SOME APPLICATIONS OF CERTAIN INTEGRAL OPERATORS INVOLVING FUNCTIONS WITH BOUNDED RADIUS ROTATIONS

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Abstract. Integral transforms map equations from their original domains into others where manipulations and solutions may be much easier than in original domains. To get back in the original environment, we use the idea of inverse of the integral transform. A function analytic and locally univalent in a given simply connected domain is said to be of bounded boundary rotation if its range has bounded boundary rotation which is defined as the total variation of the direction angle of the tangent to the boundary curve under a complete circuit.

The main objective of the present article is to study some applications of certain integral operators to functions of bounded radius rotation involving Janowski functions. We discuss some inclusion results under certain assumption on parameters involve in operators as well as in related subclasses of analytic functions. Most of these results are best possible. We also relate our findings with the existing literature of the subjects.

1. Introduction

Let $\mathcal{H}(U)$ represent the class of all analytic functions $f$ defined in the open unit disk $U := \{z \in \mathbb{C} : |z| < 1\}$ and for a positive integer $n$ and $a \in \mathbb{C}$, let

$$\mathcal{H}[a, n] := \{f \in \mathcal{H}(U) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots\}.$$ 

Also, the subclass $\mathcal{A}$ of the class $\mathcal{H}[a, n]$ is defined as:

$$\mathcal{A} := \{f \in \mathcal{H}[0, 1] : f'(0) = 1\}.$$  \hspace{1cm} (1.1)

The class of univalent functions is represented by $\mathcal{S}$ and it is a subclass of the class $\mathcal{A}$, whereas, $\mathcal{S}^*, \mathcal{C}, \mathcal{K}$ and $\mathcal{Q}$ are the well-known classes of starlike, convex, close-to-convex and quasi-convex functions respectively. The Hadamard product or convolution for $f$ and $g$ is given by

$$\left(f \ast g\right)(z) = z + a_2 b_2 z^2 + a_3 b_3 z^3 + \ldots \quad (z \in U),$$

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where \( f \) is defined by (1.1) and \( g(z) = z + a_2 z^2 + a_3 z^3 + \ldots \). For \( f, g \in \mathcal{H}(\mathbb{U}) \), we say that \( f \) is subordinate to \( g \) and write as \( f(z) \prec g(z) \), if there exists a Schwarz function \( w \), that is, \( w \in \mathcal{H}(\mathbb{U}) \) with \( w(0) = 0 \) and \( |w(z)| < 1 \), such that \( f(z) = g(w(z)), z \in \mathbb{U} \). Let \( \mathcal{P} \) denote the well-known class of Carathéodory functions \( p \) such that \( p \in \mathcal{H}(\mathbb{U}) \), with \( p(0) = 1 \) and \( \operatorname{Re} p(z) > 0 \), \( z \in \mathbb{U} \). Also \( \mathcal{P}(\rho) \) represents the class of Carathéodory functions \( p \) such that \( p \in \mathcal{H}(\mathbb{U}) \) with \( p(0) = 1 \) and \( \operatorname{Re} p(z) > \rho, (0 \leq \rho < 1, z \in \mathbb{U}) \). For details, see [5]. Using subordination defined above, Janowski [6] introduced the class \( \mathcal{P}[A, B] \) for \(-1 \leq B < A \leq 1 \). A function \( p \) analytic in \( \mathbb{U} \) such that \( p(0) = 1 \) belongs to the class \( \mathcal{P}[A, B] \), if \( p(z) = \frac{1 + Aw(z)}{1 + Bw(z)} \), where \( w \) is a Schwarz function. Geometrically, the image \( p(\mathbb{U}) \) of \( p \in \mathcal{P}[A, B] \) lies inside the open unit disk centered on the real axis with diameter ends at \( p(-1) \) and \( p(1) \). Clearly \( \mathcal{P}[A, B] \subset \mathcal{P}(\frac{1-A}{1+B}) \). For a general bilinear transformation \( \frac{1+Az}{1+Bz}, A_1 \in \mathbb{C}, B_1 \in [-1, 0] \) and \( A_1 \neq B_1 \), the class \( \mathcal{P}[A_1, B_1] \) consists of functions \( p \) with \( p(0) = 1 \) and \( p(z) < \frac{1+Az}{1+Bz}, \) see [16].

