FUZZY VOLterra INTEGRAL EQUATIONS
WITH INFINITE DELAY

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Abstract. In this paper, we study the existence and uniqueness of solutions for a class of fuzzy Volterra integral equations with infinite delay by using the method of successive approximations.

1. Introduction


In this paper, we consider the fuzzy Volterra integral equation with infinite delay of the form

\[ x'(t) = h(t, x(t)) + \int_{-\infty}^{t} q(t, s, x(s)) \, ds, \quad t \in T = (-\infty, \infty) \]

where \( h: T \times E^n \to E^n \) and \( q: T \times T \times E^n \to E^n \) are levelwise continuous and satisfy the generalized Lipschitz conditions.

Basic Assumption: For each \( t_0 \in T \), there exists a nonempty convex subset \( B(t_0) \) of the space of continuous functions \( \phi: T_1 = (-\infty, t_0] \to E^n \) such that \( \phi \in B(t_0) \) implies

\[ \int_{-\infty}^{t_0} q(t, s, \phi(s)) \, ds := Q(t, t_0, \phi) \]

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is continuous on $T_2 = [t_0, \infty)$. For a given $t_0 \in T$ and a continuous initial function $\phi : T_1 \to E^n$, we seek a continuous solution $x(t, t_0, \phi)$ satisfying (1) for $t \in [t_0, t_0 + \beta)$ for some $\beta > 0$ with $x(t, t_0, \phi) = \phi(t)$ for $t \leq t_0$.

2. Preliminaries

Let $P_K(R^n)$ denote the family of all nonempty, compact, convex subsets of $R^n$. Addition and scalar multiplication in $P_K(R^n)$ are defined as usual. Let $A$ and $B$ be two nonempty bounded subsets of $R^n$. The distance between $A$ and $B$ is defined by the Hausdorff metric

$$d(A, B) = \max \left\{ \sup_{a \in A} \inf_{b \in B} ||a - b||, \sup_{b \in B} \inf_{a \in A} ||a - b|| \right\},$$

where $|| \cdot ||$ denotes the usual Euclidean norm in $R^n$. Then it is clear that $(P_K(R^n), d)$ becomes a complete metric space [9].

Let $I = [0, 1] \subseteq R$ be a compact interval and let $E^n$ denote the set of all $u : R^n \to I$ such that $u$ satisfies the following conditions.

(i) $u$ is normal, that is, there exists an $x_0 \in R^n$ such that $u(x_0) = 1$,

(ii) $u$ is fuzzy convex,

(iii) $u$ is upper semicontinuous,

(iv) $|u|^\alpha = \text{cl}\{x \in R^n : u(x) > 0\}$ is compact.

For $0 < \alpha \leq 1$, denote $|u|^\alpha = \{x \in R^n : u(x) \geq \alpha\}$. Then from (i)-(iv) it follows that the $\alpha$-level set $|u|^\alpha \in P_K(R^n)$ for all $0 \leq \alpha \leq 1$.

If $g : R^n \times R^n \to R^n$ is a function, then using Zadeh’s extension principle, we can extend $g$ to $E^n \times E^n \to E^n$ by the equation

$$\tilde{g}(u, v)(z) = \sup_{z = g(x, y)} \min\{u(x), v(y)\}.$$

It is well known that $|\tilde{g}(u, v)|^\alpha = g(|u|^\alpha, |v|^\alpha)$ for all $u, v \in E^n$, $0 \leq \alpha \leq 1$ and any continuous function $g$. Furthermore, we have $|u + v|^\alpha = |u|^\alpha + |v|^\alpha$ and $|ku|^\alpha = k|u|^\alpha$, where $k \in R$.

