ON GLOBAL DOMINATING-$\chi$-COLORING OF GRAPHS

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Abstract. Let $G$ be a graph. Among all $\chi$-colorings of $G$, a coloring with the maximum number of color classes that are global dominating sets in $G$ is called a global dominating-$\chi$-coloring of $G$. The number of color classes that are global dominating sets in a global dominating-$\chi$-coloring of $G$ is defined to be the global dominating-$\chi$-color number of $G$, denoted by $gd_{\chi}(G)$. This concept was introduced in [5]. This paper extends the study of this notion.

1. Introduction

By a graph $G = (V, E)$, we mean a connected, finite, non-trivial, undirected graph with neither loops nor multiple edges. The order and size of $G$ are denoted by $n$ and $m$ respectively. For graph theoretic terminology we refer to Chartand and Lesniak [3].

A subset $D$ of vertices is said to be a dominating set of $G$ if every vertex in $V$ either belongs to $D$ or is adjacent to a vertex in $D$. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of $G$. A subset $D$ of vertices is said to be a global dominating set of $G$ if $D$ is a dominating set of both $G$ and $\overline{G}$; that is, every vertex outside $D$ has a neighbour as well as a non-neighbour in $D$. The global domination number $\gamma_g(G)$ is the minimum cardinality of a global dominating set of $G$.

A proper coloring of a graph $G$ is an assignment of colors to the vertices of $G$ in such a way that no two adjacent vertices receive the same color. Since all colorings in this paper are proper colorings, we simply call a proper coloring a coloring. A coloring in which $k$ colors are used is a $k$-coloring. The chromatic number of $G$, denoted by $\chi(G)$, is the minimum integer $k$ for which $G$ admits a $k$-coloring. In a given coloring of the vertices of a graph $G$, a set consisting of all those vertices assigned the same color is called a color class. If $\mathcal{C}$ is a coloring of $G$ with the color classes $U_1, U_2, \ldots, U_t$, then we write $\mathcal{C} = \{U_1, U_2, \ldots, U_t\}$. Among all $\chi$-colorings of $G$, let $\mathcal{C}$ be chosen to have a color class $U$ that dominates as many vertices of $G$ as possible. If there is a vertex in $G$ not dominated by $U$, then deleting such a vertex from...
its color class and adding it to the color class $U$ produces a new minimum vertex-coloring that contains a color class which dominates more vertices than $U$, a contradiction. Hence the color class $U$ dominates $G$. Thus we have the following observation first observed in [1].

**Observation 1.1.** Every graph $G$ contains a $\chi$-coloring with the property that at least one color class is a dominating set in $G$.

Motivated by Observation 1.1, Arumugam et al. [1] defined the dominating-$\chi$-color number, which they called dom-color number, as follows. Among all $\chi$-colorings of $G$, a coloring with the maximum number of color classes that are dominating sets in $G$ is called a dominating-$\chi$-coloring of $G$. The number of color classes that are dominating sets in a dominating-$\chi$-coloring of $G$ is defined to be the dominating-$\chi$-color number of $G$, denoted by $d_{\chi}(G)$. This parameter has been further studied in [2] and [4].

In [5], the notion of dominating-$\chi$-coloring was extended to the notion of global dominating sets in the name of global dominating-$\chi$-coloring. Among all $\chi$-colorings of $G$, a coloring with the maximum number of color classes that are global dominating sets in $G$ is called a global dominating-$\chi$-coloring of $G$. The number of color classes that are global dominating sets in a global dominating-$\chi$-coloring of $G$ is defined to be the global dominating-$\chi$-color number of $G$ and is denoted by $gd_{\chi}(G)$. Certainly, for any graph $G$, we have $d_{\chi}(G) \geq gd_{\chi}(G)$. In this paper, we discuss the parameter $gd_{\chi}$ for unicyclic graph and also prove some realization theorems associated with some relations among $gd_{\chi}$, $d_{\chi}$ and $\chi$.

We need the following theorems.

