ON COEFFICIENTS OF $\alpha$-SHAPIRO SHIELDS FUNCTIONS

T. Y. PETER CHERN AND HASI WULAN

Abstract. We investigate the Taylor coefficients of $\alpha$-Shapiro Shields functions and some applications to $\alpha$-Bloch functions and Bergman functions are given.

1. Introduction

Let $\mathcal{D} = \{ z : |z| < 1 \}$ be the unit disk in the complex plane $\mathbb{C}$. A function $f$ analytic in $\mathcal{D}$ is called to be $\alpha$-Shapiro Shields ($\alpha > 0$) [4, p.224] if $f$ satisfies

$$\sup \{(1 - |z|)^\alpha |f(z)| : z \in \mathcal{D}\} < \infty$$

(1.1)

The set of all $\alpha$-Shapiro Shields functions is denoted by $S^{-\alpha}$. Let $S^{-\alpha}_0$ be the set of all functions $f$ analytic in $\mathcal{D}$ (little $\alpha$-Shapiro Shields functions) such that

$$(1 - |z|)^\alpha |f(z)| \to 0, \quad |z| \to 1^{-}.$$  

A function $f$ analytic in $\mathcal{D}$ is called an $\alpha$-Bloch function ($\alpha > 0$) [5] if $f' \in S^{-\alpha}$, denoted by $f \in B^\alpha$. A function $f$ analytic in $\mathcal{D}$ is called a little $\alpha$-Bloch function ($\alpha > 0$) if $f' \in S^{-\alpha}_0$, denoted by $f \in B^\alpha_0$.

If a function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ belongs to a certain $S^{-\alpha}$ space, what can be said about its Taylor coefficients $a_n$? Ideally, one would like to find a condition on the $a_n$ which is both sufficient and necessary for $f$ to be in $S^{-\alpha}$. But the general situation is much more complicated, and no complete answer is available. In this paper, under an argument condition (see expression (2.1)) on coefficients, a sufficient and necessary condition on coefficients for functions to be $\alpha$-Shapiro Shields is proved (see Theorem 2.1). An example shows that the argument condition cannot be substantially relaxed. In Section 3 we obtain two sufficient conditions and one necessary condition on coefficients of $\alpha$-Shapiro Shields functions, and show by examples that any one of these conditions cannot be both sufficient and necessary. However, if we consider the special case, Hadamard gap series, a sufficient and necessary condition on coefficients for functions to be $\alpha$-Shapiro Shields is easily obtained (see Theorem 3.1). In the last section we mention some applications to $\alpha$-Bloch functions and Bergman functions.

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2. A Sufficient and Necessary Condition on Coefficients of $\alpha$-Shapiro Shields Functions

**Theorem 2.1.** Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \). If there is a \( \theta \in [0, 2\pi) \) such that the argument condition:

\[
\theta \leq \arg a_n \leq \theta + \frac{\pi}{2}
\]

holds for each \( n \in \mathbb{N} \), then for \( \alpha > 0 \) the following are equivalent:

1. \( f \in S^{-\alpha} \).
2. For each nonnegative integer \( J \) we have

\[
\sum_{k=J}^{n} k^J |a_k| = O(n^{J+\alpha}).
\]

3. There exists a nonnegative integer \( J \) such that (2.2) holds.

**Proof.** We first prove that (1) implies (2). There is no loss of generality in assuming that \( \theta = 0 \). Assume \( f \in S^{-\alpha} \), we have

\[
\sum_{k=0}^{\infty} a_k z^k = O((1 - |z|)^{-\alpha})
\]

and

\[
\left| \sum_{k=0}^{\infty} \bar{a}_k z^k \right| = |f(\bar{z})| = O((1 - |z|)^{-\alpha}).
\]