A functions \( p \) analytic in \( \mathbb{U} \) such that \( p(0) = 1 \) belongs to the class \( \mathcal{P}_k[A, B] \), if and only if

\[
p(z) = \frac{1}{2} \int_{-\pi}^{\pi} \frac{1 + Aze^{-it}}{1 + Bze^{-it}} \, d\varphi(t) \quad (-1 \leq B < A \leq 1, k \geq 2, z \in \mathbb{U}),
\]

where \( \varphi \) is a real valued function of bounded variation on \([-\pi, \pi]\) that satisfies the conditions \( \int_{-\pi}^{\pi} d\varphi(t) = 2 \) and \( \int_{-\pi}^{\pi} |d\varphi(t)| \leq 2 \) or equivalently, \( p \in \mathcal{P}_k[A, B] \) if and only if there exist \( p_1, p_2 \in \mathcal{P}[A, B] \) such that

\[
p(z) = \left( \frac{k}{4} + \frac{1}{2} \right) p_1(z) - \left( \frac{k}{4} - \frac{1}{2} \right) p_2(z) \quad (k \geq 2, z \in \mathbb{U}). \tag{1.2}
\]

From (1.2), we note that \( \mathcal{P}_k[1 - 2\rho, -1] = \mathcal{P}(\rho) \), see [20] and \( \mathcal{P}_k[1, -1] = \mathcal{P}_k \), for reference, see [23]. The classes \( \mathcal{V}_k[A, B] \) and \( \mathcal{R}_k[A, B] \) are related to the class \( \mathcal{P}_k[A, B] \) can be defined as:

\[
f \in \mathcal{V}_k[A, B] \iff \left( \frac{zf'(z)}{f'(z)} \right)' \in \mathcal{P}_k[A, B] \quad \text{and} \quad f \in \mathcal{R}_k[A, B] \iff \frac{zf'(z)}{f(z)} \in \mathcal{P}_k[A, B].
\]

For various related classes, we refer [1, 2, 3, 8, 9, 12, 13, 14, 15, 24]. For \( f \in \mathcal{A} \), we consider the following one parameter families of integral operators

\[
I^\mu_{\eta} f(z) = \frac{\Gamma(\mu+1)}{\Gamma(\mu)z^{\mu}} \int_0^z t^{(\mu-1)} \left( \log \frac{z}{t} \right)^{(\mu-1)} f(t) \, dt,
\]

\[
\mathfrak{S}^\mu_{\eta} f(z) = \left( \frac{\mu+\eta}{\eta} \right) \frac{\mu+1}{\eta} \int_0^z t^{(\mu-1)} \left( 1 - \frac{t}{z} \right)^{(\mu-1)} f(t) \, dt,
\]

and \( J^\mu_{\eta} f(z) = \frac{\eta+1}{z^\eta} \int_0^z t^{(\eta-1)} f(t) \, dt \), where \( \mu \geq 0, \eta > -1, z \in \mathbb{U}, \Gamma \) denotes the familiar gamma function and \( \left( \frac{\mu}{\eta} \right) = \frac{\Gamma(\mu+1)}{\Gamma(\mu+1)} \). The operator \( I^\mu_{\eta} \) was studied by Patel et al. [22] and the operators \( I^\mu_{\eta}, J^\mu_{\eta} \) were introduced by Jung et al. [7]. Later Owa [21] investigated certain properties of the operators \( I^\mu_{\eta} \) and \( \mathfrak{S}^\mu_{\eta} \). For \( \eta > -1 \), the operator \( J^\mu_{\eta} \) is the generalized Bernardi-Libera-Livingston integrals operator studied by Owa and Srivasra [19] and Srivastava and Owa [25].
For $\eta = 1, 2, \ldots$, the operator $J_\eta$ was studied by Bernardi [25] and $J_1$ is the Libera integral operator [10]. Using the integral representation of gamma and beta functions, it can be shown that for $f \in \mathcal{A}$:

\[
I_\eta^\mu f(z) = z + \sum_{n=2}^{\infty} \left( \frac{\eta + 1}{\eta + n} \right)^{\alpha} a_n z^n, \tag{1.3}
\]

\[
\Im_\eta^\mu f(z) = \left( \frac{\mu + \eta}{\eta} \right) z_2 F_1(1, \eta, \mu + \eta; z) \ast f(z), \tag{1.4}
\]

