ON SOME NEW INEQUALITIES OF HERMITE-HADAMARD TYPE FOR \( m \)-CONVEX FUNCTIONS

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Abstract. Some new inequalities for \( m \)-convex functions are obtained.

1. Introduction

In [71], G.H. Toader defined the \( m \)-convexity, an intermediate between the usual convexity and starshaped property.

In the first part of this section we shall present properties of \( m \)-convex functions in a similar manner to convex functions.

The following concept has been introduced in [71] (see also [34]).

Definition 1. The function \( f : [0, b] \rightarrow \mathbb{R} \) is said to be \( m \)-convex, where \( m \in [0, 1] \), if for every \( x, y \in [0, b] \) and \( t \in [0, 1] \) we have:

\[
    f(tx + m(1 - t)y) \leq tf(x) + m(1 - t)f(y).
\]

Denote by \( K_m(b) \) the set of the \( m \)-convex functions on \( [0, b] \) for which \( f(0) \leq 0 \).

Remark 1. For \( m = 1 \), we recapture the concept of convex functions defined on \( [0, b] \) and for \( m = 0 \) we get the concept of starshaped functions on \( [0, b] \). We recall that \( f : [0, b] \rightarrow \mathbb{R} \) is starshaped if

\[
    f(tx) \leq tf(x) \quad \text{for all } t \in [0, 1] \text{ and } x \in [0, b].
\]

The following lemmas hold [71].

Lemma 1. If \( f \) is in the class \( K_m(b) \), then it is starshaped.

Proof: For any \( x \in [0, b] \) and \( t \in [0, 1] \), we have:

\[
    f(tx) = f(tx + m(1 - t) \cdot 0) \leq tf(x) + m(1 - t)f(0) \leq tf(x).
\]
Lemma 2. If \( f \) is \( m \)-convex and \( 0 \leq n < m \leq 1 \), then \( f \) is \( n \)-convex.

Proof. If \( x, y \in [0, b] \) and \( t \in [0,1] \), then
\[
f(tx + n (1 - t)y) = f \left( tx + m (1 - t) \left( \frac{n}{m} \right) y \right) 
\leq tf(x) + m (1 - t) f \left( \left( \frac{n}{m} \right) y \right) 
\leq tf(x) + m (1 - t) \frac{n}{m} f(y) 
= tf(x) + n (1 - t) f(y)
\]
and the lemma is proved.

As in paper [48] due to V. G. Miheșan, for a mapping \( f \in K_m (b) \) consider the function
\[
p_{a,m} (x) := \frac{f(x) - m f(a)}{x - m}
\]
defined for \( x \in [0, b] \setminus \{ ma \} \), for fixed \( a \in [0, b] \), and
\[
r_m (x_1, x_2, x_3) := \frac{\begin{vmatrix} 1 & 1 & 1 \\ mx_1 & x_2 & x_3 \\ m f(x_1) & f(x_2) & f(x_3) \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ mx_1 & x_2 & x_3 \\ m^2 x_1^2 & x_2^2 & x_3^2 \end{vmatrix}},
\]
where \( x_1, x_2, x_3 \in [0, b] \), \( (x_2 - mx_1) (x_3 - mx_1) > 0, x_2 \neq x_3 \).

The following theorem holds [48].

Theorem 1. The following assertions are equivalent:
1°. \( f \in K_m (b) \);
2°. \( p_{a,m} \) is increasing on the intervals \( [0, ma) \), \( (ma, b] \) for all \( a \in [0, b] \);
3°. \( r_m (x_1, x_2, x_3) \geq 0 \).

Proof. 1° \( \Rightarrow \) 2°. Let \( x, y \in [0, b] \). If \( ma < x < y \), then there exists \( t \in (0, 1) \) such that
\[
x = ty + m (1 - t) a.
\]
(1.3)
We thus have
\[
p_{a,m} (x) = \frac{f(x) - m f(a)}{x - ma} 
= \frac{f(\frac{ty + m (1 - t) a - m f(a)}{t (y - ma)} \leq \frac{f(y) - m f(a)}{y - ma} 
= p_{a,m} (y).
\]
If \( y < x < ma \), there also exists \( t \in (0,1) \) for which (1.3) holds.