**Theorem 1.2** ([2]). For any graph $G$, we have $d_{\chi}(G) \leq \delta(G) + 1$.

**Theorem 1.3** ([5]). For any graph $G$, we have $gd_{\chi}(G) \leq \delta(G) + 1$.

**Theorem 1.4** ([5]). If $G$ is a graph of order $n \geq 2$, then $gd_{\chi}(G) \leq \frac{n-\chi(G)\times s(G)}{\gamma(G)-s(G)}$, where $s(G)$ denotes the minimum cardinality of any color class in any $\chi$-coloring of $G$.

**Theorem 1.5** ([5]). If $G$ is a graph with $\Delta(G) = n - 1$, then $gd_{\chi}(G) = 0$.

2. $gd_{\chi}$ for unicyclic graphs

Throughout the paper, by a unicyclic graph, we mean a connected unicyclic graph that is not a cycle. Now, in view of Theorem 1.3, for a graph with minimum degree 1, the value of global dominating $\chi$ - color number is at most 2. In particular, for a unicyclic graph $G$, $gd_{\chi}(G) \leq 2$. So, the family of unicyclic graphs can be classified into three classes namely graphs with $gd_{\chi} = 0$ ; graphs with $gd_{\chi} = 1$ and graphs with $gd_{\chi} = 2$. This section determines these classes of graphs. For this purpose, we describe the following families.
(i) Let $G_1$ be the class of all connected unicyclic graphs obtained from a cycle of length 4 by attaching at least one pendant edge at exactly two adjacent vertices of the cycle. A graph in this family is given in Figure 1(a).

(ii) Let $G_2$ be the collection of all connected unicyclic graphs obtained from a cycle of length 4 by attaching at least one pendant edge at each of two non adjacent vertices of the cycle. A graph lying in this family is given in Figure 1(b).

(iii) Let $G_3$ be the collection of all connected unicyclic graphs obtained from a cycle of length 4 by attaching at least one pendant edge at each of any three vertices of the cycle. A graph lying in this family is given in Figure 1(c).

(iv) Let $G_4$ be the collection of all connected unicyclic graphs with the cycle $C = (v_1, v_2, v_3, v_4, v_1)$ that are constructed as follows. Attach $r \geq 0$ pendant edges at $v_1$, $s \geq 0$ pendant edges at $v_3$. Also, attach $t \geq 1$ pendant edges at $v_2$, say $x_1, x_2, \ldots, x_t$ are the corresponding pendant vertices adjacent to $v_2$. Finally, for each $i \in \{1, 2, \ldots, t\}$, attach $t_i$ pendant vertex at the vertex $x_i$ with the condition that $t_1 \geq 1$ and $t_j \geq 0$ for all $j \neq 1$. A graph lying in this family is given in Figure 1(d).

(v) Let $G_5$ be the family of connected unicyclic graphs obtained from a triangle by attaching at least one pendant edge at exactly one vertex of the triangle.

![Figure 1: (a) A graph in $G_1$, (b) A graph in $G_2$, (c) A graph in $G_3$, (d) A graph in $G_4$.](image)

Theorem 2.1. Let $G$ be a unicyclic graph with even cycle $C$. If $C$ is of length at least 6, then $gd_{\chi}(G) = 2$.

Proof. Certainly $\chi(G) = 2$. Let $\{V_1, V_2\}$ be the $\chi$-coloring of $G$. Obviously, both $V_1$ and $V_2$ are dominating sets of $G$. It is enough to verify that $V_1$ and $V_2$ are global dominating sets of $G$. Since the length of the cycle $C$ is at least 6, it follows that each of $V_1$ and $V_2$ contains at least three vertices of $G$ lying on $C$. However, every vertex of $G$ has at most two neighbours on $C$; this means that every vertex of $V_1$ has a non-neighbour in $V_2$ and every vertex of $V_2$ has a non-neighbour in $V_1$. Thus $V_1$ and $V_2$ are global dominating sets of $G$. \qed
Theorem 2.2. Let $G$ be a unicyclic graph whose cycle is of length 4. Then $\text{gd}_\chi(G) = 0$ if and only if $G \in \mathcal{G}_1$.

