RINGS WITH GENERALIZED COMMUTATORS IN THE NUCLEI

Dedicated to my father on his 85th birthday

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Abstract. Let \( R \) be a prime weakly Novikov ring and \( T_k = \left[[\ldots[[R, R], R], \ldots, R], R\right] \) where \( k \) is a positive integer. We prove that if \( T_k \subseteq N_l \cap N_r \) or \( T_k \subseteq N_m \cap N_r \), then \( R \) is associative or \( T_k = 0 \). Moreover, if \( T_k \) is contained in two of the three nuclei, and \( k = 2 \) or \( k = 3 \) then the same conclusions hold. We also consider such rings with derivations. Some similar results of weakly M-rings are obtained.

1. Introduction

Let \( R \) be a nonassociative ring. We shall denote the associator and commutator by \( (x, y, z) = (xy)z - x(yz) \) and \( [x, y] = xy - yx \) for all \( x, y, z \) in \( R \) respectively. In any ring \( R \), one has the following nuclei:

\[
N_l = \{ n \in R | (n, R, R) = 0 \} \text{ - left nucleus,} \\
N_m = \{ n \in R | (R, n, R) = 0 \} \text{ - middle nucleus,} \\
N_r = \{ n \in R | (R, R, n) = 0 \} \text{ - right nucleus.}
\]

A ring \( R \) is called simple if \( R \) is the only nonzero ideal of \( R \). Thus, \( R^2 = R \). A ring \( R \) is called semiprime if the only ideal of \( R \) which squares to zero is the zero ideal. A ring \( R \) is called prime if the product of any two nonzero ideals of \( R \) is nonzero. Note that each associator and commutator are linear in each argument. Thus \( N_l, N_m \) and \( N_r \) are additive subgroups of \( (R, +) \). If \( S \) is a nonempty subset of a ring \( R \), then the ideal of \( R \) generated by \( S \) is \( \langle S \rangle \). A ring \( R \) is called weakly Novikov [4] if \( R \) satisfies the following identity.

\[
(w, x, yz) = y(w, x, z) \quad \text{for all } w, x, y, z \text{ in } R. \quad (1)
\]

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An additive mapping $d$ on a ring $R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y$ in $R$. For any ring $R$, let $T_k = [[[[R, R], R], \ldots], R, R]$ where $k$ is a positive integer. Note that $T_2 = [R, R]$ and $T_3 = [[R, R], R]$. We also note that $[R, T_k] = [T_k, R] \subseteq T_k$, where $k$ is a positive integer. Obviously, we have the following identities.

$$T_k + T_k R = T_k + RT_k$$ for all positive integers $k$. \hfill (2)

$$d(R) + d(R)R = d(R) + Rd(R).$$ \hfill (3)

$$d((x, y, z)) = (d(x), y, z) + (x, d(y), z) + (x, y, d(z))$$ for all $x, y, z$ in $R$. \hfill (4)

$$S(x, y, z) = (x, y, z) + (y, z, x) + (z, x, y) = [xy, z] + [yz, x] + [zx, y]$$ for all $x, y, z$ in $R$. \hfill (5)

We shall use the Teichmüller identity

$$(wx, y, z) - (w, xy, z) + (w, x, yz) = w(x, y, z) + (w, x, y)z$$ for all $w, x, y, z$ in $R$, \hfill (6)

which is valid in every ring.

As a consequence of (6), we have that $N_l, N_m$ and $N_r$ are associative subrings of $R$. Suppose that $n \in N_l$. Then with $w = n$ in (6) we obtain

$$(nx, y, z) = n(x, y, z)$$ for all $x, y, z$ in $R$ and $n$ in $N_l$. \hfill (7)

Suppose that $m \in N_r$. Then with $z = m$ in (6) we get

$$(w, x, zm) = (w, x, y)m$$ for all $w, x, y$ in $R$ and $m$ in $N_r$. \hfill (8)

Suppose that $j \in N_l \cap N_m$. Then with $x = j$ in (6) we have

$$(wj, y, z) = (w, jy, z)$$ for all $w, y, z$ in $R$ and $j$ in $N_l \cap N_m$. \hfill (9)

