TWIN SIGNED ROMAN DOMATIC NUMBERS IN DIGRAPHS

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Abstract. Let D be a finite simple digraph with vertex set V(D). A twin signed Roman dominating function on the digraph D is a function \( f : V(D) \to \{-1, 1, 2\} \) satisfying the conditions that (i) \( \sum_{x \in N^{-}[v]} f(x) \geq 1 \) and \( \sum_{x \in N^{+}[v]} f(x) \geq 1 \) for each \( v \in V(D) \), where \( N^{-}[v] \) (resp. \( N^{+}[v] \)) consists of \( v \) and all in-neighbors (resp. out-neighbors) of \( v \), and (ii) every vertex \( u \) for which \( f(u) = -1 \) has an in-neighbor \( v \) and an out-neighbor \( w \) for which \( f(v) = f(w) = 2 \). A set \( \{f_1, f_2, \ldots, f_d\} \) of distinct twin signed Roman dominating functions on \( D \) with the property that \( \sum_{i=1}^{d} f_i(v) \leq 1 \) for each \( v \in V(D) \), is called a twin signed Roman dominating family (of functions) on \( D \). The maximum number of functions in a twin signed Roman dominating family on \( D \) is the twin signed Roman domatic number of \( D \), denoted by \( d_{sR}^{\ast}(D) \). In this paper, we initiate the study of the twin signed Roman domatic number in digraphs and we present some sharp bounds on \( d_{sR}^{\ast}(D) \). In addition, we determine the twin signed Roman domatic number of some classes of digraphs.

1. Introduction

Let \( D \) be a finite simple directed graph with vertex set \( V(D) \) and arc set \( A(D) \) (briefly \( V \) and \( A \)). The integers \( n = n(D) = |V(D)| \) and \( m = m(D) = |A(D)| \) are the order and the size of the digraph \( D \). A digraph without directed cycles of length 2 is an oriented graph. If \( uv \) is an arc of \( D \), then we also write \( u \to v \), and we say that \( v \) is an out-neighbor of \( u \) and \( u \) is an in-neighbor of \( v \). For every vertex \( v \), we denote the set of in-neighbors and out-neighbors of \( v \) by \( N^{-}(v) = N^{-}_D(v) \) and \( N^{+}(v) = N^{+}_D(v) \), respectively. Let \( N^{-}_D[v] = N^{-}[v] = N^{-}(v) \cup \{v\} \) and \( N^{+}_D[v] = N^{+}[v] = N^{+}(v) \cup \{v\} \). We write \( d^{+}(v) = d^{+}_D(v) \) for the outdegree of a vertex \( v \) and \( d^{-}(v) = d^{-}_D(v) \) for its indegree. The minimum and maximum indegree and minimum and maximum outdegree of \( D \) are denoted by \( \delta^{-}(D) = \delta^{-} \), \( \Delta^{-}(D) = \Delta^{-} \), \( \delta^{+}(D) = \delta^{+} \) and \( \Delta^{+}(D) = \Delta^{+} \), respectively. A digraph \( D \) is \( r \)-out-regular if \( \delta^{+}(D) = \Delta^{+}(D) = r \). In addition, let \( \delta = \delta(D) = \min[\delta^{+}(D), \delta^{-}(D)] \) and \( \Delta = \Delta(D) = \max[\Delta^{+}(D), \Delta^{-}(D)] \) be the minimum and maximum degree of \( D \), respectively. A digraph \( D \) is called regular or \( r \)-regular if \( \delta(D) = \Delta(D) = r \). If \( X \subseteq V(D) \), then \( D[X] \) is the subdigraph induced by \( X \). If \( X \subseteq V(D) \) and \( v \in V(D) \), then

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A(X, v) is the set of arcs from X to v. We denote by A(X, Y) the set of arcs from a subset X to a subset Y. We denote by $D^{-1}$ the digraph obtained from $D$ by reversing the arcs of $D$. For a real-valued function $f: V \rightarrow \mathbb{R}$ the weight of $f$ is $\omega(f) = \sum_{v \in V} f(v)$, and for $S \subseteq V$, we define $f(S) = \sum_{v \in S} f(v)$, so $\omega(f) = f(V)$. Consult [7] for the notation and terminology which are not defined here.

