GENERALIZED SEQUENCE SPACE $F(X, r)$

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Abstract. In this paper, we define and study vector valued sequence space $F(X, r)$. Few topological properties, inclusion relation, boundedness properties of subset are studied for this class.

1. Introduction

Rosier [3] has developed the theory of vector valued sequence space relative to scalar valued sequence space by introducing and studying a composite space $\Lambda\{E\}$. Barnes and Roy [1] also studied the boundedness in a topological linear space. Maddox [2] and Simons [4] used the idea of a sequence of a strictly positive numbers $p = (p_k)$ (not necessarily bounded in general) to generalize the classical spaces $c$, $c_0$, $\ell_\infty$ and strongly summable sequence spaces $w_0$, $w$, $w_\infty$. In the present note, we introduce a more generalized space using a sequence $r = (r_k)$ of strictly positive real numbers which includes the corresponding work of Maddox [2], Simons [4] and Rosier [3].

2. Notions/Terminology

Throughout this paper, we consider $X$ as a locally convex Hausdorff space equipped with a topology $T$ generated by the family $P$ of continuous seminorms $p_u$ on $X$ given by

$$p_u(z) = \sup_f \{|f(z)| : f \in u^0, z \in X\}$$

where $u^0$ is the polar of $u \in u(X)$ and $u(X)$ is the fundamental system of absolutely convex closed neighborhoods at origin of $X$. We denote the topological dual of $X$ by $X'$ which is always equipped with strong topology $\beta(X', X)$ generated by the family $p' = (p_{B^0})$ of seminorms $p_{B^0}$ given by

$$p_{B^0}(f) = \sup \{|f(z)| : z \in B, f \in X'\}$$

where $B$ is bounded subset of $X$ and $B^0$ is the polar of $B$. 

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3. The Space $F(X, r)$

Let $F$ be a normal Banach space over the field of complex sequences with monotone norm $\| \cdot \|_F$ and having Schauder basis $(e_k)$ where $e_k = (0, 0, \ldots, 1, 0, \ldots)$ with 1 in the $k$-th place. Further, let $r = (r_k)$ be a sequence of positive real numbers such that $0 < r_k \leq 1$. We define

$$F(X, r) = \{ x = (x_k) : x_k \in X \text{ for each } k \text{ and for every } u \in u(X), (p_u^k(x_k)) \in F \}. \quad (3.1)$$

For $x = (x_k) \in F(X, r)$, we define

$$g_u(x) = \| (p_u^k(x_k)) \|_F. \quad (3.2)$$

One can easily verify that the space $(F(X, r), d_u)$ is a pseudometric space under the pseudometric $d_u$ given by

$$d_u(x, y) = g_u(x - y) \text{ where } x = (x_k), \ y = (y_k) \in F(X, r) \text{ and } u \in u(X). \quad (3.3)$$

Consider the space $(F(X, r), V_{g_u})$ where the topology induced by $V_{g_u}$ is the supremum of the topologies induced by all the paranorms $g_u, u \in u(X)$. This means that

"a net $(x^n)$ converges to $x = (x_k)$ in $V_{g_u}$ if and only if $(x^n)$ converges to $x = (x_k)$ in each $g_u, u \in u(X)$". \quad (3.4)

4. In this section, we study some topological properties and obtain some inclusion relations for the space $F(X, r)$.

**Theorem 4.1.** $F(X, r)$ is paranormed space under the paranorm $g_u$ given by (3.2).

Proof is straightforward, so we omit it.

**Theorem 4.2.** If $X$ is complete, then $(F(X, r), V_{g_u})$ is complete.

The proof is straightforward, so we omit it.

**Theorem 4.3.** Let $0 < r_k \leq s_k \leq 1$ for all $k$ and $x = (x_k) \in F(X, r)$. Let $A = \{ k : p_u(x_k) \geq 1 \}$ and $B = \{ k : p_u(x_k) < 1 \}$. Then

(i) $F(X, r) \subseteq F(X, s)$ if $n'(A) < \infty$,

(ii) $F(X, s) \subseteq F(X, r)$ if $n'(B) < \infty$,

where $n'(A)$ and $n'(B)$ denote the number of indices in $A$ and $B$ respectively and $F(X, s)$ is defined accordingly as in Section 3.