**Definition 1.** Let $A$ be the associator ideal of a ring $R$.

Ordinary by (6) $A$ can be characterized as all finite sums of associators and left multiples of associators. In view of (1) it suffices to take all finite sums of associators if $R$ is a weakly Novikov ring. Hence, in this case $A = (R, R, R)$. In the paper, we consider rings with generalized commutators in the nuclei. There had been other results concerning rings in which $[R, R] \subseteq N_l$. For example Thedy [5], Kleinfeld [1], Kleinfeld and Kleinfeld [2] as well as Kleinfeld and Smith [3].

**Definition 2.** For any ring $R$, let $V_k = T_k + RT_k$ for all positive integers $k$. 

2. Results of Weakly Novikov Rings

**Lemma 1.** If $R$ is a weakly Novikov ring, then $RN_r \subseteq N_r$ and $A \cdot N_r = (R, R, R) \cdot N_r = 0$.

**Proof.** Let $z \in N_r$ and $w, x, y \in R$. Then by (8) and (1), we have $(w, x, y)z = (w, x, y) = y(w, x, z) = 0$. Thus, we get $A \cdot N_r = (R, R, R) \cdot N_r = 0$ and $RN_r \subseteq N_r$, as desired.

By (2) and the result of [8], we have the

**Lemma 2.** If $R$ is a ring such that $T_k$ is contained in two of the three nuclei, then $V_k$ is an ideal of $R$ for every positive integer $k$.

In the sequel, for the convenience we denote $T_k$ and $V_k$ by $T$ and $V$ respectively.

**Theorem 1.** If $R$ is a prime weakly Novikov ring such that $T \subseteq N_1 \cap N_r$ or $T \subseteq N_m \cap N_r$, then $R$ is associative or $T = 0$.

**Proof.** Using $T \subseteq N_r$ and Lemma 1, we get

$$A \cdot V = A \cdot (T + RT) = 0.$$  

(10)

By Lemma 2 and the primeness of $R$, (10) implies $A = 0$ or $V = 0$. Thus, $R$ is associative or $T = 0$.

**Lemma 3.** If $R$ is a weakly Novikov ring such that $T \subseteq N_1 \cap N_m$, then

$$(R, R, T)R = 0$$  

(11)

**Proof.** Note that $[R, T] = [T, R] \subseteq T$. Using this, the hypotheses, (6),(1),(9) and (7), for all $y \in T$, and $w, x, z \in R$ we have $(w, x, y)z = w(x, y, z) + (w, x, y)z = (wx, y, z) - (wx, y, z) + (w, x, yz) = -(w, x, y)z - (w, yx, z) + y(w, x, z) = -(yw, x, z) + y(w, x, z) = 0$. Hence, we get $(R, R, T)R = 0$, as desired.

**Theorem 2.** Let $R$ be a prime weakly Novikov ring such that $T \subseteq N_1 \cap N_m$. If $S(x, y, z) \in N_m$ for all $x, y, z$ in $R$, or $[T, (R, R, R)] = 0$, then $R$ is associative or $T = 0$.

**Proof.** Assume that $S(x, y, z) \in N_m$ for all $x, y, z$ in $R$. Using this, (5) and the hypotheses, for all $x \in T$ and $y, z \in R$ we get $(y, z, x) = (x, y, z) + (y, z, x) + (z, x, y) = S(x, y, z) \in N_m$. Thus, $(R, R, T) \subseteq N_m$. Applying this, (1) and (11), we have $(R, R, RT)R = R(R, R, T) \cdot R = R \cdot (R, R, T)R = 0$. Combining this with (11) results in

$$(R, R, V)R = 0.$$  

(12)

Assume that $[T, (R, R, R)] = 0$. Using this, (1), (11) and (6), and noting that $[T, R] \subseteq T$, for all $w, x, y, t \in R$, and $z \in T$ we have $(w, x, y)t = z(w, x, y) \cdot t = (w, x, y)zt =
(w, x, [z, y])t + (w, x, yz)t = w(x, y, z) + (w, x, y)t = w(x, y, z) + (w, x, y)z - (wx, y, z)t = w(x, y, z) + (w, x, y)z - t. Combining this with (11), we also obtain (12).

