)W - P(U, R(X, Y)V)W - P(U, V)R(X, Y)W = 0. \tag{3.6} \]
Setting \( X = U = \xi \) in (3.6) we have,
\[ R(\xi, Y)P(\xi, V)W - P(R(\xi, Y)\xi, V)W - P(\xi, R(\xi, Y)V)W - P(\xi, V)R(\xi, Y)W = 0. \tag{3.7} \]
Making use of (3.3) and (3.5) we get
\[ R(\xi, Y)P(\xi, V)W = k[g(Y, P(\xi, V)W)\xi - \eta(P(\xi, V)W)Y] \\
-2[g(h^'Y, P(\xi, V)W)\xi - \eta(P(\xi, V)W)h^'Y] \]
\[ k((k + 1)g(V, W)\eta(Y)\xi - g(h'V, W)\eta(Y)\xi + 2\eta(W)g(Y, h'V)\xi \\
- (k + 1)\eta(V)\eta(W)\eta(Y)\xi - (k + 1)g(V, W)Y + g(h'V, W)Y \\
+ (k + 1)\eta(V)\eta(W)Y - 2\eta(W)g(h'Y, h'V)\xi \\
- (k + 1)g(V, W)h'Y + g(h'V, W)h'Y + (k + 1)\eta(V)\eta(W)h'Y \]  

(3.8)

for any vector fields \( Y, V, W \) on \( M^{2n+1} \).

With the help of (3.3) and (3.5) we obtain

\[ P(R(\xi, Y)\xi, V)W = k\eta(Y)P(\xi, V)W - kP(Y, V)W + 2P(h'Y, V)W \\
= k(k + 1)g(V, W)\eta(Y)\xi - kg(h'V, W)\eta(Y)\xi + 2k\eta(Y)\eta(W)h'Y \\
- k(k + 1)\eta(Y)\eta(V)\eta(W)\xi - kP(Y, V)W + 2P(h'Y, V)W \]  

(3.9)

for any vector fields \( Y, V, W \) on \( M^{2n+1} \).

Similarly, it follows from (3.3) and (3.5) that

\[ P(\xi, R(\xi, Y)\xi)W = -k\eta(V)P(\xi, Y)W + 2\eta(V)P(\xi, h'Y)W \\
= -k(k + 1)g(Y, W)\eta(V)\xi + kg(h'Y, W)\eta(V)\xi - 2k\eta(V)\eta(W)h'Y \\
+ 2(k + 1)g(h'Y, W)\eta(V)\xi + 2(k + 1)g(Y, W)\eta(V)\xi \\
- 4(k + 1)\eta(V)\eta(W)Y + (k + 1)(k + 2)\eta(Y)\eta(V)\eta(W)\xi \]  

(3.10)

for any vector fields \( Y, V, W \) on \( M^{2n+1} \).

Again using (3.3) and (3.5) we obtain

\[ P(\xi, V)R(\xi, Y)W = k(k + 1)g(Y, W)\eta(V)\xi - k(k + 1)g(Y, V)\eta(W)\xi \\
+ 2(k + 1)g(h'Y, V)\eta(W)\xi + kg(h'V, Y)\eta(W)\xi - 2g(h'V, h'Y)\eta(W)\xi \\
+ 2kg(Y, W)h'V - 2k\eta(Y)\eta(W)h'V - 4g(h'Y, W)h'V \\
- k(k + 1)g(Y, W)\eta(V)\xi + k(k + 1)\eta(Y)\eta(W)\eta(V)\xi \]  

(3.11)

for any vector fields \( Y, V, W \) on \( M^{2n+1} \).