Therefore,

\[
\sum_{k=0}^{\infty} \text{Re}(a_k) z^k = O((1 - |z|)^{-\alpha}),
\]

where \( 0 \leq \text{Re}(a_k) \) since \( 0 \leq \arg a_n \leq \frac{\pi}{2}, k = 0, 1, \ldots \). Similarly

\[
\sum_{k=0}^{\infty} \text{Im}(a_k) z^k = O((1 - |z|)^{-\alpha}),
\]

where \( 0 \leq \text{Im}(a_k), k = 0, 1, \ldots \). Thus

\[
\sum_{k=0}^{\infty} |a_k| z^k = O((1 - |z|)^{-\alpha}).
\]

For each positive integer \( n \), we choose \( z = 1 - 1/n \). Then

\[
\sum_{k=0}^{n} |a_k|(1 - 1/n)^k = O(n^\alpha).
\]
For $0 \leq k \leq n$, we have
\[(1 - 1/n)^k \geq (1 - 1/n)^n \to 1/e, \quad n \to \infty.
\]
Therefore,
\[\sum_{k=0}^{n} |a_k| = O(n^\alpha), \quad n \to \infty. \tag{2.3}\]

For each nonnegative integer $J$, and $n \geq J$, it follows from (2.3) that
\[\sum_{k=J}^{n} k^J |a_k| \leq n^J \sum_{k=J}^{n} |a_k| = O(n^{\alpha + J}).\]

Hence (2) holds.

(2) implies (3) is obvious.

Finally we show that (3) implies (1). We may suppose that (2.2) holds for a fixed nonnegative integer $J$. Then there exists a constant $C$ such that
\[\sum_{k=J}^{n} k^J |a_k| \leq C n^{\alpha + J}, \quad n \in \mathbb{N}. \tag{2.4}\]

Here and elsewhere constants are denoted by $C$ which is positive and may indicate different from one occurrence to the next. Since
\[\frac{1}{(1 - |z|^2)^\lambda} = \sum_{n=0}^{\infty} \frac{\Gamma(n + \lambda)}{n! \Gamma(\lambda)} |z|^n, \quad \lambda > 0,
\]
and $\frac{\Gamma(n + \lambda)}{n!} \approx n^{\lambda - 1}$ as $n \to \infty$ by Stirling’s formula, it follows from (2.4) that
\[|f^{(J)}(z)| \leq C \sum_{n=J}^{\infty} n^J |a_n| |z|^{n-J} \sum_{n=J}^{\infty} |z|^n (1 - |z|) \]
\[= C (1 - |z|) \sum_{n=J}^{\infty} \left( \sum_{k=J}^{n} k^J |a_k| \right) |z|^{n-J} \]
\[\leq C (1 - |z|) \sum_{n=J}^{\infty} n^{\alpha + J} |z|^{n-J} \]
\[\leq C (1 - |z|)^{-(\alpha + J)}. \tag{2.5}\]

By successive integration this shows that $f$ is $\alpha$-Shapiro Shields. This completes the proof of Theorem 2.1.

It follows from Theorem 2.1 that

**Corollary 2.2.** Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with $a_n \geq 0$ and $\alpha > 0$. Then the following statements are equivalent:
(1) \( f \in S^{-\alpha} \).

(2) For each nonnegative integer \( J \) we have

\[
\sum_{k=J}^{n} k^J a_k = O(n^{J+\alpha}).
\] (2.6)

(3) There exists a nonnegative integer \( J \) such that (2.6) holds.

**Corollary 2.3.** Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \). If there exists a \( \theta \in [0, 2\pi) \) such that the argument condition:

\[
\theta \leq \arg a_n \leq \theta + \frac{\pi}{2}
\]

holds for each \( n \in \mathbb{N} \), then for \( \alpha > 0 \) the following are equivalent:

(1) \( f \in B^\alpha \).

(2) For each nonnegative integer \( J \) we have

\[
\sum_{k=J}^{n} k^{J+1} |a_k| = O(n^{J+\alpha}).
\] (2.7)

(3) There exists a nonnegative integer \( J \) such that (2.7) holds.

The following example shows that the argument condition (2.1) can not be substantially relaxed in Theorem 2.1. Let us first state an earlier result which is due to Hardy; see [2].