**Proof.** Let $n'(A) < \infty$ and let $x = (x_k) \in F(X, r)$. Define two sequences $(y_k)$ and $(z_k)$ as

$$y_k = \begin{cases} x_k & \text{if } p_u(x_k) \geq 1, \\ \theta & \text{if } p_u(x_k) < 1, \end{cases}, \quad z_k = \begin{cases} \theta & \text{if } p_u(x_k) \geq 1, \\ x_k & \text{if } p_u(x_k) < 1, \end{cases} \quad (4.1)$$
(where $\theta$ is the zero element of $X$).

Clearly from (4.1) it follows that

$$p^s_u(y_k) \geq p^r_u(y_k) \quad \text{and} \quad p^s_u(z_k) \leq p^r_u(z_k) \tag{4.2}$$

for each $k$. But we can find an integer $n_k$ such that

$$p^s_u(y_k) \leq n_k p^r_u(y_k) \leq M p^r_u(y_k). \tag{4.3}$$

where $M = \max n_k (k \in n'(A))$.

Since $F$ is normal space, so (4.2) and (4.3) imply that $x = (x_k) \in F(X, s)$. Hence

$$F(X, r) \subseteq F(X, s) \tag{4.4}$$

If $n'(B) < \infty$, then on the similar lines as used in above. Theorem 4.3 (i) we can prove that

$$F(X, s) \subseteq F(X, r). \tag{4.5}$$

This completes the proof.

**Theorem 4.4.** Let $0 < r_k \leq 1$ for each $k$. If $\sum_{k=1}^{\infty} N^{\xi_k} < \infty$, for some integer $N(> 1)$ (where $\xi_k$ is the conjugate index of $r_k$ i.e. $(1/r_k) + (1/\xi_k) = 1$ and number of indices in $A = n'(A) < \infty$, (see Theorem 4.3), then $F(X, r) = F(X)$, where $F(X)$ is the vector space of all $X$-valued sequences $x = (x_k)$ such that sequence of scalars $(p_u(x_k)) \in F$, for each $u \in u(X)$.

**Proof.** Given $n'(A) < \infty$. So by Theorem 4.3

$$F(X, r) \subseteq F(X). \tag{4.6}$$

Conversely, let $x = (x_k) \in F(X)$ and define two sequences $y = (y_k)$ and $z = (z_k)$ as in Theorem 4.3. Since

$$p_u(y_k) \geq 1 (k \in A) \quad \text{and} \quad p_u(z_k) < 1 (k \in B),$$

we have

$$p^s_u(y_k) \leq p_u(y_k) = p_u(x_k) \quad (k \in A) \tag{4.7}$$

and on the same lines as used by Simons [4, Theorem 3, p.427] we have

$$p^s_u(z_k) \leq p_u(z_k)(1 + N \log N) = p_u(x_k)(1 + N \log N). \tag{4.8}$$

Since $F$ is normal space and

$$p^s_u(x_k) = p^s_u(y_k) + p^s_u(z_k),$$

so we have $x = (x_k) \in F(X, r)$. Therefore

$$F(X) \subseteq F(X, r). \tag{4.9}$$
Hence from (4.6) and (4.9), we have $F(X,r) = F(X)$.

5. This section deals with the results related to the boundedness properties of subset of $F(X,r)$.

Let $R$ be a normal subset of $F$ and $u \in u(X)$. We define

$$[R,u] = \{ x = (x_k) \in F(X,r) : (p^r_u(x_k)) \in R \}.$$  

**Theorem 5.1.** Let $\inf r_k > 0$ and $0 < r_k \leq 1$. Then following statements are equivalent:
(i) subset $[R,u]$ of $F(X,r)$ is metrically bounded;
(ii) $[R,u]$ is bounded.

Using the same procedure as in Theorem 6 of Simons [4], proof follows.

**Remark 5.2.** In Theorem 5.1 the condition $\inf r_k > 0$ is not needed while proving (ii) $\Rightarrow$ (i).

Now we investigate the bounded set in $F(X,r)$ when $\lim r_k = 0$. We define $M_k[R,u] = \sup p^r_u(x_k)\|e_k\|_F$, where sup is taken over $k$-th component of $x = (x_k) \in [R,u]$.