**Theorem 2.1.** ([8]) If $u \in E^n$, then

(i) $|u|^\alpha \in P_K(R^n)$ for $0 \leq \alpha \leq 1$,

(ii) $|u|^\alpha \subset |u|^\beta$ for $0 \leq \alpha_1 \leq \alpha_2 \leq 1$, and

(iii) If $\{\alpha_k\} \subset [0, 1]$ is a nondecreasing sequence converging to $\alpha > 0$, then

$$|u|^\alpha = \bigcap_{k=1}^{\infty} |u|^\alpha_k.$$
Conversely, if \( \{A^\alpha : 0 \leq \alpha \leq 1\} \) is a family of subset \( A \) of \( \mathbb{R}^n \) satisfying (i)-(iii), then there exists a \( u \in E^n \) such that 
\[
|u|^\alpha = A^\alpha \text{ for } 0 < \alpha \leq 1
\]
and
\[
|u|^0 = \bigcup_{0 \leq \alpha \leq 1} A^\alpha \subset A^0.
\]

Define the metric \( D : E^n \times E^n \to \mathbb{R}^+ \cup \{0\} \) by 
\[
D(u, v) = \sup_{0 \leq \alpha \leq 1} d([u]^\alpha, [v]^\alpha),
\]
where \( d \) is the Hausdorff metric defined in \( P_K(R^n) \).

The following definitions are given in [5].

**Definition 2.1.** A mapping \( F : I \to E^n \) is strongly measurable, if for all \( \alpha \in [0, 1] \) the set-valued mapping \( F_\alpha : I \to P_K(\mathbb{R}^n) \) defined by \( F_\alpha(t) = [F(t)]^\alpha \) is Lebesgue measurable when \( P_K(\mathbb{R}^n) \) has the topology induced by the Hausdorff metric \( d \).

**Definition 2.2.** A mapping \( F : I \to E^n \) is called levelwise continuous at \( t_0 \in I \) if the set-valued mapping \( F_\alpha(t) = [F(t)]^\alpha \) is continuous at \( t = t_0 \) with respect to the Hausdorff metric \( d \) for all \( \alpha \in [0, 1] \).

**Definition 2.3.** A mapping \( F : I \to E^n \) is called integrably bounded if there exists an integrable function \( h \) such that \( \|x\| \leq h(t) \) for every \( x \in F_0(t) \).

**Definition 2.4.** The integral of a fuzzy mapping \( F : I \to E^n \) is defined levelwise by 
\[
\int_I [F(t)]^\alpha dt = \int_I F_\alpha(t) dt
\]
The set of all \( \int_I f(t) dt \) such that \( f : I \to \mathbb{R}^n \) is a measurable selection for \( F_\alpha \) for all \( \alpha \in [0, 1] \).

**Theorem 2.2.** ([2]) If \( F : I \to E^n \) is strongly measurable and integrably bounded, then \( F \) is integrable.

It is known that 
\[
\left[ \int_I F(t) dt \right]^0 = \int_I F_0(t) dt.
\]

**Theorem 2.3.** Let \( F, G : I \to E^n \) be integrable and \( \lambda \in \mathbb{R} \). Then
\[
(i) \int_I (F(t) + G(t)) dt = \int_I F(t) dt + \int_I G(t) dt,
(ii) \int_I \lambda F(t) dt = \lambda \int_I F(t) dt,
(iii) D(F, G) \text{ is integrable},
(iv) D(\int_I F(t) dt, \int_I G(t) dt) \leq \int_I D(F(t), G(t)) dt.
\]

**Definition 2.5.** A mapping \( F : I \to E^n \) is called differentiable at \( t_0 \in I \) if, for any \( \alpha \in [0, 1] \), the set-valued mapping \( F_\alpha(t_0) = [F(t)]^\alpha \) is Hukuhara differentiable at \( t_0 \) with \( DF_\alpha(t_0) \) and the family \( \{DF_\alpha(t_0) : \alpha \in [0, 1]\} \) define a fuzzy number \( F(t_0) \in E^n \).
If $F : I \rightarrow E^n$ is differentiable at $t_0 \in I$, then we say that $F'(t_0)$ is the fuzzy derivative of $F(t)$ at the point $t_0$.

**Theorem 2.4.** Let $F : I \rightarrow E^n$ be differentiable. Denote $F_\alpha(t) = [f_\alpha(t), g_\alpha(t)]$. Then $f_\alpha$ and $g_\alpha$ are differentiable and $[F'(t)]^\alpha = [f'_\alpha(t), g'_\alpha(t)]$.