Finally, using (3.8)–(3.11) we have from (3.7)

\[ kP(Y, V)W - 2P(h'Y, V)W + kg(h'V, Y)\eta(W)\xi + 2(k + 1)g(V, W)h'Y \\
- k(k + 1)g(V, W)Y + kg(h'V, W)Y + (k^2 + 5k + 4)\eta(V)\eta(W)Y \\
- 2g(h'Y, h'V)\eta(W)\xi - 2(k + 1)^2\eta(Y)\eta(V)\eta(W)\xi \\
- 2g(h'V, W)h'Y - 2\eta(V)\eta(W)h'Y + (k^2 - k - 2)g(Y, W)\eta(V)\xi \\
- (3k + 2)g(h'Y, W)\eta(V)\xi + k(k + 1)g(Y, V)\eta(W)\xi \]
\[-2(k + 1)g(h^'Y, V)\eta(W)\xi - 2kg(Y, W)h^'V + 4g(h^'Y, W)h^'V = 0 \] (3.12)

for any vector fields \(Y, V, W\) on \(M^{2n+1}\). Letting \(Y, W \in [\lambda]'\) and \(V \in [-\lambda]'\) and applying Lemma 3.3 we have

\[P(Y, V)W = (k + 1 - \lambda)g(Y, W)V\]
\[P(h^'Y, V)W = (\lambda + 1)(k + 1)g(Y, W)V.\] (3.13)

By using (3.13) and noticing \(Y, W \in [\lambda]'\) and \(V \in [-\lambda]'\) it follows from (3.12) that

\[[k(k + 1 - \lambda) - 2(\lambda + 1)(k + 1) + 2\lambda k - 4\lambda^2]g(Y, W)V = 0.\] (3.14)

Using the relationship \(\lambda = \pm \sqrt{-k-1}\) in (3.14) we get

\[\lambda(\lambda + 1)^2(\lambda - 1) = 0.\] (3.15)

If \(\lambda = 0\), then \(k = -1\) and consequently from (2.5) \(h^' = 0\), which contradicts our hypothesis \(h^' \neq 0\). Then it follows from (3.15) that \(\lambda^2 = 1\) and hence \(k = -2\). Without losing generality we may choose \(\lambda = 1\). Then we can write from Lemma 3.3

\[R(X_\lambda, Y_\lambda)Z_\lambda = -4[g(Y_\lambda, Z_\lambda)X_\lambda - g(X_\lambda, Z_\lambda)Y_\lambda],\]
\[R(X_{-\lambda}, Y_{-\lambda})Z_{-\lambda} = 0\]

for any \(X_\lambda, Y_\lambda, Z_\lambda \in [\lambda]'\) and \(X_{-\lambda}, Y_{-\lambda}, Z_{-\lambda} \in [-\lambda]'\). Also it follows from Lemma 3.1 that \(K(X, \xi) = -4\) for any \(X \in [\lambda]'\) and \(K(X, \xi) = 0\) for any \(X \in [-\lambda]'\). Again from Lemma 3.1 we see that \(K(X, Y) = -4\) for any \(X, Y \in [\lambda]'; K(X, Y) = 0\) for any \(X, Y \in [-\lambda]'\) and \(K(X, Y) = 0\) for any \(X \in [\lambda]', Y \in [-\lambda]'\). As is shown in [5] that the distribution \([\xi] + [\lambda]'\) is integrable with totally geodesic leaves and the distribution \([-\lambda]'\) is integrable with totally umbilical leaves by \(H = -(1 - \lambda)\xi\), where \(H\) is the mean curvature vector field for the leaves of \([-\lambda]'\) immersed in \(M^{2n+1}\). Here \(\lambda = 1\), then two orthogonal distributions \([\xi] + [\lambda]'\) and \([-\lambda]'\) are both integrable with totally geodesic leaves immersed in \(M^{2n+1}\). Then we can say that \(M^{2n+1}\) is locally isometric to \(\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n\). This completes the proof of our theorem. \(\square\)

Since \(R \cdot R = 0\) implies \(R \cdot P = 0\), we have the following:

**Corollary 3.1.** A semisymmetric almost Kenmotsu manifold \(M^{2n+1}(n > 1)\) with \(\xi\) belonging to the \((k, \mu)'\)-nullity distribution and \(h^' \neq 0\) is locally isometric to the Riemannian product of an \((n + 1)\)-dimensional manifold of constant sectional curvature \(-4\) and a flat \(n\)-dimensional manifold.