and

\[
J_\eta f(z) = z + \sum_{v=2}^{\infty} \left( \frac{\eta + 1}{\eta + v} \right) a_v z^v, \tag{1.5}
\]

where $\mu \geq 0$, $\eta > -1, z \in \mathbb{U}$. By virtue of (1.3), (1.4), (1.5) and $\mu \geq 0$, $\eta > -1$, we have the following identities:

\[
z \left( I_\eta^\mu f(z) \right)' = (\eta + 1) I_\eta^{\mu-1} f(z) - \eta I_\eta^\mu f(z), \quad z \in \mathbb{U} \tag{1.6}
\]

and

\[
z \left( \Im_\eta^\mu f(z) \right)' = (\mu + \eta) \Im_\eta^{\mu-1} f(z) - (\mu + \eta - 1) \Im_\eta^\mu f(z). \tag{1.7}
\]

2. Preliminaries

To establish our main results, we will use the following lemmas.

**Lemma 1.** Let $\delta, \gamma$ with $\Re(\delta + \gamma) > 0$ and let $A_1 \in \mathbb{C}$, $B_1 \in [-1, 0]$ satisfy either

\[
\Re \left[ \delta (1 + A_1 B_1) + \gamma (1 + B_1^2) \right] \geq \left| \delta A_1 + \overline{\delta} B_1 + B_1 (\gamma + \overline{\gamma}) \right|,
\]

when $B_1 \in (-1, 0)$ or $\delta (1 + A_1) > 0$ and $\Re \left[ \delta (1 - A_1) + 2 \gamma \right] \geq 0$, when $B_1 = -1$. If $p(0) = 1$ satisfies

\[
p(z) + \frac{zp'(z)}{\delta p(z) + \gamma} < \frac{1 + A_1 z}{1 + B_1 z} \quad (z \in \mathbb{U}), \tag{2.1}
\]

then $p(z) < q(z) < \frac{1 + A_1 z}{1 + B_1 z}, z \in \mathbb{U}$, where $q(z) + \frac{zq'(z)}{\delta q(z) + \gamma} = \frac{1 + A_1 z}{1 + B_1 z}$ $(z \in \mathbb{U})$ has univalent solution in $\mathbb{U}$ given by

\[
q(z) = \begin{cases} 
\delta \int_0^1 \left[ \frac{1 + B_1 t z}{1 + B_1 z} \right] \delta^\left( \frac{A_1}{\pi} - 1 \right) t^{(\delta + \gamma - 1)} dt \end{cases}^{-1} - \frac{\gamma}{\delta}, \quad B_1 \neq 0 \tag{2.2}
\]

\[
\left( \delta \int_0^1 e^{\delta A_1 (t-1) z} t^{(\delta + \gamma - 1)} dt \right)^{-1} - \frac{\gamma}{\delta}, \quad B_1 = 0.
\]

In addition $q$ is the best dominant of (2.1).

**Lemma 2.** Let $\alpha$ be a positive measure on the unit interval $I = [0, 1]$. Let $g(t, z)$ be a function analytic in $\mathbb{U}$ for each $t \in I$ and integrable in $t$ for each $z \in \mathbb{U}$ and for almost all $t \in I$. Suppose that $\Re(g(t, z)) > 0$ in $\mathbb{U}$, $g(t, -r)$ is real for $r$ and $\Re \left( \frac{1}{g(t, z)} \right) \geq \frac{1}{g(t, -r)}$ for $|z| \leq r < 1$ and $t \in I$. If $g(z) = \int_I g(t, z) d\alpha(t)$, then $\Re \left( \frac{1}{g(z)} \right) \geq \frac{1}{g(-r)}, |z| \leq r$. 


Lemma 3. For real or complex numbers \( a, b, c \) \((c \neq 0, -1, -2, \ldots)\) and \( \text{Re} \, c > \text{Re} \, b > 0 \), we have
\[
\int_0^1 t^{b-1} (1-t)^{c-b-1} (1-tz)^{-a} \, dt = \frac{\Gamma(b) \Gamma(c-b)}{\Gamma(c)} _2F_1(a, b, c; z) \quad (z \in \mathbb{U}),
\] (2.3)
where
\[
_2F_1(a, b, c; z) = (1-z)^{-a} _2F_1\left(a, c-b, c; \frac{z}{1-z}\right)
\] (2.4)
and \( _2F_1(a, b, c; z) = _2F_1(b, a, c; z) \).