Proof. Let $C = (v_1, v_2, v_3, v_4, v_1)$ and let $(X, Y)$ be the $\chi$-coloring of $G$. Assume that $v_1, v_3 \notin X$ and $v_2, v_4 \notin Y$. Obviously, both $X$ and $Y$ are dominating sets of $G$. Now, suppose $\text{gd}_\chi(G) = 0$. Then both $X$ and $Y$ can not be global dominating sets. Therefore there exist vertices $x \in X$ and $y \in Y$ such that $x$ is adjacent to all the vertices of $Y$ and $y$ is adjacent to all the vertices of $X$. Since $G$ is unicyclic, each of $x$ and $y$ must lie on $C$, say $x = v_1$ and $y = v_2$. Again, as $G$ is unicyclic, the vertex $v_4$ is not adjacent to any vertex of $X$ other than $v_1$ and $v_3$. Similarly, the vertex $v_3$ is not adjacent to any vertex of $Y$ other than $v_2$ and $v_4$. Further, a vertex of $X - \{v_1, v_3\}$ can not be adjacent with any vertex of $Y - \{v_2, v_4\}$ and similarly a vertex of $Y - \{v_2, v_4\}$ can not be adjacent with any vertex of $X - \{v_1, v_3\}$; for otherwise a cycle distinct from $C$ will get formed. That is, $v_2$ is the only neighbour in $Y$ for each vertex of $X - \{v_1, v_3\}$ and $v_1$ is the only neighbour in $X$ for each vertex of $Y - \{v_2, v_4\}$. Thus the vertices of $G$ outside $C$ are pendant and therefore $G \in \mathcal{G}_1$. The converse is an easy verification.

Theorem 2.3. Let $G$ be a unicyclic graph whose cycle is of length 4. Then $\text{gd}_\chi(G) = 1$ if and only if $G \in \bigcup_{i=2}^4 \mathcal{G}_i$.

Proof. Let $\{V_1, V_2\}$ be the $\chi$-coloring of $G$. Assume that $V_2$ is a global dominating set of $G$ and $V_1$ is not. Also, assume that $v_1, v_3 \in V_1$ and $v_2, v_4 \in V_2$. As $V_1$ is not a global dominating set, there is a vertex $x \in V_2$ that is adjacent to all the vertices of $V_1$. As discussed in the proof of Theorem 2.2, $x$ must lie on $C$. But $V_2$ is a global dominating set. Therefore, every vertex of $V_1$ has a non-neighbour in $V_2$ and so the set $B = V_2 - \{v_2, v_4\} \neq \phi$. Since $v_2$ is adjacent to every vertex of $V_1$, every vertex in $B$ is a pendant vertex of $G$. Now, let $A$ be the set of neighbours of $v_2$ in $V_1$ other than $v_1$ and $v_3$. If $A = \phi$, then $N(v_1) \cap B \neq \phi$ and $N(v_3) \cap B \neq \phi$ and $[N(v_1) \cup N(v_3)] \cap B = B$. Thus $G \in \mathcal{G}_1$.

Suppose $A \neq \phi$. Now, if the vertices in $A$ are pendant, then $N(v_1) \cap B \neq \phi$, $N(v_3) \cap B \neq \phi$ and $[N(v_1) \cup N(v_3)] \cap B = B$ so that $G \in \mathcal{G}_2$. So, the remaining case is that $A \neq \phi$ and $A$ has a vertex $u$ with $\text{deg} \ u \geq 2$. That is, $u$ has a neighbour in $B$, say $w$. Note that the vertex $w$ is a non-neighbour of both $v_1$ and $v_3$ as $u$ is pendant. But however the vertices $v_1$ and $v_3$ may have neighbours in $B$ and thus $G \in \mathcal{G}_3$. Now, it is not difficult to see that if $G \in \bigcup_{i=2}^3 \mathcal{G}_i$, then $\text{gd}_\chi(G) = 1$.

Lemma 2.4. If $\text{gd}_\chi(G) = 0$, then $d_\chi(G) \geq 2$.