Using (1) and (12), we see that < (R, R, T) >= (R, R, V). By the semiprimeness of R, (12), implies (R, R, V) = 0. Hence, V ⊆ N_r.

Consequently, T ⊆ N. By Theorem 1, R is associative or T = 0.

In [3], Kleinfeld and Smith had proved that if R is a prime left alternative ring with [R, R] ⊆ N_l and characteristic # 2, 3 then R is associative. A linearization of the left alternative identity shows that N_l = N_m. We have the similar result for the weakly Novikov ring case.

**Theorem 3.** If R is a prime weakly Novikov ring such that [R, R] is contained in two of the three nuclei, then R is associative or commutative.

In the latter case, N_r = 0 or R is associative.

**Proof.** In view of Theorem 1, we may assume [R, R] ⊆ N_l ∩ N_m. Let B = [R, R] + R[R, R]. By Lemma 2, B is an ideal of R. Using Lemma 3, we get

\[(R, R, [R, R])R = 0.\]  

(13)

Applying (5) and [R, R] ⊆ N_l ∩ N_m, for all x, y, z ∈ R we have S(x, y, z) = (x, y, z) + (y, z, x) + (z, x, y) ∈ N_l ∩ N_m. Let x ∈ [R, R]. Then we get (y, z, x) ∈ N_l ∩ N_m. Thus, we obtain (R, R, [R, R]) ⊆ N_l ∩ N_m. Using this and (13), we have R(R, [R, R])R = R · (R, R, [R, R])R = 0.

Hence, applying this, (1) and (13), and noting that B is an ideal of R, we obtain that (R, R, B) · R = 0 and < (R, R, [R, R]) >= (R, R, B). Thus, by the semiprimeness of R we get < (R, R, B) > = 0 and so [R, R] ⊆ N_r. By Theorem 1, R is associative or commutative.

Assume that R is commutative. Thus we have N_r R = RN_r ⊆ N_r and A · N_r = 0 by Lemma 1. Hence N_r is an ideal of R. By the primeness of R, A · N_r = 0 implies A = 0 or N_r = 0.

By Theorem 3, we obtain the

**Corollary 1.** If R is a prime weakly Novikov ring such that [R, R] is contained in two of the three nuclei with N_r ≠ 0 or [R, R] ≠ 0, then R is associative, that is N_r = R.

In the sequel, for the convenience we denote V_5 by D.

**Lemma 4.** If R is a weakly Novikov ring such that [R, R], R] ⊆ N_l ∩ N_m then < (R, R, D) > · (R, R, R) = 0, where < (R, R, D) > = (R, R, D) + (R, R, D)R + R · (R, R, D)R.

**Proof.** Let D = [[R, R], R] + R[[R, R], R] and [[R, R], R] ⊆ N_l ∩ N_m. By Lemma 3, we obtain

\[(R, R, [[R, R], R])R = 0.\]  

(14)

Thus (14) implies

\[(R, R, [[R, R], R]) ⊆ N_l.\]  

(15)
Assume that \( y \in [[R, R], R] \) and \( w, x, z, u, v, t \in R \). Using (14), the hypotheses and (5) we have \( z(w, x, y) = [z, (w, x, y)] = [z, S(w, x, y)] \in [[R, R], R] \subseteq N_r \cap N_m \) and so by (1) twice we get (\( w, x, [z, y] + y(w, x, z) = (w, x, [z, y]) + (w, x, y) = (w, x, z) = z(w, x, y) \in N_r \cap N_m \). Applying these, (1) and (15) we obtain the following two inclusions.

\[
(R, R, R[[R, R], R]) = R(R, R, [[R, R], R]) \subseteq N_r \cap N_m. \tag{16}
\]

\[
[[R, R], R]A = [[R, R], R](R, R, R) \subseteq N_r. \tag{17}
\]

Then (17) implies

\[
[[R, R], R]A \cdot R = [[R, R], R] \cdot AR \subseteq [[R, R], R]A \subseteq N_r. \tag{18}
\]