**Theorem 2.5.** Let $F : I \rightarrow E^n$ be differentiable and assume that the derivative $F'$ is integrable over $I$. Then, for each $s \in I$, we have

$$F(s) = F(a) + \int_a^s F'(t) \, dt.$$  

**Definition 2.6.** A mapping $f : I \times E^n \rightarrow E^n$ is called levelwise continuous at point $(t_0, x_0) \in I \times E^n$ provided, for any fixed $\alpha \in [0, 1]$ and arbitrary $\epsilon > 0$, there exists a $\delta(\epsilon, \alpha) > 0$ such that

$$d([f(t, x)]^\alpha, [f(t_0, x_0)]^\alpha) < \epsilon$$

whenever $|t - t_0| < \delta(\epsilon, \alpha)$ and $d([x]^\alpha, [x_0]^\alpha) < \delta(\epsilon, \alpha)$ for all $t \in I, x \in E^n$.

### 3. Main results

Assume that $h : T_0 \times E^n \rightarrow E^n$ and $q : T_0 \times T_0 \times E^n \rightarrow E^n$ are levelwise continuous, where $T_0 = \{t \in T : t_0 \leq t < t_0 + \beta\}$. Consider the fuzzy Volterra integral equation (1) where $\phi(t_0) \in E^n$. We denote $J = T_0 \times B(\phi(t_0), b)$ and $J_0 = T_0 \times T_0 \times B(\phi(t_0), b)$ where $a > 0$, $b > 0$, $\phi(t_0) \in E^n$, and

$$B(\phi(t_0), b) = \{x \in E^n : D(x, \phi(t_0)) \leq b\}.$$

**Definition 3.1.** A mapping $x : T_0 \rightarrow E^n$ is a solution to the problem (1) if it is levelwise continuous and satisfies the integral equation

$$x(t) = \phi(t_0) + \int_{t_0}^t h(s, x(s)) \, ds + \int_{t_0}^t \int_{t_0}^u q(u, s, x(s)) \, ds \, du + \int_{t_0}^t Q(u, t_0, \phi) \, du, \quad \text{for all } t \in T_0.$$

Assume that the following conditions hold.

(A) $h : J \rightarrow E^n$ is levelwise continuous and for any pair $(t, x), (t, y) \in J$ and $\alpha \in [0, 1]$, we have

$$d([h(t, x)]^\alpha, [h(t, y)]^\alpha) \leq k_h d([x]^\alpha, [y]^\alpha),$$

where $k_h$ is a given constant.

(B) $q : J_0 \rightarrow E^n$ is levelwise continuous and for any pair $(t, s, x_1, x_2) \in J_0$, $-b \leq s \leq t \leq b$ and $\alpha \in [0, 1]$, we have

$$d([q(t, s, x_1)]^\alpha, [q(t, s, x_2)]^\alpha) \leq k_q \left[ d([x_1]^\alpha, [x_2]^\alpha) \right],$$

where $k_q$ is a given constant.
(C) Let \( K = \max\{k_0, k_q\} \) be such that \( 0 < K < 1 \).

**Theorem 3.1.** If the conditions (A)–(C) hold, then there exists a unique solution \( x = x(t) \) of (1) defined on the interval \( t_0 \leq t < t_0 + \beta \).

**Proof.** Let \( 0 < L < \beta \) be given. Therefore \( t_0 \leq t \leq t_0 + L \).

Let

\[
\delta = \min \left\{ L, \sqrt{\left( \frac{M + M_2}{M_1} \right)^2 + \frac{2b}{M_1} - \frac{(M + M_2)}{M_1}} \right\},
\]

where \( M = D(h(t, \phi(t_0)), \hat{0}), \hat{0} \in E^n \), such that \( \hat{0} = 1 \) for \( t = 0 \) and 0 otherwise, and for any \( (t, x) \in J \) and \( M_1 = D(q(u, s, \phi(t_0)), \hat{0}) \) for any \( (u, s, \phi(t_0)) \in J_0 \), and \( M_2 = D(Q(u, t_0, \phi(t_0)), \hat{0}) \) for any \( (u, t_0, \phi(t_0)) \in J_0 \).