For the proof of the above lemmas, we refer [11].

Lemma 4. Let \( p \in \mathcal{P}(\varrho) \) with \( p(0) = 1, 0 \leq \varrho < 1 \). Then
\[
\text{Re} \, p(z) > 2\varrho - 1 + \frac{2(1-\varrho)}{1+|z|} \quad (z \in \mathbb{U}).
\]

Lemma 5. Let \( p_i \in \mathcal{P}(\varrho_i) : p_i(0) = 1, 0 \leq \varrho_i < 1, i = 1, 2 \). Then \( p_1 \ast p_2 \in \mathcal{P}(\varrho_3) \) for \( \varrho_3 = 1 - 2(1-\varrho_1)(1-\varrho_2) \). This result is best possible.

For reference of Lemma 4 and Lemma 5, see [21] and [26] respectively.

3. Main results

Theorem 1. Let \( f \in \mathcal{A} \) be such that
\[
\frac{\Im \, \mu \eta^{-2} f(z)}{\Im \, \mu \eta^{-1} f(z)} \in \mathcal{P}_k[A_1, B_1] \quad (z \in \mathbb{U}),
\] (3.1)
for \(-1 \leq B_1 \leq 0, A_1 \neq B_1, k \geq 2, \mu \geq 2, \eta > -1 \) and \( \mu + \eta - 1 > 0 \). Then
\[
\frac{\Im \, \mu \eta^{-1} f(z)}{\Im \, \mu \eta f(z)} \in \mathcal{P}_k[A_1^*, B_1], A_1^* = \frac{(\mu + \eta - 1)A_1 + B_1}{\mu + \eta} \quad (z \in \mathbb{U}).
\]

(i) Further, if \( f \in \mathcal{A} \) satisfies (3.1), then we have
\[
\frac{\Im \, \mu \eta^{-1} f(z)}{\Im \, \mu \eta f(z)} < \frac{1}{(\mu + \eta)Q(z)} = \tilde{q}(z),
\]
where
\[
Q(z) = \begin{cases} 
\int_0^1 \left[ \frac{1 + B_1 tz}{1 + B_1 z} \right]^{(\mu + \eta - 1)A_1^*-1} t^{(\mu + \eta - 1)} \, dt, & B_1 \neq 0 \\
\int_0^1 \left[ \frac{1 + B_1 tz}{1 + B_1 z} \right]^{(\mu + \eta - 1)} t^{(\mu + \eta - 1)} \, dt, & B_1 = 0
\end{cases}
\] (3.2)
(ii) Moreover, if \(-1 \leq B_1 < 0\), and \(B_1 < \text{Re} A_1 \leq \min \left\{ \frac{-2B_1}{\mu + \eta - 1}, \frac{\mu + \eta - B_1}{\mu + \eta - 1} \right\}\), then for \(f \in \mathcal{A}\) satisfying (3.1), we have \(\frac{\mathfrak{H}_{\mu}^{-1} f(z)}{\mathfrak{H}_{\mu} f(z)} \in \mathcal{P}(p_1)\), where

\[
\theta_1 = \left[ 2 F_1 \left( 1, \frac{(\mu + \eta - 1)(B_1 - A_1)}{B_1}, \mu + \eta + 1; \frac{B_1}{B_1 - 1} \right) \right]^{-1}. \tag{3.3}
\]

This result is best possible.

**Proof.** Let

\[
\frac{\mathfrak{H}_{\mu}^{-1} f(z)}{\mathfrak{H}_{\mu} f(z)} = p(z) = \left( \frac{k}{4} + \frac{1}{2} \right) p_1(z) - \left( \frac{k}{4} - \frac{1}{2} \right) p_2(z) \quad (z \in \mathbb{U}), \tag{3.4}
\]

where \(p\) is analytic in \(\mathbb{U}\) with \(p(0) = 1\). On logarithmic differentiation of (3.4) and using (1.7), we obtain

\[
\frac{\mathfrak{H}_{\mu}^{-2} f(z)}{\mathfrak{H}_{\mu}^{-1} f(z)} = \frac{(\mu + \eta) p(z) - 1}{(\mu + \eta - 1)} + \frac{zp'(z)}{(\mu + \eta - 1)p(z)} \quad (z \in \mathbb{U}). \tag{3.5}
\]