The above corollary have been proved by Wang and Liu [14].
Next we consider an almost Kenmotsu manifold with $\xi$ belonging to the $(k, \mu)'$-nullity distribution and $h' \neq 0$ satisfying the curvature condition $P \cdot S = 0$. Then $(P(X, Y) \cdot S)(U, V) = 0$ for all vector fields $X, Y, U, V$, which implies

$$S(P(X, Y)U, V) + S(U, P(X, Y)V) = 0$$

(3.16)

for any vector fields $X, Y, U, V$ on $M^{2n+1}$.

Putting $X = U = \xi$ in (3.16) we have,

$$S(P(\xi, Y)\xi, V) + S(\xi, P(\xi, Y)V) = 0.$$  

(3.17)

Making use of (3.4) and (3.5) the above equation implies

$$S(h'Y, V) + nk(k + 1)g(Y, V) - nkg(h'Y, V) = 0$$

(3.18)

for any vector fields $Y, V$ on $M^{2n+1}$.

Substituting $Y = h'Y$ in (3.18) and using (2.5) we obtain

$$(k + 1)(-S(Y, V) + nk\eta(Y)\eta(V) + nkg(h'Y, V) + nkg(Y, V)) = 0$$

(3.19)

for any vector fields $Y, V$ on $M^{2n+1}$.

Again from Lemma 3.2 we have

$$S(Y, V) = -2ng(Y, V) + 2n(k + 1)\eta(Y)\eta(V) - 2ng(h'Y, V)$$

(3.20)

for any vector fields $Y, V$ on $M^{2n+1}$.

Making use of (3.20) we obtain from (3.19)

$$(k + 1)(k + 2)(g(Y, V) + g(h'Y, V) - \eta(Y)\eta(V)) = 0.$$  

(3.21)

Letting $Y, V \in [\lambda]'$ in (3.21) implies that

$$(k + 1)(k + 2)(1 + \lambda)g(Y, V) = 0.$$  

(3.22)

Using the relation $\lambda = \pm \sqrt{-k - 1}$ in (3.22) we have

$$\lambda^2(\lambda + 1)^2(\lambda - 1) = 0.$$  

(3.23)

Suppose $\lambda = 0$, then $k = -1$ and hence it follows from (2.5) that $h' = 0$, which contradicts our hypothesis $h' \neq 0$. Then from (3.23) we have $\lambda^2 = 1$ and hence $k = -2$. Without losing the generality, we may choose $\lambda = 1$. Then we can write from Lemma 3.3

$$R(X_\lambda, Y_\lambda)Z_\lambda = -4[g(Y_\lambda, Z_\lambda)X_\lambda - g(X_\lambda, Z_\lambda)Y_\lambda],$$
\[ R(X_{-\lambda}, Y_{-\lambda})Z_{-\lambda} = 0 \]

for any \( X_{\lambda}, Y_{\lambda}, Z_{\lambda} \in [\lambda]' \) and \( X_{-\lambda}, Y_{-\lambda}, Z_{-\lambda} \in [-\lambda]' \). Also it follows from Lemma 3.1 that \( K(X, \xi) = -4 \) for any \( X \in [\lambda]' \) and \( K(X, \xi) = 0 \) for any \( X \in [-\lambda]' \). Again from Lemma 3.1 we see that \( K(X, Y) = -4 \) for any \( X, Y \in [\lambda]' \); \( K(X, Y) = 0 \) for any \( X, Y \in [-\lambda]' \) and \( K(X, Y) = 0 \) for any \( X \in [\lambda]', Y \in [-\lambda]' \). As is shown in [5] that the distribution \([\xi] \oplus [\lambda]' \) is integrable with totally geodesic leaves and the distribution \([-\lambda]' \) is integrable with totally umbilical leaves by \( H = -(1-\lambda)\xi \), where \( H \) is the mean curvature vector field for the leaves of \([-\lambda]' \) immersed in \( M^{2n+1} \). Here \( \lambda = 1 \), then two orthogonal distributions \([\xi] \oplus [\lambda]' \) and \([-\lambda]' \) are both integrable with totally geodesic leaves immersed in \( M^{2n+1} \). Then we can say that \( M^{2n+1} \) is locally isometric to \( \mathbb{H}^{n+1}(-4) \times \mathbb{R}^n \). By the above discussions we can state the following:

**Theorem 3.2.** Let \((M^{2n+1}, \phi, \xi, \eta, g)(n > 1)\) be an almost Kenmotsu manifold with \( \xi \) belonging to the \((k, \mu)'\)-nullity distribution and \( h' \neq 0 \). If the manifold satisfies the curvature condition \( P \cdot S = 0 \), then the manifold is locally isometric to the Riemannian product of an \((n + 1)\)-dimensional manifold of constant sectional curvature \(-4\) and a flat \( n \)-dimensional manifold.

**4. \( \xi \) belongs to the \((k, \mu)\)-nullity distribution**

In this section we deal with almost Kenmotsu manifolds of which \( \xi \) belonging to the \((k, \mu)\)-nullity distribution.

From (1.2) we obtain

\[ R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY], \quad (4.1) \]

where \( k, \mu \in \mathbb{R} \). Before proving our main results in this section we state the following:

**Lemma 4.1** (Theorem 4.1 of [5]). Let \( M \) be an almost Kenmotsu manifold of dimension \( 2n + 1 \). Suppose that the characteristic vector field \( \xi \) belonging to the \((k, \mu)\)-nullity distribution. Then \( k = -1, h = 0 \) and \( M \) is locally a warped product of an open interval and an almost Kähler manifold.

In view of Lemma 4.1 it follows from (4.1) that

\[ R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \quad (4.2) \]

\[ R(\xi, X)Y = -g(X, Y)\xi + \eta(Y)X, \quad (4.3) \]

\[ S(X, \xi) = -2n\eta(X) \quad (4.4) \]

for any vector fields \( X, Y \) on \( M^{2n+1} \).
Applying (4.3) and (4.4) in (1.5) we have the following

\[ P(\xi, Y)Z = -g(Y, Z)\xi - \frac{1}{2n}S(Y, Z)\xi \]  \hspace{1cm} (4.5)

for any vector fields \( Y, Z \) on \( M^{2n+1} \). We can state our main theorem as follows:

**Theorem 4.1.** An almost Kenmotsu manifold \( (M^{2n+1}, \phi, \xi, \eta, g) \) with \( \xi \) belonging to the \( (k, \mu) \)-nullity distribution is Weyl projective semisymmetric if and only if the manifold is of constant curvature \(-1\).

**Proof.** Let \( M^{2n+1} \) be a Weyl projective semisymmetric almost Kenmotsu manifold with \( \xi \) belonging to the \( (k, \mu) \)-nullity distribution. Therefore \( (R(X, Y)\cdot P)(U, V)W = 0 \) for all vector fields \( X, Y, U, V, W \), which implies

\[ R(X, Y)P(U, V)W - P(R(X, Y)U, V)W - P(U, R(X, Y)V)W - P(U, V)R(X, Y)W = 0. \]  \hspace{1cm} (4.6)

Substituting \( X = U = \xi \) in (4.6) we obtain

\[ R(\xi, Y)P(\xi, V)W - P(R(\xi, Y)\xi, V)W - P(\xi, R(\xi, Y)V)W - P(\xi, V)R(\xi, Y)W = 0. \]  \hspace{1cm} (4.7)

Making use of (4.3) and (4.5) we have

\[ R(\xi, Y)P(\xi, V)W = g(V, W)\eta(Y)\xi + \frac{1}{2n}S(V, W)\eta(Y)\xi - g(V, W)Y - \frac{1}{2n}S(V, W)Y \]  \hspace{1cm} (4.8)

for any vector field \( Y, V, W \) on \( M^{2n+1} \).

Similarly using (4.3) and (4.5) we obtain

\[ P(R(\xi, Y)\xi, V)W = P(Y, V)W + g(V, W)\eta(Y)\xi + \frac{1}{2n}S(V, W)\eta(Y)\xi \]  \hspace{1cm} (4.9)

for any vector field \( Y, V, W \) on \( M^{2n+1} \).