Proof. Suppose $\text{gd}_\chi(G) = 0$ and $d_\chi(G) = 1$. Consider a $\chi$-coloring $\{V_1, V_2, \ldots, V_\chi\}$ of $G$ such that $V_1$ is a dominating set of $G$. As $\text{gd}_\chi(G) = 0$, $V_1$ can not be a global dominating set of $G$. Therefore, there exists a vertex $v$ such that $v$ is adjacent to every vertex of $V_1$. Assume
without loss of generality that \( v \in V_2 \). Certainly, no \( V_i (2 \leq i \leq \chi) \), is a dominating set and in particular \( V_2 \) is not a dominating set. So, there are vertices in \( V - V_2 \) that are not dominated by any vertex of \( V_2 \); let \( S \) be the set of those vertices. Clearly \( S \subseteq V - V_2 \). Also, as \( v \) is adjacent to each vertex of \( V_1 \), it follows that \( S \subseteq V - V_1 \) and thus \( S \subseteq V - (V_1 \cup V_2) \). Now, if \( D \) is an independent dominating set of the subgraph \( \langle S \rangle \) induced by \( S \), then \( V_2 \cup D \) is an independent dominating set of \( G \). Therefore \( \{V_1, V_2 \cup D, V_3 - V_3, V_4 - V_4, \ldots, V_\chi - V_\chi \} \), where \( V_i = V_i \cap D \) for all \( i \in \{3, 4, \ldots, \chi \} \) is a \( \chi \)-coloring of \( G \) in which both \( V_1 \) and \( V_2 \cup D \) are dominating sets of \( G \), a contradiction to the assumption that \( d_\chi (G) = 1 \).

**Corollary 2.5.** If \( d_\chi (G) = 1 \), then \( gd_\chi (G) = 1 \).

Let us now concentrate on the unicyclic graphs with odd cycle.

**Theorem 2.6.** Let \( G \) be a unicyclic graph with odd cycle \( C \). If all the vertices on \( C \) are support vertices, then \( gd_\chi (G) = 1 \).

**Proof.** Let \( C = (v_1, v_2, \ldots, v_n, v_1) \), where each \( v_i \) is support. In view of Corollary 2.5, it is enough to prove that \( d_\chi (G) = 1 \). As in Observation 1.1, \( d_\chi (G) \geq 1 \). For the other inequality, we need to prove that every \( \chi \)-coloring of \( G \) has exactly one color class that is a dominating set of \( G \). On the contrary, assume that \( G \) has a \( \chi \)-coloring \( \{V_1, V_2, V_3 \} \) of \( G \) with \( V_1 \) and \( V_2 \) are dominating sets of \( G \). It is clear that if \( x \) is a support vertex of \( G \), then a dominating set of \( G \) must contain either \( x \) or all its pendant neighbours. Here \( V_1 \) and \( V_2 \) are assumed to be dominating sets and therefore all the support vertices and the pendant vertices of \( G \) must be contained in \( V_1 \cup V_2 \). In particular, \( \{v_1, v_2, \ldots, v_n\} \) is a subset of \( V_1 \cup V_2 \); this is possible only when \( n \) is even. But \( n \) is odd and thus exactly one color class of any \( \chi \)-coloring of \( G \) can be a dominating set of \( G \). This completes the proof.

**Theorem 2.7.** Let \( G \) be a unicyclic graph with odd cycle \( C \). If the length of \( C \) is at least 7 with the property that not all the vertices on \( C \) are supports, then \( gd_\chi (G) = 2 \).