From (3.4), we can write

\[
\frac{(\mu + \eta) p(z) - 1}{(\mu + \eta - 1)} = \left( \frac{k}{4} + \frac{1}{2} \right) \frac{(\mu + \eta) p_1(z) - 1}{(\mu + \eta - 1)} - \left( \frac{k}{4} - \frac{1}{2} \right) \frac{(\mu + \eta) p_2(z) - 1}{(\mu + \eta - 1)}.
\]

Let

\[
\frac{(\mu + \eta) p(z) - 1}{(\mu + \eta - 1)} = \psi(z) = \left( \frac{k}{4} + \frac{1}{2} \right) \psi_1(z) - \left( \frac{k}{4} - \frac{1}{2} \right) \psi_2(z) \quad (z \in \mathbb{U}),
\]

where \(\psi_i(z) = \frac{(\mu + \eta) p_i(z) - 1}{(\mu + \eta - 1)}\), \(i = 1, 2, z \in \mathbb{U}\). Thus (3.5) becomes

\[
\frac{\mathfrak{H}_{\mu}^{-2} f(z)}{\mathfrak{H}_{\mu}^{-1} f(z)} = \psi(z) + \frac{z \psi'(z)}{(\mu + \eta - 1) \psi(z) + 1} \quad (z \in \mathbb{U}). \tag{3.6}
\]

On simplification as in [12] and [17], we have

\[
\psi(z) + \frac{z \psi'(z)}{D \psi(z) + 1} = \frac{k + 2}{4} \left( \psi_1(z) + \frac{z \psi_1'(z)}{D \psi_1(z) + 1} \right) - \frac{k - 2}{4} \left( \psi_2(z) + \frac{z \psi_2'(z)}{D \psi_2(z) + 1} \right),
\]

where \(D = \mu + \eta - 1\). From (3.6), it is clear that

\[
\psi_i(z) + \frac{z \psi_i'(z)}{(\mu + \eta - 1) \psi_i(z) + 1} \in \mathcal{P}[A_1, B_1] \text{ for } i = 1, 2.
\]

Now using Lemma 1 for \(\delta = \mu + \eta - 1\), \(\gamma = 1: \delta + \gamma = \mu + \eta > 0\), we have

\[
\psi_i(z) < q(z) < \frac{1 + A_1 z}{1 + B_1 z} \quad (z \in \mathbb{U}), \tag{3.7}
\]

where \(q\) is the best dominant and is given by (2.2) for \(\delta = \mu + \eta - 1\) and \(\gamma = 1\). Again from (3.7), we have

\[
p_i(z) < \frac{1}{(\mu + \eta) Q(z)} \quad (z \in \mathbb{U}),
\]
where $Q$ is given by (3.2). Next we show that

$$\inf_{|z| < 1} \{\Re q(z)\} = q(-1). \quad (3.8)$$

Setting $a = (\mu + \eta - 1) \frac{(b_1 - A)}{B_1}$, $b = \mu + \eta$ and $c = \mu + \eta + 1$ so that $\mu + \eta + 1 > \mu + \eta > 0$ in (3.2) and by using (2.3), (3.6) and (3.7), we see that for $B_1 \neq 0$

$$Q(z) = (1 + B_1 z)^a \int_0^1 t^{b-1} (1 + B_1 t z)^{-a} d t = \frac{\Gamma(b)}{\Gamma(c)} z \binom{1}{1} t(1, 1, c; \frac{B_1 z}{B_1 z + 1}). \quad (3.9)$$

To prove (3.8), we have to show that

$$\Re \left( \frac{1}{Q(z)} \right) \geq \frac{1}{Q(-1)} \quad (z \in \mathbb{U}).$$