Combined (15) with (16) results in

\[
(R, R, D) \subseteq N_r. \tag{19}
\]

Using (1), (17), (7) and (18), we have (\( w, x, y z)(u, v, t) = y(w, x, z) \cdot (u, v, t) = (y(w, x, z) \cdot u, v, t) = 0 \). Hence applying this, (2) and (14) we obtain

\[
(R, R, D)A = (R, R, D)(R, R, R) = 0. \tag{20}
\]

Then by (20), (19), (7) and (1) we get \( 0 = (R, R, D)(R, R, R) = (R, R, D)R, R, R) \) and \( 0 = (R, R, D)(R, R, R) = (R, R, D)R, R, R) \). Thus, by these, (14) and (1) we have

\[
R(R, R, [[R, R], R]) \cdot R = (R, R, D)R \subseteq N_r \cap N_m. \tag{21}
\]

Let \( x \in R(R, R, [[R, R], R]) \) and \( w, y, z \in R \). Then by (16) and (1) we get \( x \in N_r \cap N_m \) and \( wx \in (R, R, D) \). Hence by (9) and (19), we obtain (\( w, x, y, z = (w, x, y, z) = 0 \). Combined this, (1), (14) and (21) results in

\[
(R, R, D)R \subseteq N. \tag{22}
\]

Using (1), (19) and (22) we see that \( <(R, R, D) >= (R, R, D) + (R, R, D)R + R \cdot (R, R, D)R \).

Combined (19) with (20) results in

\[
(R, R, D)R \cdot A = (R, R, D) \cdot RA \subseteq (R, R, D)A = 0. \tag{23}
\]

Applying (22) and (23), we get \( \{R \cdot (R, R, D)R\} \cdot A = R \cdot \{R, R, D)R \cdot A \} = 0 \). Thus using this, (20) and (23), we have \( <(R, R, D) > \cdot A = 0 \), as desired.

\textbf{Theorem 4.} If \( R \) is a prime weakly Novikov ring such that \( [[R, R], R] \) is contained in two of the three nuclei, then \( R \) is associative of \( [[R, R], R] = 0 \).

\textbf{Proof.} In view of Theorem 1, we may assume \( [[R, R], R] \subseteq N_r \cap N_m \).

Let \( D = [[R, R], R] + R[[R, R], R] \). Then by Lemma 4 we obtain \( <(R, R, D) > \cdot A = 0 \), where \( <(R, R, D) >= (R, R, D) + (R, R, D)R + R \cdot (R, R, D)R \). By the semiprimeness
of $R$, this implies $< (R,R,D) > = 0$. Hence $[[R,R],R] \subseteq N_r$. Thus by Theorem 1, $R$ is associative or $[[R,R],R] = 0$.

By Theorem 4, we have the

**Corollary 2.** If $R$ is a prime weakly Novikov ring such that $[[R,R],R]$ is contained in two of the three nuclei with $[[R,R],R] \neq 0$, then $R$ is associative.

The following is very easy.

**Remark 1.** If $R$ is a simple weakly Novikov ring such that $T \subseteq N_r$, then $R$ is associative or $T = 0$.

**Proof.** Assume that $A = (R,R,R)$. By Lemma 1, we have $RT = AT = 0$. Thus, we get $TR = [T,R] \subseteq T$. Hence, we see that $< T > = T$. By the simplicity of $R$, we obtain $T = 0$, as desired.

**Remark 2.** If $R$ is a semiprime weakly Novikov ring such that $(R,R,R) \subseteq N_l$ or $(R,R,R) \subseteq N_r$ then $R$ is associative.

**Proof.** We see that the associator ideal $A$ of $R$ is all finite sums of associators. Assume that $(R,R,R) \subseteq N_l$. Then by this and (7), for all $w \in (R,R,R)$ and $x,y,z \in R$ we get $w(x,y,z) = (wx,y,z) \in (A,R,R) = 0$.

Thus, we have $(R,R,R)(R,R,R) = 0$ and so $A^2 = 0$.