**Proof.** As we know \( gd_\chi (G) \leq 2 \) and so in order to prove the theorem it is enough if we are able to come up with a \( \chi \)-coloring of \( G \) where two color classes are global dominating sets. Here we provide such a coloring as follows. Let \( C = (v_1, v_2, \ldots, v_n, v_1) \). Assume that \( v_1 \) is not a support vertex of \( G \). Consider the \( \chi \)-coloring \( \{V_1, V_2\} \) of the tree \( G - v_1 v_n \). Assume that \( v_1 \in V_1 \). Then \( v_n \in V_1 \). Now, take \( \mathcal{C} = \{V_1 - \{v_1\}, V_2, \{v_1\}\} \). Then \( \mathcal{C} \) is a \( \chi \)-coloring of \( G \). We prove that \( V_1 - \{v_1\} \) and \( V_2 \) are global dominating sets of \( G \). Note that both \( V_1 \) and \( V_2 \) are dominating sets of \( G - v_1 v_n \). Therefore, obviously \( V_2 \) is a dominating set of \( G \) as well. Further, the set \( V_2 - \{v_1\} \) also serves as a dominating set of \( G \) as \( v_1 \) is not a support. So, \( V_1 - \{v_1\} \) and \( V_2 \) are dominating sets of \( G \). Also, as the length of \( C \) is at least 7, it follows that each of \( V_1 - \{v_1\} \) and \( V_2 \) contains at least three vertices of \( G \) lying on \( C \). But every vertex of \( G \) can have at most two neighbours
on $C$. So, every vertex of $G$ will have a non-neighbour in each of $V_1 - \{v_1\}$ and $V_2$ and therefore these two sets are global dominating sets of $G$. Thus $\mathcal{C}$ is a $\chi$-coloring of $G$ where $V_1 - \{v_1\}$ and $V_2$ are global dominating sets of $G$ as desired. \hfill \Box

**Theorem 2.8.** Let $G$ be a unicyclic graph whose cycle is of length 5. Then $gd_{\chi}(G)$ is either 1 or 2.

**Proof.** Let $C = (v_1, v_2, v_3, v_4, v_5, v_1)$. Since $G$ is a unicyclic graph, at least one of $v_1$, $v_2$, $v_3$, $v_4$ and $v_5$ has degree at least 3. Let it be $v_1$. Consider a neighbour $u$ of $v_1$ outside $C$. Let $T = G - v_1 v_5$. Then $\{V_1, V_2\}$ be a $\chi$-coloring of $G$. Note that the vertices $u$, $v_2$ and $v_4$ belong to the same color class, say $V_1$. Then $v_1$, $v_3$ and $v_5$ belong to $V_2$. Certainly, $\{V_1, V_2 - \{v_5\}, \{v_5\}\}$ is a $\chi$-coloring of $G$. We now claim that $V_1$ is a global dominating set of $G$. Clearly $V_1$ is a dominating set of $G$. Consider an arbitrary vertex $x$ of $G$. If $x \in N[u]$, then $v_4$ is a non-neighbour of $x$. If $x \notin N[u]$, then $u$ is a non-neighbour of $x$ and so $V_1$ is a global dominating set of $G$. Hence $gd_{\chi}(G) \geq 1$.

By an extreme vertex in a unicyclic graph $G$; we mean a vertex $v$ on the cycle $C$ of $G$ with the property that $v$ is adjacent to a vertex outside $C$ where degree is at least two. Let $w$ be a vertex of $G$ with $\text{deg}\ w \geq 3$. A branch of $G$ at $w$ is a maximal subtree $T$ of $G$ containing an edge outside $C$ that is incident at $w$ such that $w$ is a pendant vertex in $T$.

**Theorem 2.9.** Let $G$ be a unicyclic graph whose cycle $C$ is of length exactly 3. Then $gd_{\chi}(G) = 0$ if and only if $G \in \mathcal{G}_5$.