A signed Roman dominating function (abbreviated SRDF) on $D$ is defined as a function $f: V \rightarrow \{-1, 1, 2\}$ such that (i) $f(N^-[v]) = \sum_{x \in N^-[v]} f(x) \geq 1$ for each vertex $v \in V$ and (ii) every vertex $u$ for which $f(u) = -1$ has an in-neighbor $v$ for which $f(v) = 2$. The signed Roman domination number $\gamma_{sR}(D)$ of $D$ is the minimum weight of an SRDF on $D$. A $\gamma_{sR}(D)$-function is a signed Roman dominating function $f$ on $D$ of weight $\gamma_{sR}(D)$. The signed Roman domination number of a digraph was introduced by Sheikholeslami and Volkmann in [5] and has been studied in [5, 6].

In [6], a set $\{f_1, f_2, \ldots, f_d\}$ of distinct signed Roman dominating functions on $D$ with the property that $\sum_{i=1}^{d} f_i(v) \leq 1$ for each $v \in V(D)$, is called a signed Roman dominating family (of functions) on $D$. The maximum number of functions in a signed Roman dominating family (SRD family) on $D$ is the signed Roman domatic number of $D$, denoted by $d_{sR}(D)$.

In [2], a signed Roman dominating function of $D$ is called a twin signed Roman dominating function (briefly TSRDF) if it also is a signed Roman dominating function of $D^{-1}$, i.e., $f(N^+[v]) \geq 1$ for every $v \in V$ and every vertex $u$ for which $f(u) = -1$ has an out-neighbor $v$ for which $f(v) = 2$. The twin signed Roman domination number for a digraph $D$ is $\gamma^*_{sR}(D) = \min\{\omega(f) \mid f \text{ is an TSRDF of } D\}$. A $\gamma^*_{sR}(D)$-function is a twin signed Roman dominating function on $D$ of weight $\gamma^*_{sR}(D)$. Since every TSRDF of $D$ is an SRDF on both $D$ and $D^{-1}$ and since the constant function 1 is an TSRDF of $D$, we have

$$\max\{\gamma_{sR}(D), \gamma_{sR}(D^{-1})\} \leq \gamma^*_{sR}(D) \leq n. \quad (1)$$

A set $\{f_1, f_2, \ldots, f_d\}$ of distinct twin signed Roman dominating functions on $D$ with the property that $\sum_{i=1}^{d} f_i(v) \leq 1$ for each $v \in V(D)$, is called a twin signed Roman dominating family (of functions) on $D$. The maximum number of functions in a twin signed Roman dominating family (TSRDF family) on $D$ is the twin signed Roman domatic number of $D$, denoted by $d^*_{sR}(D)$. The twin signed Roman domatic number is well-defined and

$$d^*_{sR}(D) \geq 1 \quad (2)$$

for all digraphs $D$, since the set consisting of the TSRDF with constant value 1 forms an TSRDF family on $D$. Since every TSRDF family of $D$ is an SRD family on both $D$ and $D^{-1}$, we have

$$d^*_{sR}(D) \leq \min\{d_{sR}(D), d_{sR}(D^{-1})\}. \quad (3)$$
In this paper, we initiate the study of the twin signed Roman domatic number in digraphs and we present some sharp bounds on $d_{sR}^*(D)$. In addition, we determine the twin signed Roman domatic number of some classes of digraphs.