Then we have:

\[
p_{a,m}(x) = \frac{f(x) - mf(a)}{x - ma} \\
= \frac{mf(a) - f(ty + m(1-t)a)}{ma - ty - m(1-t)a} \\
\geq \frac{mf(a) - tf(y) + m(1-t)f(a)}{t(ma - y)} \\
= \frac{f(y) - mf(a)}{y - ma} \\
= p_{a,m}(y).
\]

\( 2^o \Rightarrow 3^o \). A simple calculation shows that

\[
r_m(x_1, x_2, x_3) = \frac{p_{x_1,m}(x_3) - p_{x_2,m}(x_2)}{x_3 - x_2}.
\]

Since \( p_{x_1,m} \) is increasing on the intervals \([0, mx_1), (mx_1, b]\), one obtains

\( r_m(x_1, x_2, x_3) \geq 0. \)

\( 3^o \Rightarrow 1^o \). Let \( x_1, x_2 \in [0, b] \) and let \( x_2 = tx_3 + m(1-t)x_1, \ t \in (0,1) \). Obviously \( mx_1 < x_2 < x_3 \) or \( x_3 < x_2 < mx_1 \), hence

\[
r_m(x_1, x_2, x_3) = \frac{tf(x_3) + m(1-t)f(x_1) - f(tx_3 + m(1-t)x_1)}{t(1-t)(x_3 - mx_1)^2}
\]

from where we obtain (1.1), i.e., \( f \in K_m(b) \).

The following corollary holds for starshaped functions.

**Corollary 1.** Let \( f : [0, b] \to \mathbb{R} \). The following statements are equivalent

(i) \( f \) is starshaped;

(ii) The mapping \( p(x) := \frac{f(x)}{x} \) is increasing on \((0, b]\).

The following lemma is also interesting in itself.

**Lemma 3.** If \( f \) is differentiable on \([0, b]\), then \( f \in K_m(b) \) if and only if:

\[
\begin{align*}
f'(x) &\geq \frac{f(x)-mf(y)}{x-my} \quad \text{for} \ x > my, \ y \in (0, b), \\
f'(x) &\leq \frac{f(x)-mf(y)}{x-my} \quad \text{for} \ 0 \leq x < my, \ y \in (0, b).
\end{align*}
\]

**Proof.** The mapping \( p_{y,m} \) is increasing on \((my, b]\) iff \( p_{y,m}'(x) \geq 0, \) which is equivalent with the condition (1.4).

**Corollary 2.** If \( f \) is differentiable in \([0, b]\), then \( f \) is starshaped iff \( f'(x) \geq \frac{f(x)}{x} \) for all \( x \in (0, b]. \)
The following inequalities of Hermite-Hadamard type for \(m\)-convex functions hold [34].

**Theorem 2.** Let \( f : [0, \infty) \to \mathbb{R} \) be a \( m\)-convex function with \( m \in (0, 1] \). If \( 0 \leq a < b < \infty \) and \( f \in L_1 [a, b] \), then one has the inequality:

\[
\frac{1}{b-a} \int_a^b f(x) \, dx \leq \min \left\{ \frac{f(a) + mf \left( \frac{b}{m} \right)}{2}, \frac{f(b) + mf \left( \frac{a}{m} \right)}{2} \right\}.
\]

(1.5)

**Proof.** Since \( f \) is \( m\)-convex, we have

\[ f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y), \text{ for all } x, y \geq 0, \]

which gives:

\[ f(ta + (1-t)b) \leq tf(a) + m(1-t) f \left( \frac{b}{m} \right) \]

and

\[ f(tb + (1-t)a) \leq tf(b) + m(1-t) f \left( \frac{a}{m} \right) \]

for all \( t \in [0,1] \). Integrating on \([0,1]\) we obtain

\[
\int_0^1 f(ta + (1-t)b) \, dt \leq \frac{[f(a) + mf \left( \frac{b}{m} \right)]}{2}
\]

and

\[
\int_0^1 f(tb + (1-t)a) \, dt \leq \frac{[f(b) + mf \left( \frac{a}{m} \right)]}{2}.
\]

However,

\[
\int_0^1 f(ta + (1-t)b) \, dt = \int_0^1 f(tb + (1-t)a) \, dt = \frac{1}{b-a} \int_a^b f(x) \, dx
\]

and the inequality (1.5) is obtained.