**Proof.** Let $C = (v_1, v_2, v_3, v_1)$. Assume $gd_{\chi}(G) = 0$. We first prove that $G$ has no extreme vertex. On the contrary, assume that $G$ has an extreme vertex; let it be $v_1$. Choose a vertex $x$ in a branch of $G$ at $v_1$ such that $d(v_2, x) = 3$. Consider the $\chi$-coloring $\mathcal{C} = \{V_1, V_2\}$ of the tree $G - v_1 v_2$. As the distance between $v_2$ and $x$ in $G$ is 3, the distance between them in $G - v_1 v_2$ is 4 and therefore they both belong to the same color class in $\mathcal{C}$, say $V_1$. Therefore $v_3 \in V_2$ and $v_1 \in V_1$. We now prove that there is a $\chi$-coloring of $G$ in which at least one color class is a global dominating set of $G$. If $v_1$ is not a support vertex, then consider the $\chi$-coloring $\{V_1 - \{v_1\}, V_2, \{v_1\}\}$ of $G$. On the other hand, if $v_1$ is a support vertex, the consider the $\chi$-coloring $\{(V_1 - \{v_1\}) \cup U, V_2 - U, \{v_1\}\}$ of $G$, where $U$ is the set of all pendant neighbours of $v_1$ (Note that $U$ is a subset of $V_2$ in $\mathcal{C}$). Also remain that both $x$ and $v_2$ belong to $V_1$. We now prove that $V_1 - \{v_1\}$ and $(V_1 - \{v_1\}) \cup U$ are global dominating sets of $G$. Clearly both are dominating sets of $G$. Now, choose an arbitrary vertex $y$ in $G$. If $y \in N[v_2]$, then $x$ is a non-neighbour of $y$ in $V_1$. If $y \notin N[v_2]$, then $v_2$ is a non-neighbour of $y$ in $V_1$. This proves the result and so $gd_{\chi}(G) \geq 1$, a contradiction. Therefore $G$ has no extreme vertex. That is, every vertex outside $C$ is a pendant vertex and every vertex on $C$ is either a support vertex or it is of degree exactly two.
Now, suppose exactly two vertices on $C$ are support vertices, say $v_2$ and $v_3$. Then $\{S \cup \{v_1\}, \{v_2\}, \{v_3\}\}$, where $S$ is the set of all pendant vertices of $G$, is a $\chi$-coloring of $G$ in which $S \cup \{v_1\}$ is a global dominating set of $G$ and so $gd_\chi(G) \geq 1$, a contradiction. Suppose all the three vertices on $C$ are support vertices. Then by Theorem 2.6, $gd_\chi(G) = 1$, again a contradiction. Hence the result. The converse follows from Theorem 1.5.

3. Realization Theorems

**Theorem 3.1.** For given integers $k$ and $l$ with $0 \leq l \leq k$, there exists a uniquely - $k$ - colorable graph $G$ with $gd_\chi(G) = l$.

![Figure 2: A uniquely colorable graph with $gd_\chi = 2$ and $\chi = 4$.](image)

**Proof.** For $l = 0$, take $G = K_k$. Assume $l \geq 1$. Then the required graph $G$ is obtained from the complete $k$ - partite graph with parts $V_1, V_2, \ldots, V_k$ where $V_i = \{u_i, v_i\}$, for all $i \in \{1, 2, \ldots, k\}$. Introducing $2l$ new vertices $x_1, x_2, \ldots, x_l, y_1, y_2, \ldots, y_l$. For each $i \in \{1, 2, \ldots, l\}$, join the vertex $x_i$ to each vertex of $u_j$, where $j \neq i$ and $1 \leq j \leq k$ ; and join the vertex $y_i$ to each vertex of $v_j$, where $j \neq i$ and $1 \leq j \leq k$. Let $G$ be the resultant graph. For $l = 2$ and $k = 4$, the graph $G$ is given in Figure 2. From the construction of $G$, it is clear that $G$ is a uniquely - $k$ - colorable graph and $\delta(G) = l - 1$. One can easily verify that $\mathcal{C} = \{V_1 \cup \{x_1, y_1\}, V_2 \cup \{x_2, y_2\}, \ldots, V_l \cup \{x_l, y_l\}, V_{l+1}, \ldots, V_k\}$ is a $\chi$-coloring of $G$ in which $V_1 \cup \{x_1, y_1\}, V_2 \cup \{x_2, y_2\}, \ldots, V_l \cup \{x_l, y_l\}$ are global dominating sets of $G$. Therefore $gd_\chi(G) \geq l$. Since $\delta(G) = l - 1$ and by Theorem 1.3, we have $gd_\chi(G) \leq l$. Thus $gd_\chi(G) = l$.

**Theorem 3.2.** For given integers $a, b$ and $c$ with $0 \leq a \leq b \leq c$, there exists a graph $G$ for which $gd_\chi(G) = a$, $d_\chi(G) = b$ and $\chi(G) = c$ except when $a = 0$ and $b = 1$.