A signed Roman dominating function (SRDF) on a graph $G = (V(G), E(G))$ is defined in [1] as a function $f : V(G) \rightarrow \{-1,1,2\}$ such that $f(N[v]) = \sum_{x \in N[v]} f(x) \geq 1$ for each vertex $v \in V$, where $N[v]$ is the closed neighborhood of $v$, and every vertex $u$ for which $f(u) = -1$ is adjacent to a vertex $v$ for which $f(v) = 2$. The weight of an SRDF $f$ on $G$ is $\omega(f) = \sum_{v \in V(G)} f(v)$. The signed Roman domination number $\gamma_{sR}(G)$ of $G$ is the minimum weight of an SRDF on $G$. A set $\{f_1, f_2, \ldots, f_d\}$ of distinct SRDF on $G$ with the property that $\sum_{i=1}^d f_i(v) \leq 1$ for each $v \in V(G)$, is called in [4] a signed Roman dominating family (of functions) on $G$. The maximum number of functions in a signed Roman dominating family on $G$ is the signed Roman domatic number of $G$, denoted by $d_{sR}(G)$.

An orientation of a graph $G$ is an assignment of orientations to its edges. The associated digraph $D(G)$ of a graph $G$ is obtained by replacing each edge of $G$ by a pair of two mutually opposite oriented edges. The definitions imply the next observation immediately.

Observation 1. If $G$ is a graph and $D(G)$ its associated digraph, then $\gamma_{sR}(G) = \gamma_{sR}^*(D(G))$ and $d_{sR}(G) = d_{sR}^*(D(G))$.

We make use of the following results in this paper.

Observation 2. ([1]) If $K_n$ is the complete graph of order $n \geq 1$, then $\gamma_{sR}(K_n) = 1$, unless $n = 3$ in which case $\gamma_{sR}(K_n) = 2$.

Observation 3. ([4]) If $K_n$ is the complete graph of order $n \geq 1$, then $d_{sR}(K_n) = n$, unless $n = 3$ in which case $d_{sR}(K_n) = 1$.

Observations 1, 2 and 3 lead to the next results immediately.

Observation 4. If $K_n^*$ is the complete digraph of order $n \geq 1$, then $\gamma_{sR}^*(K_n^*) = 1$, unless $n = 3$ in which case $\gamma_{sR}^*(K_n^*) = 2$.

Observation 5. If $K_n^*$ is the complete digraph of order $n \geq 1$, then $d_{sR}^*(K_n^*) = n$, unless $n = 3$ in which case $d_{sR}^*(K_n^*) = 1$.

If $n \geq 4$ and $\{f_1, f_2, \ldots, f_n\}$ is a signed Roman dominating family of functions on $K_n^*$, then we conclude from

$$n = n \cdot 1 \leq \sum_{i=1}^n \omega(f_i) = \sum_{i=1}^n \sum_{v \in V(K_n^*)} f_i(v) = \sum_{v \in V(K_n^*)} \sum_{i=1}^n f_i(v) \leq \sum_{v \in V(K_n^*)} 1 = n$$

that $\omega(f_i) = 1$ and so $f_i$ is a $\gamma_{sR}(K_n^*)$-function for each $i$. It follows that each $f_i$ assigns 2 to some vertex of $K_n^*$.
Observation 6. ([3]) If $K_{p,p}$ is the complete bipartite graph of order $2p$, then $\gamma_{sR}(K_{p,p}) = 4$ when $p \geq 3$.

Using Observations 1 and 6, we obtain the next result.

Observation 7. If $K^*_{p,p}$ is the complete bipartite digraph of order $2p$, then $\gamma^*_{sR}(K^*_{p,p}) = 4$ when $p \geq 3$.

Observation 8. ([2]) If $C_n$ is an oriented cycle of order $n \geq 2$, then $\gamma^*_{sR}(C_n) = n/2$ when $n$ is even and $\gamma^*_{sR}(C_n) = (n + 3)/2$ when $n$ is odd.

Observation 9. ([6]) If $D$ is a digraph, then $d_{sR}(D) \leq \delta^-(D) + 1$.

Observation 10. ([6]) Let $D$ be an $r$-out-regular digraph of order $n$ such that $r \geq 1$. If $n \not\equiv 0 \pmod{(r + 1)}$, then $d_{sR}(D) \leq r$.

Inequality (3) and Observation 10 imply the next corollary.