**Theorem 5.3.** Assuming that $\lim r_k = 0$. Then a set $[R,u]$ is bounded in $F(X,r)$ if and only if
(i) $M_k[R,u]$ is bounded for all $k \geq 1$,
(ii) Given any $\varepsilon > 0$, there exists an integer $m$ such that $\| \sum_{k=m}^{\infty} p^r_u(x_k)e_k\|_F < \varepsilon$, for all $x = (x_k) \in [R,u]$.

Proof of the theorem is omitted as it can be proved using the same procedure as adopted by Barnes [1, Theorem 2.1].

**Theorem 5.4.** Assuming that $\lim r_k = 0$. Then, if $[R,u]$ is bounded, then $[R,u]$ is totally bounded.

**Proof.** Since $[R,u]$ is bounded and $r_k \to 0 \ (k \to \infty)$ so for given $\varepsilon(> 0)$ by Theorem 5.3 (ii)

$$\| \sum_{k \geq k_0}^{\infty} p^r_u(x_k)e_k\|_F < \varepsilon/2.$$ 

Let $X^{k_0} = \prod_{i=1}^{k_0} X_i$ = product of $X_i$, where $X_i = X$, $1 \leq i \leq k_0$ and $P_{k_0} : F(X,r) \to x^{k_0}$ such that $P_{k_0}(x) = (x_1, x_2, \ldots, x_{k_0})$, where $x = (x_k) \in F(X,r)$, it is easy to see that seminorm $p_u$ of $X$ induces a pseudometric

$$d_{u,k}(P_{k_0}(x), P_{k_0}(y)) = \sum_{k=1}^{k_0} P_u(x_k - y_k)$$ on $X^{k_0}$.
where \( x = (x_k) \) and \( y = (y_k) \) \( \in F(X, r) \) which is equivalent to a pseudometric \( d'_{u, k_0} \) where
\[
d'_{u, k_0}(P_{k_0}(x), P_{k_0}(y)) = \sum_{k=1}^{k_0} p_u^r(x_k - y_k) \text{ on } X^{k_0}.
\]
Since projection map \( p_{k_0} \) is continuous, so \( p_{k_0}[R, u] \) is bounded. So \( P_{k_0}[R, u] \) is totally bounded. Hence
\[
p_{k_0}[R, u] \subset \bigcup_{i=1}^{m} S(z^i, \varepsilon/ (2m_0))
\]
where \( m_0 > \max_{1 \leq k \leq k_0} \|e_k\|_F \), and \( S(z^i, \varepsilon/(2m_0)) = \{ P_k x_0 : d'_{u, k_0}(z^i, P_k x) < \varepsilon/(2m_0) \} \).
\[
z^i = (z^i_1, z^i_2, \ldots, z^i_{k_0}), 1 \leq i \leq m.
\]
Now let
\[
D_{k_0} = \{ b : b = (z^i_1, z^i_2, \ldots, z^i_{k_0}, \theta, \theta, \ldots), 1 \leq i \leq m \}
\]
where \( \theta \) is zero element of \( X \). Clearly \( D_{k_0} \) is a finite set and \( D_{k_0} \subset [R, u] \). If \( x = (x_k) \in [R, u] \), then \( P_{k_0} x \in P_{k_0}[R, u] \). But \( p_{k_0}[R, u] \) is totally bounded as shown above, so there exists
\[
b = (z^i_1, z^i_2, \ldots, z^i_{k_0}, 0, 0, \ldots) \in D_{k_0}
\]
for some \( i \) such that
\[
d'_{u, k_0}(p_{k_0} x, b) = \sum_{k=1}^{k_0} p_u^r(x_k - z^i_k) < \varepsilon/(2m_0).
\]
Now consider
\[
d_u(x, b) \leq \sum_{k=1}^{k_0} p_u^r(x_k - z^i_k)\|e_k\|_F + \sum_{k \geq k_0} \infty p_u^r(x_k)\|e_k\|_F
\]
< \( m_0(\varepsilon/2m_0) + \varepsilon/2 = \varepsilon \).
Since \( x \in [R, u] \) is arbitrary and \( D_{k_0} \) is finite, it follows that \([R, u] \) is totally bounded.

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References


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