**Proof.** If $a, b$ and $c$ are integers with $gd_\chi(G) = a$, $d_\chi(G) = b$ and $\chi(G) = c$, then by Lemma 2.4, we have $b \geq 2$ when $a = 0$. Conversely, suppose $a, b$ and $c$ are integers with $0 \leq a \leq b \leq c$ and $b \geq 2$ when $a = 0$. We construct the required graph $G$ as follows.

**Case 1.** $a = 0$. 


Then by assumption \( b \geq 2 \). Consider the complete graph \( K_c \) on \( c \) vertices with the vertex set \( \{v_1, v_2, \ldots, v_c\} \). Introduce a vertex \( u \) and join it to each of the vertices \( v_2, v_3, \ldots, v_b \) by an edge. For \( a = 0 \), \( b = 4 \) and \( c = 5 \), the graph \( G \) is illustrated in Figure 3. Clearly \( \chi(G) = c \). Since \( \Delta(G) = n - 1 \), it follows from Theorem 1.5 that \( g_d \chi(G) = 0 \). Further, \( \{\{v_1, u\}, \{v_2\}, \{v_3\}, \ldots, \{v_c\}\} \) is a \( \chi \)-coloring of \( G \) where \( \{v_1, u\}, \{v_2\}, \{v_3\}, \ldots, \{v_b\} \) are dominating sets of \( G \) so that \( d_\chi(G) \geq b \).

The inequality \( d_\chi(G) \leq b \) follows from Theorem 1.2 as \( \delta(G) = b - 1 \). Thus \( d_\chi(G) = b \).

**Case 2.** \( a \geq 1 \).

Here, consider a complete \( c \) - partite graph \( H = K_{2, 2, \ldots, 2} \) with parts \( V_1, V_2, \ldots, V_c \) where \( V_i = \{x_i, y_i\} \) for all \( i \in \{1, 2, \ldots, c\} \). Introduce \( 2a \) new vertices; let them be \( u_1, u_2, \ldots, u_a, v_1, v_2, \ldots, v_a \). For each \( i \in \{1, 2, \ldots, a\} \), join the vertex \( u_i \) to each vertex of the set \( \{x_j : j \neq i \text{ and } 1 \leq j \leq b\} \). Similarly, for each \( i \in \{1, 2, \ldots, a\} \), join the vertex \( v_i \) to each vertex of the set \( \{y_j : j \neq i \text{ and } 1 \leq j \leq b\} \). Let \( G \) be the resultant graph. For \( a = 2 \), \( b = 4 \) and \( c = 5 \), the graph \( G \) is illustrated in Figure 4. Clearly, \( \chi(G) = c \). Now, consider the \( \chi \)-coloring \( \mathcal{C} = \{V_1 \cup \{u_1, v_1\}, V_2 \cup \{u_2, v_2\}, \ldots, V_a \cup \{u_a, v_a\}, V_a+1, V_a+2, \ldots, V_c\} \) of \( G \). It is easy to verify that for each \( i \in \{1, 2, \ldots, a\} \), the set \( V_i \cup \{u_i, v_i\} \) is a global dominating set of \( G \) and for each \( j \in \{a+1, a+2, \ldots, b\} \), the set \( V_j \) is a dominating set of \( G \). Hence \( d_\chi(G) \geq b \) and \( g_d \chi(G) \geq a \). By Theorem 1.2, we have \( d_\chi(G) \leq b \) as \( \delta(G) = b - 1 \) and thus \( d_\chi(G) = b \). We now need to verify that \( g_d \chi(G) \leq a \). Now, clearly the set \( \{u_1, x_1, y_1, v_1\} \) is a global dominating set of \( G \) with minimum cardinality so that \( \gamma_g(G) = 4 \). Also \( s(G) = 2 \). Therefore by Theorem 1.4, we have \( g_d \chi(G) \leq \frac{2a+2c-2c}{2} = a \). Hence \( g_d \chi(G) = a \). \( \square \)
Figure 4: A graph with $gd_x = 2$, $d_x = 4$ and $\chi = 5$.

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