Corollary 11. Let $D$ be an $r$-out-regular digraph of order $n$ such that $r \geq 1$. If $n \not\equiv 0 \pmod{(r + 1)}$, then $d^*_{sR}(D) \leq r$.

2. Properties of the twin signed Roman domatic number

In this section we present basic properties of $d^*_{sR}(D)$ and sharp bounds on the twin signed Roman domatic number of digraphs. Using Observation 9 and (3), we obtain our first bound on $d^*_{sR}(D)$.

Proposition 12. If $D$ is a digraph, then $d^*_{sR}(D) \leq \delta(D) + 1$.

Observation 5 shows that Proposition 12 is sharp. Inequality (2) and Proposition 12 imply the next corollary immediately.

Corollary 13. If $D$ is a digraph with $\delta(D) = 0$, then $d^*_{sR}(D) = 1$.

As we observed in (3), $d^*_{sR}(D) \leq d_{sR}(D)$. Now, we show that the difference $d_{sR}(D) - d^*_{sR}(D)$ can be arbitrarily large.

Theorem 14. For every positive integer $k \geq 3$, there exists a digraph $D$ such that

$$d_{sR}(D) - d^*_{sR}(D) \geq k.$$
**Proof.** Let \( k \geq 3 \) be an integer, and let \( D \) be the digraph obtained from two copies of \( K_{k+1}^* \), say \( G_1, G_2 \), by adding a new vertex \( x \) and adding arcs going from every vertex in \( V(G_1) \cup V(G_2) \) into \( x \). Since \( d^+(x) = 0 \), we deduce from Corollary 13 that \( d^*_{sR}(D) = 1 \).

Let \( \{f_1, f_2, \ldots, f_{k+1}\} \) be a signed Roman dominating family on the digraph \( G_1 \), and let \( \{g_1, g_2, \ldots, g_{k+1}\} \) be a signed Roman dominating family on \( G_2 \). As we note after Observation 5, each \( f_i \) assigns 2 to some vertex of \( G_1 \) and each \( g_j \) assigns 2 to some vertex of \( G_2 \). For \( 1 \leq i \leq k+1 \), define \( h_i : V(D) \rightarrow \{-1, 1, 2\} \) by \( h_i(x) = -1 \), \( h_i(u) = f_i(u) \) if \( u \in V(G_1) \) and \( h_i(u) = g_i(u) \) if \( u \in V(G_2) \). Clearly, \( \{h_1, h_2, \ldots, h_{k+1}\} \) is a signed Roman dominating family of \( D \) and hence \( d_{sR}(D) \geq k+1 \). Thus \( d_{sR}(D) - d^*_{sR}(D) \geq k \), and the proof is complete. \( \square \)

**Theorem 15.** If \( D \) is a digraph of order \( n \), then

\[
\gamma^*_{sR}(D) \cdot d^*_{sR}(D) \leq n.
\]

Moreover, if \( \gamma^*_{sR}(D) \cdot d^*_{sR}(D) = n \), then for each TSRD family \( \{f_1, f_2, \ldots, f_d\} \) on \( D \) with \( d = d^*_{sR}(D) \), each function \( f_i \) is a \( \gamma^*_{sR}(D) \)-function and \( \sum_{i=1}^{d} f_i(v) = 1 \) for each \( v \in V(D) \).

**Proof.** Let \( \{f_1, f_2, \ldots, f_d\} \) be an TSRD family on \( D \) with \( d = d^*_{sR}(D) \) and let \( v \in V(D) \). Then

\[
d \cdot \gamma^*_{sR}(D) = \sum_{i=1}^{d} \gamma^*_{sR}(D) \leq \sum_{i=1}^{d} \sum_{v \in V(D)} f_i(v) = \sum_{v \in V(D)} \sum_{i=1}^{d} f_i(v) \leq \sum_{v \in V(D)} 1 = n. \tag{4}
\]