Again from (3.2) and (3.9) for $-1 \leq B_1 < 0$ such that

$$B_1 < \Re A_1 \leq \min \left\{ \frac{-2B_1}{\mu + \eta - 1}, \frac{\mu + \eta - B_1}{\mu + \eta - 1} \right\},$$

we have

$$Q(z) = \int_0^1 g(t, z) d \alpha(t) \quad (z \in \mathbb{U}),$$

where $g(t, z) = \frac{1 + B_1 z}{1 + (1 - t) B_1 z}$ and

$$d \alpha(t) = \frac{\Gamma(b)}{\Gamma(c) (c - b)} t^{a-1} (1 - t)^{c-a-1} d t$$

is positive measure on $[0, 1]$. For $-1 \leq B_1 < 0$, we find that $\Re g(t, z) > 0$, where $g(t, -r)$ is real for $|z| \leq r < 1$, $t \in [0, 1]$ and

$$\Re \left( \frac{1}{g(t, z)} \right) = \Re \left( \frac{1 + (1 - t) B_1 z}{1 + B_1 z} \right) \geq \frac{1}{1 - B_1 r} = \frac{1}{g(t, -r)}.$$

Thus by making use of Lemma 2 and letting $r \to 1^-$, we prove that $\Re \frac{1}{Q(z)} \geq \Re \frac{1}{Q(-1)}$, $z \in \mathbb{U}$. In the case $A_1 = g_0 = \min \left\{ \frac{-2B_1}{\mu + \eta - 1}, \frac{\mu + \eta - B_1}{\mu + \eta - 1} \right\}$, we have the required result by taking $A_1 \to g_0^+$. The result is best possible because of best dominant property of $q$.

For special cases, we refer, [15, 19].

**Theorem 2.** Let $\mu \geq 2$, $\eta > -1$ and $\mu + \eta > 1$. If $f \in \mathcal{A}$ satisfies

$$\mathcal{S}_1^{\mu-2} f(z) \in \mathcal{P}_{k}[A_1, B_1] \quad (z \in \mathbb{U}),$$

for $A_1 \in \mathcal{C}$, $-1 \leq B_1 < 0$, $A_1 \neq B_1$ and $B_1 < \Re A_1 \leq \min \left\{ \frac{-2B_1}{\mu + \eta - 1}, \frac{\mu + \eta - B_1}{\mu + \eta - 1} \right\}$, then $\mathcal{S}_1^{\mu} f \in R_k(g_0)$, where $g_0 = (\mu + \eta)(1 - g_1)$ and $g_1$ is defined by (3.3).
Proof. From (1.7), we have
\[
1 + \frac{1}{\mu + \eta} \left\{ \frac{z (\mathfrak{I}_\mu f(z))'}{\mathfrak{I}_\mu f(z)} - 1 \right\} = \frac{\mathfrak{I}_\eta^{\mu-2} f(z)}{\mathfrak{I}_\eta^{\mu-1} f(z)} \quad (z \in \mathbb{U}).
\]
By applying Theorem 1, we have
\[
1 + \frac{1}{\mu + \eta} \left\{ \frac{z (\mathfrak{I}_\mu f(z))'}{\mathfrak{I}_\mu f(z)} - 1 \right\} \in \mathcal{P}_k(\varrho_1),
\]
where \(\varrho_1\) is defined by (3.3). This yields the required result.

Theorem 3. Let \(f \in \mathcal{A}\) satisfies
\[
\frac{\mathfrak{I}_\eta^{\mu-2} f(z)}{\mathfrak{I}_\eta^{\mu-1} f(z)} \in \mathcal{P}_k[\mathcal{A}_1, B_1], \quad (k \geq 2, \mu \geq 2, \eta > -1, z \in \mathbb{U}),
\]
for \(A_1 \in \mathbb{C}, B_1 \in [-1, 0], A_1 \neq B_1\) and \(\mu + \eta - 1 > 0\), then
\[
\frac{\mathfrak{I}_\eta^{\mu-1} f(z)}{\mathfrak{I}_\eta^{\mu} f(z)} < \frac{1}{(\eta + 1)Q(z)} = \widetilde{q}(z) \in \mathcal{P}_k[\mathcal{A}_1, B_1] \quad (z \in \mathbb{U}),
\]
where \(Q(z) = \int_0^1 \left[ \frac{1 + B_1 t}{1 + B_1 z} \right]^{(\eta + 1)} \left[ \frac{A_1}{B_1} \right]^{(\eta + 1) - 1} t^\eta \, dt\) for \(B_1 \neq 0\) and \(Q(z) = \int_0^1 t^\eta e^{(\eta + 1)A_1 (t^\eta - 1)z} \, dt\) when \(B_1 = 0\). Also \(q\) is the best dominant of (3.2). Furthermore, if \(-1 \leq \text{Re} A_1 \leq 1\), then
\[
\frac{\mathfrak{I}_\eta^{\mu-1} f(z)}{\mathfrak{I}_\eta^{\mu} f(z)} \in \mathcal{P}_k(\varrho) \quad (z \in \mathbb{U}),
\]
where \(\varrho = \frac{1}{z \mathfrak{I}_1 \left[ \frac{1}{B_1}, 1, \frac{1}{B_1} + 1, \frac{1}{B_1} - 1, \frac{1}{B_1} + 2, \frac{1}{B_1} - 2 \right]}\). The result is best possible.