Again, it follows from (4.3) and (4.5) that

\[ P(\xi, R(\xi, Y)V)W = -g(Y, W)\eta(V)\xi - \frac{1}{2n}S(Y, W)\eta(V)\xi \]  \hspace{1cm} (4.10)

for any vector field \( Y, V, W \) on \( M^{2n+1} \).

Finally, using (4.3) and (4.5) we have

\[ P(\xi, V)R(\xi, Y)W = -g(V, Y)\eta(W)\xi - \frac{1}{2n}S(V, Y)\eta(W)\xi \]  \hspace{1cm} (4.11)

for any vector field \( Y, V, W \) on \( M^{2n+1} \).
Substituting (4.8)–(4.11) into (4.7) gives

\[ P(Y, V)W = -g(V, W)Y - \frac{1}{2n}S(V, W)Y + g(Y, W)\eta(V)\xi + \frac{1}{2n}S(Y, W)\eta(V)\xi \]
\[ + g(V, Y)\eta(W)\xi + \frac{1}{2n}S(V, Y)\eta(W)\xi \] (4.12)

for any vector field \( Y, V, W \) on \( M^{2n+1} \).

In view of (1.5) and (4.12) we obtain

\[ R(Y, V)W = -g(V, W)Y + g(Y, W)\eta(V)\xi + \frac{1}{2n}S(Y, W)\eta(V)\xi \]
\[ + g(V, Y)\eta(W)\xi + \frac{1}{2n}S(V, Y)\eta(W)\xi - \frac{1}{2n}S(Y, W)W. \] (4.13)

Contracting \( Y \) in (4.13) it follows that

\[ S(V, W) = -2ng(V, W) \] (4.14)

for any vector field \( V, W \) on \( M^{2n+1} \).

Taking account of (4.14) we have from (4.13)

\[ R(Y, V)W = -[g(V, W)Y - g(Y, W)V], \] (4.15)

that is, the manifold is of constant curvature \(-1\).

Conversely, if the manifold is of constant curvature \(-1\) then obviously Weyl projective semisymmetry follows. This completes the proof. \( \Box \)

Since \( R \cdot R = 0 \) implies \( R \cdot P = 0 \), we have the following:

**Corollary 4.1.** An almost Kenmotsu manifold \((M^{2n+1}, \phi, \xi, \eta, g)\) with \( \xi \) belonging to the \((k, \mu)\)-nullity distribution is semisymmetric if and only if the manifold is of constant curvature \(-1\).

The above corollary have been proved by Wang and Liu [14].

Let \( M^{2n+1} \) be an almost Kenmotsu manifold with \( \xi \) belonging to the \((k, \mu)\)-nullity distribution satisfying the curvature condition \( P \cdot S = 0 \). Then \( (P(X, Y) \cdot S)(U, V) = 0 \) for all vector fields \( X, Y, U, V \), which implies

\[ S(P(X, Y)U, V) + S(U, P(X, Y)V) = 0 \] (4.16)

for any vector fields \( X, Y, U, V \) on \( M^{2n+1} \).

Setting \( X = U = \xi \) in (4.16) we have,

\[ S(P(\xi, Y)\xi, V) + S(\xi, P(\xi, Y)V) = 0. \] (4.17)
Using (4.4) and (4.5) we obtain from (4.17)

$$\eta(P(\xi, Y)V) = 0. \quad (4.18)$$

In view of (4.5) and (4.18) it follows that

$$S(Y, V) = -2ng(Y, V), \quad (4.19)$$

which implies that the manifold is an Einstein manifold.

Conversely, let the manifold be an Einstein manifold of the form (4.19). Then it is obvious that $P \cdot S = 0$. This leads to the following:

**Theorem 4.2.** *An almost Kenmotsu manifold $M^{2n+1}$ with $\xi$ belonging to the $(k, \mu)$-nullity distribution satisfies the curvature condition $P \cdot S = 0$ if and only if the manifold is an Einstein one.*

**References**


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