Another result of this type which holds for differentiable functions is embodied in the following theorem [34].

**Theorem 3.** Let \( f : [0, \infty) \to \mathbb{R} \) be a \( m\)-convex function with \( m \in (0, 1] \). If \( 0 \leq a < b < \infty \) and \( f \) is differentiable on \((0, \infty)\), then one has the inequality:

\[
\frac{f(mb)}{m} - \frac{b-a}{2} f' (mb) \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{(b-ma) f(b) - (a-mb) f(a)}{2(b-a)}.
\]

(1.6)
**Proof.** Using Lemma 3, we have for all $x, y \geq 0$ with $x \geq my$ that
\[(x - my) f'(x) \geq f(x) - mf(y). \tag{1.7}\]
Choosing in the above inequality $x = mb$ and $a \leq y \leq b$, then $x \geq my$ and
\[(mb - my) f'(mb) \geq f(mb) - mf(y).\]
Integrating over $y$ on $[a, b]$, we get
\[m \frac{(b-a)^2}{2} f'(mb) \geq (b-a) f(mb) - m \int_a^b f(y) \, dy, \]
thus proving the first inequality in (1.6).
Putting in (1.7) $y = a$, we have
\[(x - ma) f'(x) \geq f(x) - mf(a), \quad x \geq ma. \]
Integrating over $x$ on $[a, b]$, we obtain the second inequality in (1.6).

**Remark 2.** The second inequality from (1.6) is also valid for $m = 0$. That is, if $f : [0, \infty) \to \mathbb{R}$ is a differentiable starshaped function, then for all $0 \leq a < b < \infty$ one has:
\[\frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{bf(b) - af(a)}{2(b-a)},\]
which also holds from Corollary 2.

2. The New Results

We will now point out some new results of the Hermite-Hadamard type.

**Theorem 4.** Let $f : [0, \infty) \to \mathbb{R}$ be a $m$-convex function with $m \in (0,1]$ and $0 \leq a < b$. If $f \in L_1 [a, b]$, then one has the inequalities
\[f \left( \frac{a+b}{2} \right) \leq \frac{1}{b-a} \int_a^b \frac{f(x) + mf \left( \frac{x}{m} \right)}{2} \, dx \leq \frac{m+1}{4} \left[ \frac{f(a) + f(b)}{2} + m \cdot \frac{f \left( \frac{a}{m} \right) + f \left( \frac{b}{m} \right)}{2} \right]. \tag{2.1}\]

**Proof.** By the $m$-convexity of $f$ we have that
\[f \left( \frac{x+y}{2} \right) \leq \frac{1}{2} \left[ f(x) + mf \left( \frac{y}{m} \right) \right] \]
for all $x, y \in [0, \infty)$. 

If we choose \( x = ta + (1 - t) b, \ y = (1 - t) a + tb \), we deduce

\[
 f\left(\frac{a+b}{2}\right) \leq \frac{1}{2} \left[ f(ta + (1 - t) b) + m f \left( (1 - t) \cdot \frac{a}{m} + t \cdot \frac{b}{m} \right) \right]
\]

for all \( t \in [0,1] \).

Integrating over \( t \in [0,1] \) we get

\[
f\left(\frac{a+b}{2}\right) \leq \frac{1}{2} \left[ \int_0^1 f(ta + (1 - t) b) \, dt + m \int_0^1 f \left( (1 - t) \cdot \frac{a}{m} + t \cdot \frac{b}{m} \right) \, dt \right]. \tag{2.2}
\]

Taking into account that

\[
 \int_0^1 f(ta + (1 - t) b) \, dt = \frac{1}{b-a} \int_a^b f(x) \, dx,
\]

and

\[
 \int_0^1 f \left( t \cdot \frac{a}{m} + (1 - t) \cdot \frac{b}{m} \right) \, dt = \frac{m}{b-a} \int_{a/m}^{b/m} f(x) \, dx = \frac{1}{b-a} \int_a^b f \left( \frac{x}{m} \right) \, dx,
\]

we deduce from (2.2) the first part of (2.1).