**Lemma.** Let \( f(z) = \sum_{n=0}^{\infty} n^{-b} e^{i b z} z^n, \, 0 < a < 1 \). If \( 1 - b - a/2 > 0 \), then \( f \) is unbounded and

\[
|f(z)| = O((1 - |z|)^{(1-b-a/2)}).
\]

**Example 1.** For \( \alpha > 0 \) there exists an \( \alpha \)-Shapiro Shields function \( f \) which does not satisfy the condition (2.2) for any given nonnegative integer \( J \).

**Proof.** For \( \alpha > 0 \), we choose \( f(z) = \sum_{n=0}^{\infty} n^{-b} e^{i \pi n} z^n \), where \( 0 < a < 1 \). By Lemma, \( |f(z)| = O((1 - |z|)^{-a}) \), so \( f \in S^{-\alpha} \). However, for given nonnegative integer \( J \), we have

\[
\frac{1}{n^{\alpha+J}} \sum_{k=J}^{n} k^J |a_k| = \frac{1}{n^{\alpha+J}} \sum_{k=J}^{n} k^{J+\alpha-1+\frac{\pi}{2}}
\]

\[
= \frac{1}{n^{\alpha+J}} \left( \sum_{k=0}^{n} k^{J+\alpha-1+\frac{\pi}{2}} - \sum_{k=0}^{J-1} k^{J+\alpha-1+\frac{\pi}{2}} \right)
\]

\[
\geq \frac{1}{n^{\alpha+J}} \left( C_n^{J+\alpha+\frac{\pi}{2}} - \sum_{k=0}^{J-1} k^{J+\alpha-1+\frac{\pi}{2}} \right) \to \infty, \, n \to \infty.
\]
This example also indicates that the argument condition (2.1) cannot be substantially relaxed in Theorem 2.1.

3. Two Sufficient Conditions and One Necessary Condition on Coefficients of $\alpha$-Shapiro Shields Functions

**Theorem 3.1.** Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be analytic in $D$ and $\alpha > 0$. If the coefficient condition (2.2) is satisfied for some nonnegative integer $J$, then $f \in S^{-\alpha}$.

The proof of Theorem 3.1 is given implicitly in the proof of Theorem 2.1 in which we proved (3) $\implies$ (1) without the argument condition; hence the coefficient condition (2.2) is sufficient for a given analytic function $f$ to be $\alpha$-Shapiro Shields. However it is not necessary; see the example 1 above.

**Theorem 3.2.** Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be analytic in $D$. If $\alpha > 0$ and
\[
\sum_{k=1}^{n} k^{q-1} |a_k|^q = O(n^{\alpha q})
\]
for some $q$, $1 \leq q \leq \infty$, then $f \in S^{-\alpha}$.

**Proof.** By Theorem 2.1 and Theorem 3.1 we only need to consider the case $1 < q < \infty$. Applying Theorem 3.1 it suffices to show that $\sum_{k=1}^{n} k |a_k| = O(n^{\alpha+1})$. By Hölder inequality, for $1/p + 1/q = 1$, we have
\[
\sum_{k=1}^{n} k |a_k| \leq \left( \sum_{k=1}^{n} k^{p-1} \right)^{1/p} \left( \sum_{k=1}^{n} k^{q-1} |a_k|^q \right)^{1/q} = O(n^{1+\alpha}).
\]

**Example 2.** For $\alpha > 0$ there exists a function $f \in S^{-\alpha}$, but $f$ does not satisfy (3.1) for any $q \geq 1$.