Assume that $(R,R,R) \subseteq N_r$. Then by Lemma 1, we obtain


In either case, we have $A^2 = 0$. By the semiprimeness of $R$, this implies $A = 0$. Thus, $R$ is associative.

In view of Theorem 1 of [6], we have the

**Remark 3.** If $R$ is a semiprime weakly Novikov ring with a derivation $d$ such that $d(R) \subseteq N_r$, then $d(A) = 0$. Moreover, if $R$ is prime such that $d(R) \subseteq N_l \cap N_r$ or $d(R) \subseteq N_m \cap N_r$, then $R$ is associative or $d = 0$.

**Proof.** By the definition of $d$, $d(R) \subseteq N_r$, (8), (1) and $A = (R,R,R)$, for all $w,x,y,z,t \in R$ we get $(w,x,y)z(\delta(z)) = (w,x,yd(z)) = y(w,x,dz) = 0$, $(w,x,y)zd(\delta(z)) = 0$ and so $d(y)(w,x,z) = (w,x,dz) = (w,x,d\delta(z)) = (w,x,\delta y(\delta)) = 0$.

Let $E = d(R) + Rd(R)$. Then the above three equalities imply

$$A \cdot E = 0 \text{ and } d(R) \cdot A = 0.$$

Using (24), we have that $d(A)R \subseteq d(A)$ and $Rd(A) \subseteq d(A)$. Hence $< d(A) > = d(A)$. Applying (4), we see that $d(A) \subseteq A$. Thus by (24), $d(A) \cdot A = 0$ and so by the semiprimeness of $R$, this implies $d(A) = 0$. 

Using
Assume that $R$ is prime such that $d(R) \subseteq N_l \cap N_r$ or $d(R) \subseteq N_m \cap N_r$. Then by (3) and the result of [8], $E$ is an ideal of $R$. By the primeness of $R$, (24) implies $A = 0$ or $E = 0$. Hence, $R$ is associative or $d = 0$.

In Remark 3, if $R$ is a semiprime weakly Novikov ring with a derivation $d$ such that $d(R) \subseteq N_r$, then $d(A) = 0$. Hence, the results of [?] can be applied.

3. Results of weakly M-rings

In the sequel, we denote $T_k$ and $V_k$ by $T$ and $V$ respectively.

A ring $R$ is called a weakly M-ring if $R$ satisfies the following identity.

\[(w,xy,z) = x(w,y,z) \text{ for all } w,x,y,z \text{ in } R. \quad (25)\]

Note that if $R$ is a weakly M-ring then by (6) and (25) we obtain $A = (R,R,R)$.

**Theorem 5.** If $R$ is a prime weakly M-ring such that $T \subseteq N_l \cap N_m$ or $T \subseteq N_m \cap N_r$, then $R$ is associative or $T = 0$.

**Proof.** Note that $[T,R] \subseteq T$. Using this, $T \subseteq N_m$ and (25), for all $x \in T$ and $w,y,z,t \in R$ we have $x(w,y,z) = x(w,y,z) - y(w,x,z) = (w,xy,z) - (w,tx,\cdot z) = (w,[x,y],z) = 0$, and so $tx \cdot (w,y,z) = t \cdot x(w,y,z) = 0$. These two identities yield

\[V \cdot A = 0 \quad (26)\]

Since $V$ is an ideal of $R$, by the primeness of $R$, (26) implies $A = 0$ or $V = 0$. Hence, $R$ is associative or $T = 0$.

The following three remarks are similar to those in section 2. The proofs are also similar, so we omit it.

**Remark 4.** If $R$ is a simple weakly M-ring such that $T \subseteq N_m$, then $R$ is associative or $T = 0$.

**Remark 5.** If $R$ is a semiprime weakly M-ring such that $(R,R,R) \subseteq N_m$, then $R$ is associative.

**Remark 6.** If $R$ is a prime weakly M-ring with a derivation $d$ such that $d(R) \subseteq N_l \cap N_m$ or $d(R) \subseteq N_m \cap N_r$, then $R$ is associative or $d = 0$.

Finally, we ask if the theorem or the remark is valid for the other cases.

**References**


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