We will show that the sequence of functions defined inductively on \( [t_0, t_0 + L] \) by

\[
x_0(t) \equiv \phi(t_0), \quad t \in T_0,
\]

\[
x_n(t) = \phi(t_0) + \int_{t_0}^{t} h(s, x_{n-1}(s)) \, ds + \int_{t_0}^{t} \int_{t_0}^{u} q(u, s, x_{n-1}(s)) \, ds \, du
\]

\[
+ \int_{t_0}^{t} Q(u, t_0, \phi(t_0)) \, du, \quad n = 1, 2, 3, \ldots,
\]

(2)

From (2), it follows that, for \( n = 1 \),

\[
x_1(t) = \phi(t_0) + \int_{t_0}^{t} h(s, \phi(t_0)) \, ds + \int_{t_0}^{t} \int_{t_0}^{u} q(u, s, \phi(t_0)) \, ds \, du + \int_{t_0}^{t} Q(u, t_0, \phi(t_0)) \, du,
\]

(3)

which proves that \( x_1(t) \) is levelwise continuous on \( |t - t_0| \leq L \) and hence on \( |t - t_0| \leq \delta \). Moreover, for any \( \alpha \in [0, 1] \), we have

\[
d([x_1(t)]^\alpha, [x_0(t)]^\alpha) = d \left( \left[ \phi(t_0) + \int_{t_0}^{t} h(s, \phi(t_0)) \, ds + \int_{t_0}^{t} \int_{t_0}^{u} q(u, s, \phi(t_0)) \, ds \, du \right]^\alpha, \left[ \phi(t_0) \right]^\alpha \right)
\]

\[
\leq \int_{t_0}^{t} d([h(s, \phi(t_0))]^\alpha, \hat{0}) \, ds + \int_{t_0}^{t} \int_{t_0}^{u} d([q(u, s, \phi(t_0))]^\alpha, \hat{0}) \, ds \, du
\]

\[
+ \int_{t_0}^{t} d([Q(u, t_0, \phi(t_0))]^\alpha, \hat{0}) \, du,
\]

where \( D(h(s, \phi(t_0)), \hat{0}) \) and \( D(q(u, s, \phi(t_0)), \hat{0}) \) are levelwise continuous on \( |t - t_0| \leq L \).
and by the definition of \(D\), we get

\[
D(x_1(t), x_0(t)) \leq \int_{t_0}^{t} D(h(s, \phi(t_0)), \tilde{\theta}) ds + \int_{t_0}^{t} \int_{t_0}^{u} D(q(u, s, \phi(t_0)), \tilde{\theta}) ds du
\]

\[
+ \int_{t_0}^{t} D(Q(u, t_0, \phi(t_0)), \tilde{\theta}) du
\]

\[
\leq (M + M_2)|t - t_0| + M_1 \frac{|t - t_0|^2}{2!}
\]

\[
\leq (M + M_2)\delta + M_1 \frac{\delta^2}{2!}
\]

\[
\leq b. \tag{4}
\]

Now, assume that \(x_{n-1}(t)\) is levelwise continuous on \(|t - t_0| \leq \delta\) and that

\[
D(x_{n-1}(t), x_0(t)) \leq b.
\]

From (2), we deduce that \(x_n(t)\) is levelwise continuous on \(|t - t_0| \leq \delta\) and that

\[
D(x_n(t), x_0(t)) \leq b.
\]

Consequently, we conclude that \(x_n(t)\) consists of levelwise continuous mappings on \(|t - t_0| \leq \delta\) and that

\[
(t, x_n(t)) \in J \text{ and } (t, s, x_n(t)) \in J_0, \quad |t - t_0| \leq \delta, n = 1, 2, \ldots,
\]