By the \( m \)-convexity of \( f \) we also have

\[
 \frac{1}{2} \left[ f(ta + (1 - t) b) + m f \left( (1 - t) \cdot \frac{a}{m} + t \cdot \frac{b}{m} \right) \right] \leq \frac{1}{2} \left[ tf(a) + m(1-t) f \left( \frac{b}{m} \right) + m(1-t) f \left( \frac{a}{m} \right) + m^2 tf \left( \frac{b}{m^2} \right) \right] \tag{2.3}
\]

for all \( t \in [0,1] \).

Integrating the inequality (2.3) over \( t \) on \([0,1]\), we deduce

\[
 \frac{1}{b-a} \int_a^b f(x) + mf \left( \frac{x}{m} \right) \, dx \leq \frac{1}{2} \left[ f(a) + m f \left( \frac{b}{m} \right) + m f \left( \frac{a}{m} \right) + m^2 f \left( \frac{b}{m^2} \right) \right]. \tag{2.4}
\]

By a similar argument we can state:

\[
 \frac{1}{b-a} \int_a^b f(x) + mf \left( \frac{x}{m} \right) \, dx \leq \frac{1}{8} \left[ f(a) + f(b) + 2m \left( f \left( \frac{a}{m} \right) + f \left( \frac{b}{m} \right) \right) + m^2 \left( f \left( \frac{a}{m^2} \right) + f \left( \frac{b}{m^2} \right) \right) \right] \tag{2.5}
\]

and the proof is completed.

**Remark 3.** For \( m = 1 \), we can drop the assumption \( f \in L_1[a, b] \) and (2.1) exactly becomes the Hermite-Hadamard inequality.
The following result also holds.

**Theorem 5.** Let \( f : [0, \infty) \to \mathbb{R} \) be a \( m \)-convex function with \( m \in (0, 1] \). If \( f \in L^1[a, b] \) where \( 0 \leq a < b \), then one has the inequality:

\[
\frac{1}{m + 1} \left[ \int_a^b f(x) \, dx + \frac{mb - a}{b - ma} \int_{ma}^b f(x) \, dx \right] \leq (mb - a) \frac{f(a) + f(b)}{2}. \tag{2.6}
\]

**Proof.** By the \( m \)-convexity of \( f \) we can write:

\[
\begin{align*}
& f(ta + m(1 - t)b) \leq tf(a) + m(1 - t)f(b), \\
& f((1 - t)a + mtb) \leq (1 - t)f(a) + mtf(b), \\
& f(tb + (1 - t)ma) \leq tf(b) + m(1 - t)f(a)
\end{align*}
\]

and

\[
\begin{align*}
& f((1 - t)b + tma) \leq (1 - t)f(b) + mtf(a)
\end{align*}
\]

for all \( t \in [0, 1] \) and \( a, b \) as above.

If we add the above inequalities we get

\[
\begin{align*}
& f(ta + m(1 - t)b) + f((1 - t)a + mtb)
+ f(tb + (1 - t)ma) + f((1 - t)b + tma)
\leq f(a) + f(b) + m(f(a) + f(b)) = (m + 1)(f(a) + f(b)).
\end{align*}
\]

Integrating over \( t \in [0, 1] \), we obtain

\[
\begin{align*}
& \int_0^1 f(ta + m(1 - t)b) \, dt + \int_0^1 f((1 - t)a + mtb) \, dt \\
& + \int_0^1 f(tb + (1 - t)ma) \, dt + \int_0^1 f((1 - t)b + tma) \, dt
\leq (m + 1)(f(a) + f(b)).
\end{align*}
\]

As it is easy to see that

\[
\begin{align*}
& \int_0^1 f(ta + m(1 - t)b) \, dt = \int_0^1 f((1 - t)a + mtb) \, dt = \frac{1}{mb - a} \int_a^b f(x) \, dx
\end{align*}
\]

and

\[
\begin{align*}
& \int_0^1 f(tb + (1 - t)ma) \, dt = \int_0^1 f((1 - t)b + tma) \, dt = \frac{1}{b - ma} \int_{ma}^b f(x) \, dx
\end{align*}
\]

from (2.7) we deduce the desired result, namely, the inequality (2.6).
Remark 4. For an extensive literature on Hermite-Hadamard type inequalities, see the references enclosed.

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References

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