Proof. Set
\[
\frac{\mathfrak{I}_\eta^{\mu-1} f(z)}{\mathfrak{I}_\eta^{\mu} f(z)} = p(z) = \left( \frac{k}{4} + \frac{1}{2} \right) p_1(z) - \left( \frac{k}{4} - \frac{1}{2} \right) p_2(z) \quad (z \in \mathbb{U}),
\]
where \(p\) is analytic in \(\mathbb{U}\) with \(p(0) = 1\). One logarithmic differentiation on both sides of (3.12) and using the identity (1.6), we have
\[
\frac{\mathfrak{I}_\eta^{\mu-2} f(z)}{\mathfrak{I}_\eta^{\mu-1} f(z)} = p(z) + \frac{zp'(z)}{(\eta + 1) p(z)} \quad (z \in \mathbb{U}).
\]
From (3.10), we have
\[
p(z) + \frac{zp'(z)}{(\eta + 1) p(z)} \in \mathcal{P}_k[A_1, B_1].
\]
Proof. Let \( B \) be the best dominant of \((3.13)\) and is given by \((2.2)\), this proves \((3.11)\). Proceeding as in Theorem 1, the remaining part of Theorem 3 follows. \(\Box\)

For \( \eta = 1, A_1 = 1 - 2\varrho, B_1 = -1, 0 \leq \varrho < 1 \) and \( k = 2 \), if \( f \in \mathcal{A} \) satisfies

\[
\frac{I^{\mu-2} f(z)}{I^{\mu-1} f(z)} \in \mathcal{P}(\varrho), \quad 0 \leq \varrho < 1 \quad (z \in \mathbb{U}),
\]

then

\[
\frac{I^{\mu-1} f(z)}{I^{\mu} f(z)} \in \mathcal{P}(\varrho_1), \text{ where } \varrho_1 = \frac{1}{2 F_1(1,4(1-\varrho),3; \frac{1}{2})}.
\]

This improves the result of Owa [18].

**Theorem 4.** Let \( \mu > 1, \eta > -1, \lambda < 1 \) and \(-1 \leq B_i < A_i \leq 1 \) for \( i = 1, 2 \). If

\[
(1 - \lambda) \frac{3^{\mu-1} f_i(z)}{z} + \lambda \frac{3^\mu f_i(z)}{z} \in \mathcal{P}_{k}[A_i,B_i], \quad i = 1, 2, \quad k \geq 2 \quad (z \in \mathbb{U}),
\]

then

\[
(1 - \lambda) \frac{3^{\mu-1} F(z)}{z} + \lambda \frac{3^\mu F(z)}{z} \in \mathcal{P}_k(1 - 2\varrho, -1) \quad (z \in \mathbb{U}),
\]

where \( F = f_1 * f_2 \) \(\) and \( \varrho = 1 - \frac{4(A_1 - B_1)(A_2 - B_2)}{(1-B_1)(1-B_2)} \left(1 - \chi \int_0^1 \frac{\mu^{x-1}}{1+u} \, du \right) \) for \( \chi = \frac{\mu + \eta}{1 - \lambda} \). The result is sharp when \( B_1 = B_2 = -1 \).