Let us prove that there exists a fuzzy set-valued mapping \(x : [t_0, t_0 + L] \to \mathbb{E}^n\) such that \(D(x_n(t), x(t)) \to 0\) uniformly on \(|t - t_0| \leq \delta\) as \(n \to \infty\). For \(n = 2\), from (2),

\[
x_2(t) = \phi(t_0) + \int_{t_0}^{t} h(s, x_1(s)) ds + \int_{t_0}^{t} \int_{t_0}^{u} q(u, s, x_1(s)) ds du + \int_{t_0}^{t} Q(u, t_0, \phi(t_0)) du. \tag{5}
\]

From (3) and (5), we have

\[
d([x_2(t)]^a, [x_1(t)]^a)
\]

\[
= d\left(\left[\int_{t_0}^{t} h(s, x_1(s)) ds + \int_{t_0}^{t} \int_{t_0}^{u} q(u, s, x_1(s)) ds du + \int_{t_0}^{t} Q(u, t_0, \phi(t_0)) du\right]^a, \right.
\]

\[
\left.\left[\int_{t_0}^{t} h(s, \phi(t_0)) ds + \int_{t_0}^{t} \int_{t_0}^{u} q(u, s, \phi(t_0)) ds du + \int_{t_0}^{t} Q(u, t_0, \phi(t_0)) du\right]^a\right)
\]

\[
\leq k_h \int_{t_0}^{t} d([x_1(s)]^a, [\phi(t_0)]^a) ds + k_q \int_{t_0}^{t} \int_{t_0}^{u} d([x_1(s)]^a, [\phi(t_0)]^a) ds du,
\]

So by the definition of \(D\), we have

\[
D(x_2(t), x_1(t)) \leq k_h \int_{t_0}^{t} D(x_1(s), \phi(t_0)) ds + k_q \int_{t_0}^{t} \int_{t_0}^{u} D(x_1(s), \phi(t_0)) ds du. \tag{6}
\]
Now, we can apply the first inequality (4) in the right-hand side of (6) to get
\[
D(x_2(t), x_1(t))
\leq (M + M_2)K \frac{|t - t_0|^2}{2!} + M_1 K \frac{|t - t_0|^3}{3!} + (M + M_2) K \frac{|t - t_0|^3}{3!} + M_1 K \frac{|t - t_0|^4}{4!}
\leq K \left[ (M + M_2) \frac{\delta^2}{2!} + (M + M_1 + M_2) \frac{\delta^3}{3!} + M_1 \frac{\delta^4}{4!} \right].
\]

(7)

Starting from (4) and (7), assume that
\[
D(x_n(t), x_{n-1}(t))
\leq K^{n-1} \left[ \frac{(n-1) C_0 (M + M_2) \delta^n}{n!} + \frac{[(n-1) C_1 (M + M_2) + (n-1) C_0 M_1] \delta^{n+1}}{(n+1)!} + \cdots + \frac{[(n-1) C_{n-1} (M + M_2) + (n-1) C_{n-2} M_1] \delta^{2n-1}}{(2n-1)!} + M_1 \delta^{2n} \right].
\]

(8)

and we prove that such an inequality holds for \(D(x_{n+1}(t), x_n(t))\). Indeed, from (2) and the assumptions, it follows that
\[
d([x_{n+1}(t)]^a, [x_n(t)]^a) = 
\leq k_h \int_{t_0}^t d([x_n(s)]^a, [x_{n-1}(s)]^a) ds + k_\eta \int_{t_0}^t \int_{t_0}^u d([x_n(s)]^a, [x_{n-1}(s)]^a) ds du,
\]

for any \(a \in [0, 1]\) and from the condition on \(D\), we have
\[
D(x_{n+1}(t), x_n(t)) \leq k_h \int_{t_0}^t D(x_n(s), x_{n-1}(s)) ds + k_\eta \int_{t_0}^t \int_{t_0}^u D(x_n(s), x_{n-1}(s)) ds du.
\]

According to (8), we get
\[
D(x_{n+1}(t), x_n(t))
\leq K^n \left[ \frac{n C_0 (M + M_2) \delta^n}{n!} + \frac{[n C_1 (M + M_2) + n C_0 M_1] \delta^{n+1}}{(n+1)!} + \cdots + \frac{[n C_{n-1} (M + M_2) + n C_{n-2} M_1] \delta^{2n-1}}{(2n-1)!} + M_1 \frac{\delta^{2n+2}}{(2n+2)!} \right].
\]