If \( \gamma^*_{sR}(D) \cdot d^*_{sR}(D) = n \), then the two inequalities occurring in (4) become equalities. Hence for the TSRD family \( \{f_1, f_2, \ldots, f_d\} \) on \( D \) and for each \( i \), \( \sum_{v \in V(D)} f_i(v) = \gamma^*_{sR}(D) \). Thus each function \( f_i \) is a \( \gamma^*_{sR}(D) \)-function, and \( \sum_{i=1}^{d} f_i(v) = 1 \) for each \( v \in V(D) \). \( \square \)

Observations 4 and 5 demonstrate that Theorem 15 is sharp. In [6], we have shown that \( d_{sR}(K_{p,p}^*) = \frac{p}{2} \) when \( p \geq 4 \) is an even integer with \( p \neq 6 \). Analogously, one can prove that \( d^*_{sR}(K_{p,p}^*) = \frac{p}{2} \) when \( p \geq 4 \) is an even integer with \( p \neq 6 \). Using this identity and Observation 7, we have a further example which shows the sharpness of Theorem 15.

Applying Observation 8, Proposition 12 and Theorem 15, we obtain the twin signed Roman domatic number for oriented cycles.

**Corollary 16.** Let \( C_n \) be an oriented cycle of length \( n \geq 2 \). Then \( d^*_R(C_n) = 1 \) when \( n \) is odd and \( d^*_R(C_n) = 2 \) when \( n \) is even.

**Proof.** First let \( n \) be odd. Using Observation 8 and Theorem 15, we deduce that

\[
d^*_R(C_n) = \frac{n}{\gamma^*_R(C_n)} = \frac{2n}{n+3} < 2
\]

and thus \( d^*_R(C_n) = 1 \).
Now let \( n = 2p \) be even, and let \( C_n = u_1 v_1 u_2 v_2 \ldots u_p v_p u_1 \). Define the function \( f_i : V(C_n) \to \{-1, 1, 2\} \) by \( f_1(u_i) = -1 \) and \( f_1(v_i) = 2 \) and \( f_2(u_i) = 2 \) and \( f_2(v_i) = -1 \) for \( 1 \leq i \leq p \). Then \( f_1 \) and \( f_2 \) are TSRDF on \( C_n \) such that \( f_1(x) + f_2(x) = 1 \) for each \( x \in V(C_n) \). Therefore \( d_{sR}^*(C_n) \geq 2 \). It follows from Proposition 12 that \( d_{sR}^*(C_n) \leq 2 \) and so \( d_{sR}^*(C_n) = 2 \) when \( n \) is even. \( \square \)

According to Corollary 16, the oriented cycle \( C_n \) is another example which shows the sharpness of Theorem 15, when \( n \) is even.

**Theorem 17.** If \( D \) is a digraph of order \( n \), then

\[
\gamma_{sR}^*(D) + d_{sR}^*(D) \leq n + 1
\]

with equality if and only if \( D = K^*_n \) (\( n \neq 3 \)) or \( \gamma_{sR}^*(D) = n \) and \( d_{sR}^*(D) = 1 \).

**Proof.** It follows from Theorem 15 that

\[
\gamma_{sR}^*(D) + d_{sR}^*(D) \leq \frac{n}{d_{sR}^*(G)} + d_{sR}^*(D).
\]  

(5)

According to (2) and Proposition 12, we have \( 1 \leq d_{sR}^*(G) \leq n \). Using these bounds, and the fact that the function \( g(x) = x + n/|x| \) is decreasing for \( 1 \leq x \leq \sqrt{n} \) and increasing for \( \sqrt{n} \leq x \leq n \), we observe that the maximum of \( g \) on the interval \([1, n]\) is \( n + 1 \). Therefore (5) leads to the desired bound.

If \( D = K^*_n \) (\( n \neq 3 \)), then we deduce from Observations 4 and 5 that \( \gamma_{sR}^*(D) + d_{sR}^*(D) = n + 1 \). Clearly, if \( \gamma_{sR}^*(D) = n \) and \( d_{sR}^*(D) = 1 \), then \( \gamma_{sR}^*(D) + d_{sR}^*(D) = n + 1 \).