**Proof.** We consider the function $f(z)$ in Example 1 again. So $f \in S^{-\alpha}$. However, for any $q \geq 1$, we have
\[
\sum_{k=1}^{n} k^{q-1} |a_k|^q = \sum_{k=1}^{n} k^{\alpha q-1+\frac{q}{2}} \approx n^{\alpha q+\frac{q}{2}} \neq O(n^{\alpha q}).
\]

**Theorem 3.3.** If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is a function in $S^{-\alpha}$, then we have
\[
a_n = O(n^{\alpha}), \quad n \to \infty.
\]
Proof. From Cauchy’s formula

\[ |a_n| = \left| \frac{1}{2\pi i} \int_{|z|=r} \frac{f(z)}{z^n} \, dz \right| \]
\[ \leq C \left( \frac{r^{1-n}}{(1-r)\alpha} \right)^{n-1} \]
\[ = O\left( \frac{r^{1-n}}{(1-r)\alpha} \right), \quad r \to 1^- . \]

The minimum value of the last term occurs at \( r_0 = 1 - \frac{\alpha}{n+1} \). Evaluating for this \( r_0 \), we obtain

\[ \frac{r_0^{1-n}}{(1-r_0)\alpha} = (1 - \frac{\alpha}{n+1})^{1-n}(\frac{n+1-\alpha}{\alpha})^n = O(n^n) \]

as \( n \to \infty \), and the result follows.

Example 3. There is a function \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) with (3.2), but \( f \notin S^{-\alpha} \).

Proof. It is easy to check that the function \( f(z) = \sum_{n=0}^{\infty} n^\alpha z^n \) satisfies (3.2), but

\[ \sum_{k=0}^{n} ka_k = \sum_{k=0}^{n} k^{\alpha+1} \approx n^{\alpha+2} + O(1) . \]

It shows by Corollary 2.2 that \( f \) is not \( \alpha \)-Shapiro Shields.

Although (3.2) is a necessary condition not a sufficient condition for \( f \) to be \( \alpha \)-Shapiro Shields, in case \( f \) is a Hadamard gap series we have the following result.

Theorem 3.4. Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) be analytic in \( D \) and \( \alpha > 0 \). If \( f \) has Hadamard gaps, that is

\[ n_{k+1}/n_k \geq \lambda > 1, \quad k = 0, 1, \ldots , \]

then \( f \in S^{-\alpha} \) if and only if \( f \) satisfies

\[ a_n = O(n_k^\alpha), \quad n \to \infty \] \quad (3.3)

Proof. It suffices to prove that if \( f \) satisfies (3.3), then \( f \in S^{-\alpha} \). For each large positive integer \( m \), put \( i_m = \max\{ j : n_j \leq m \} \). Then

\[ \frac{1}{m^\alpha} \sum_{n_j \leq m} |a_n| \leq \frac{C}{m^\alpha} \sum_{j=0}^{i_m} n_j^\alpha \]
\[ \leq \frac{C}{m^\alpha} \sum_{j=0}^{i_m} \lambda^{(j-i_m)} n_j^\alpha \]
\[ = O\left( \sum_{j=0}^{i_m} \lambda^{(j-i_m)} \right) = O(1) . \]
Applying Theorem 2.1, \( f \in S^{-\alpha} \).

4. Notes

1. If there is a result on the coefficients for \( \alpha \)-Shapiro Shields functions in terms of its Taylor coefficients \( a_n \) with magnitude big oh, then there is a similar result for little \( \alpha \)-Shapiro Shields functions in terms of \( a_n \) with magnitude little oh and vice versa. In this way we may establish some results on the coefficients of little \( \alpha \)-Shapiro Shields functions.

2. All results in this paper can be applied to \( \alpha \)-Bloch functions since \( f \in B^\alpha \) if and only if \( f' \in S^{-\alpha} \) for analytic functions \( f \). For example, Yamashita’s result [4] can be obtained by Theorem 3.4.

3. Let \( L^p(B) \) be the Bergman space [3] of all analytic functions \( f \) such that \( |f|^p \) is integrable in \( B \). Since \( L^p(B) \subset S^{-2/p} \) and \( S^{-\alpha} \subset L^{1/(\alpha+\varepsilon)}(B) \) for \( \varepsilon > 0 \) (see [1]), some results on the coefficients for \( \alpha \)-Shapiro Shields functions can be held for the functions of Bergman space.

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References