**Proof.** Set \( H_i(z) = (1 - \lambda) \frac{3^{\mu-1} f_i(z)}{z} + \lambda \frac{3^\mu f_i(z)}{z} \), \( i = 1, 2, \) \( z \in \mathbb{U} \). From \((3.14)\), we see that \( H_i \in \mathcal{P}_k[A_i,B_i] \subset \mathcal{P}_k \left( \frac{1 - A_i}{1 - B_i} \right) \), \( i = 1, 2, \) \( z \in \mathbb{U} \). Let

\[
H_i(z) = \left( \frac{k}{4} + \frac{1}{2} \right) h_i(z) - \left( \frac{k}{4} - \frac{1}{2} \right) p_i(z), \quad i = 1, 2 \quad (z \in \mathbb{U}),
\]

where \( h_i, \ p_i \in \mathcal{P}(\varrho_i), \) \( i = 1, 2 \). By making use of \((1.7)\) \(\) and \((3.4)\), we obtain

\[
3^\mu f_i(z) = \chi z^{1-\chi} \int_0^z t^x H_i(t) \, dt, \quad i = 1, 2, \chi = \frac{\mu + \eta}{1 - \lambda}, \quad z \in \mathbb{U}.
\]

Using above equation \((3.15)\), a simple computation shows that

\[
3^\mu F(z) = \chi z^{1-\chi} \int_0^z t^x H(t) \, dt, \quad \chi = \frac{\mu + \eta}{1 - \lambda}, \quad z \in \mathbb{U},
\]
where \( H(z) = \chi z^{1-x} \int_0^z t^{x-1}(H_1 \ast H_2)(t)\,dt \). For \( \chi = \frac{\mu+n}{1-A} \), we can also write

\[
H(z) = \frac{k+2}{4} \chi z^{1-x} \int_0^z t^{x-1}(h_1 \ast h_2)(t)\,dt - \frac{k-2}{4} \chi z^{1-x} \int_0^z t^{x-1}(p_1 \ast p_2)(t)\,dt. \tag{3.17}
\]

From Lemma 5, we have \( h_1 \ast h_2 \in \mathcal{P}(\rho_3) \) and \( p_1 \ast p_2 \in \mathcal{P}(\rho_3) \), where \( \rho_3 = 1-2(1-\rho_1)(1-\rho_2) \).

Now using (3.16), (3.17) and Lemma 4, we have

\[
\text{Re} \chi z^{1-x} \int_0^z t^{x-1}(h_1 \ast h_2)(t)\,dt > 1 - \frac{4(A_1 - B_1)(A_2 - B_2)}{(1-B_1)(1-B_2)} \left(1 - \chi \int_0^1 \frac{u^{x-1}}{1+u}\,du\right),
\]

where \( \chi = \frac{\mu+n}{1-A} \). Similarly, for \( \chi = \frac{\mu+n}{1-A} \), we can prove

\[
\text{Re} \chi z^{1-x} \int_0^z t^{x-1}(p_1 \ast p_2)(t)\,dt > 1 - \frac{4(A_1 - B_1)(A_2 - B_2)}{(1-B_1)(1-B_2)} \left(1 - \chi \int_0^1 \frac{u^{x-1}}{1+u}\,du\right).
\]

When \( B_1 = B_2 = -1 \), we consider the functions \( f_i \in \mathcal{A}, i = 1,2 \), which satisfy the condition (3.14) and is given by

\[
\mathcal{D}_\eta^\mu f_i(z) = \chi z^{1-x} \int_0^z t^{x-1} \left\{ \frac{k+2}{4} \frac{1+A_i t}{1-t} - \frac{k-2}{4} \frac{1+A_i t}{1-t} \right\} dt,
\]

where \( \chi = \frac{\mu+n}{1-A} \). It follows from (3.16) that for \( \chi = \frac{\mu+n}{1-A} \), we have

\[
H(z) = \frac{k+2}{4} \chi \int_0^1 u^{x-1} \left(1 - A_{1,2} + \frac{A_{1,2}}{1-uz}\right)\,du - \frac{k-2}{4} \chi \int_0^1 u^{x-1} \left(1 - A_{1,2} + \frac{A_{1,2}}{1-uz}\right)\,du.
\]

where \( A_{1,2} = (1+A_1)(1+A_2) \). Therefore \( H(z) \to 1-(1+A_1)(1+A_2) \left(1 - \chi \int_0^1 \frac{u^{x-1}}{1+u}\,du\right) \) for \( z \to -1^+ \) and \( \chi = \frac{\mu+n}{1-A} \). Thus the proof is completed.

\[\square\]

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