Conversely, assume that \( \gamma_{sR}^*(D) + d_{sR}^*(D) = n + 1 \). Since the maximum of \( g \) on \([1, n]\) is achieved only at 1 and \( n \), it follows from (5) that

\[
n + 1 = \gamma_{sR}^*(D) + d_{sR}^*(D) \leq \frac{n}{d_{sR}^*(G)} + d_{sR}^*(D) \leq n + 1,
\]

which implies that \( \gamma_{sR}^*(D) = n \) and \( d_{sR}^*(D) = 1 \) or \( \gamma_{sR}^*(D) = 1 \) and \( d_{sR}^*(D) = n \). If \( d_{sR}^*(D) = n \) and \( \gamma_{sR}^*(D) = 1 \), then Proposition 12 implies that \( \delta(D) = n - 1 \) and hence \( D \) is the complete digraph \( K^*_n \). Since \( \gamma_{sR}^*(D) = 1 \), we conclude from Observation 4 that \( n \neq 3 \) and so \( D = K^*_n \) (\( n \neq 3 \)). \( \square \)

If \( H \) is the disjoint union of oriented triangles, then it follows from Observation 8 and Corollary 16 that \( \gamma_{sR}^*(H) = n \) and \( d_{sR}^*(H) = 1 \). Thus, in Theorem 17, \( \gamma_{sR}^*(D) = n \) and \( d_{sR}^*(D) = 1 \) is possible.

The complement \( \overline{D} \) of a digraph \( D \) is the digraph with vertex set \( V(D) \) such that for any two distinct vertices \( u, v \) the arc \((u, v)\) belongs to \( \overline{D} \) if and only if \((u, v)\) does not belong to \( D \).

**Theorem 18.** For every digraph \( D \) of order \( n \),

\[
d_{sR}^*(D) + d_{sR}^*(\overline{D}) \leq n + 1
\]

with equality if and only if \( D = K^*_n \) or \( \overline{D} = K^*_n \) and \( n \neq 3 \).
Proof. Since \( \delta(\overline{D}) = n - 1 - \Delta(D) \), it follows from Proposition 12 that
\[
d^*_s(D) + d^*_s(\overline{D}) \leq (\delta(D) + 1) + (\delta(\overline{D}) + 1)
= (\delta(D) + 1) + (n - 1 - \Delta(D) + 1) \leq n + 1,
\]
and this is the desired inequality. If \( D \) is not regular, then \( \Delta(D) - \delta(D) \geq 1 \), and hence the above inequality chain implies the better bound \( d^*_s(D) + d^*_s(\overline{D}) \leq n \).

If \( D = K^*_n \) (\( n \neq 3 \)), then we deduce from Observation 5 and Corollary 13 that \( d^*_s(D) + d^*_s(\overline{D}) = n + 1 \).

Now assume that \( d^*_s(D) + d^*_s(\overline{D}) = n + 1 \). As seen above, this condition shows that \( D \) is an \( r \)-regular digraph. Therefore \( \overline{D} \) is \((n - r - 1)\)-regular. If \( r = 0 \) or \( r = n - 1 \), then \( D = K^*_n \) or \( \overline{D} = K^*_n \), and we obtain the desired result.

Next assume that \( 1 \leq r \leq n - 2 \) and \( 1 \leq \delta(\overline{D}) \leq n - 2 \). We assume, without loss of generality, that \( r \leq (n - 1)/2 \). If \( n \equiv 0 \) (mod\((r + 1)\)), then it follows from Corollary 11 and Proposition 12 that
\[
n + 1 = d^*_s(D) + d^*_s(\overline{D}) \leq r + (n - 1 - r + 1) = n,
\]
a contradiction. Next assume that \( n \equiv 0 \) (mod\((r + 1)\)). Then \( n = p(r + 1) \) with an integer \( p \geq 2 \).

If \( n \equiv 0 \) (mod\((n - r)\)), then it follows from Corollary 11 and Proposition 12 that
\[
n + 1 = d^*_s(D) + d^*_s(\overline{D}) \leq (r + 1) + (n - 1 - r) = n,
\]
a contradiction. Therefore assume that \( n \equiv 0 \) (mod\((n - r)\)). Then \( n = q(n - r) \) with an integer \( q \geq 2 \). Since \( r \leq (n - 1)/2 \), this leads to the contradiction
\[
n = q(n - r) \geq q \left( n - \frac{n - 1}{2} \right) = \frac{q(n + 1)}{2} \geq n + 1,
\]
and the proof is complete. \( \square \)

For some special cases we will improve Proposition 12.

**Theorem 19.** Let \( D \) be a digraph. If \( D \) has a vertex \( v \) with the property that \( d^+(v) = 2 \) or \( d^-(v) = 2 \), then \( d^*_s(D) = 1 \).

**Proof.** Assume, without loss of generality, that \( d^+(v) = 2 \). Let \( u_1 \) and \( u_2 \) be the two out-neighbors of \( v \). Using Proposition 12, we observe that \( d^*_s(D) \leq 3 \). First we show that \( d^*_s(D) \leq 2 \).

Suppose, to the contrary, that \( d^*_s(D) = 3 \), and let \( \{f, g, h\} \) be a TSRD family on \( D \). Since \( f(x) + g(x) + h(x) \leq 1 \) for each \( x \in V(D) \), we deduce that \( f(x) = -1 \) or \( g(x) = -1 \) or \( h(x) = -1 \) for each \( x \in V(D) \). In addition, if \( f(y) = 2 \) for a vertex \( y \), then \( g(y) = h(y) = -1 \). Now assume,
without loss of generality, that \( f(v) = -1 \). Then \( f(u_1) = 2 \) or \( f(u_2) = 2 \), say \( f(u_1) = 2 \). If \( f(u_2) = -1 \), then \( f(N^+[v]) = 0 \), a contradiction. Next let \( f(u_2) \geq 1 \). Then, without loss of generality, \( g(u_2) = -1 \). Since \( g(u_1) = -1 \), we obtain the contradiction \( g(N^+[v]) \leq 0 \). Thus \( d_{sR}^*(D) \leq 2 \).

Next we show that \( d_{sR}^*(D) = 1 \). Suppose, to the contrary, that \( d_{sR}^*(D) = 2 \), and let \( \{f, g\} \) be a TSRD family on \( D \). Since \( f(x) + g(x) \leq 1 \) for each \( x \in V(D) \), we deduce that \( f(x) = -1 \) or \( g(x) = -1 \) for each \( x \in V(D) \). Assume, without loss of generality, that \( f(v) = -1 \). Then \( f(u_1) = 2 \) or \( f(u_2) = 2 \), say \( f(u_1) = 2 \). If \( f(u_2) = -1 \), then \( f(N^+[v]) = 0 \), a contradiction. Next let \( f(u_2) \geq 1 \). Then \( g(u_1) = g(u_2) = -1 \), and we arrive at the contradiction \( g(N^+[v]) \leq 0 \). □

For \( r = 2 \), Theorem 19 yields to the following improvement of Corollary 11.

**Corollary 20.** If \( D \) is a 2-out-regular digraph, then \( d_{sR}^*(D) = 1 \).

**Corollary 21.** Let \( D \) be a digraph. If \( D \) has a vertex \( v \) with the property that \( d^+(v) + d^-(v) = 3 \), then \( d_{sR}^*(D) = 1 \).

**Proof.** If \( \delta(D) = 0 \), then Corollary 13 implies the desired result. Let now \( \delta(D) \geq 1 \). Since
\[ d^+(v) + d^-(v) = 3, \]
we observe that \( d^+(v) = 2 \) or \( d^-(v) = 2 \). Now we deduce from Theorem 19 that \( d_{sR}^*(D) = 1 \). □

A fan and a wheel is a graph obtained from a path and a cycle by adding a new vertex and edges joining it to all vertices of the path and cycle, respectively. Corollary 21 leads to the next result immediately.

**Corollary 22.** If \( D \) is an orientation of a fan, a wheel or a cubic graph, then \( d_{sR}^*(D) = 1 \).

**References**


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