Consequently, inequality (8) holds for \(n = 1, 2, \cdots\). We can also write
\[
D(x_n(t), x_{n-1}(t))
\leq \frac{K^n}{K} \left[ \frac{(n-1) C_0 (M + M_2) \delta^n}{n!} + \frac{[(n-1) C_1 (M + M_2) + (n-1) C_0 M_1] \delta^{n+1}}{(n+1)!} + \cdots + \frac{[(n-1) C_{n-1} (M + M_2) + (n-1) C_{n-2} M_1] \delta^{2n-1}}{(2n-1)!} + M_1 \frac{\delta^{2n}}{(2n)!} \right],
\]

(9)

for \(n = 1, 2, \cdots\), and \(|t - t_0| \leq \delta\).
Let us mention that
\[ x_n(t) = x_0(t) + [x_1(t) - x_0(t)] + \cdots + [x_n(t) - x_{n-1}(t)], \]
which implies that the sequence \( \{x_n(t)\} \) and the series
\[ x_0(t) + \sum_{n=1}^{\infty} [x_n(t) - x_{n-1}(t)] \]
have the same convergence properties. From (9), it follows that \( D(x_n(t), x_{n-1}(t)) \to 0 \) uniformly on \( |t - t_0| \leq \delta \) as \( n \to \infty \). Hence, there exists a fuzzy set-valued mapping \( x : [t_0, t_0 + L] \to E^n \) such that \( D(x_n(t), x(t)) \to 0 \) uniformly on \( |t - t_0| \leq \delta \) as \( n \to \infty \). From the assumptions, we get
\[ d([h(t, x_n(t))]^\alpha, [h(t, x(t))]^\alpha) \leq k_h d([x_n(t)]^\alpha, [x(t)]^\alpha) \]
for any \( \alpha \in [0, 1] \), and so
\[ D(h(t, x_n(t)), h(t, x(t))) \leq k_h D(x_n(t), x(t)) \to 0 \tag{10} \]
uniformly on \( |t - t_0| \leq \delta \) as \( n \to \infty \). Furthermore,
\[ d([q(t, s, x_n(s))]^\alpha, [q(t, s, x(s))]^\alpha) \leq k_q d([x_n(s)]^\alpha, [x(s)]^\alpha) \]
for any \( \alpha \in [0, 1] \), and
\[ D(q(t, s, x_n(s)), q(t, s, x(s))) \leq k_q D(x_n(s), x(s)) \to 0 \tag{11} \]
uniformly on \( |t - t_0| \leq \delta \) as \( n \to \infty \).

Taking (10) and (11) into account, from (2), we obtain,
\[ x(t) = \phi(t_0) + \int_{t_0}^{t} h(s, x(s))ds + \int_{t_0}^{t} \int_{t_0}^{u} q(u, s, x(s))dsdu + \int_{t_0}^{t} Q(u, t_0, \phi(t_0))du \]
for \( n \to \infty \),

Consequently, there is at least one levelwise continuous solution of (1).

We want to prove now that this solution is unique, that is, from
\[ y(t) = \phi(t_0) + \int_{t_0}^{t} h(s, y(s))ds + \int_{t_0}^{t} \int_{t_0}^{u} q(u, s, y(s))dsdu + \int_{t_0}^{t} Q(u, t_0, \phi(t_0))du \tag{12} \]
on \( |t - t_0| \leq \delta \), we want to show that \( D(x(t), y(t)) \equiv 0 \). Indeed, from (2) and (12), we have
\[ d([y(t)]^\alpha, [x_n(t)]^\alpha) \leq k_h \int_{t_0}^{t} d([y(s)]^\alpha, [x_{n-1}(s)]^\alpha)ds + k_q \int_{t_0}^{t} d([y(s)]^\alpha, [x_{n-1}(s)]^\alpha)dsdu \]
for any \( \alpha \in [0, 1], \ n = 1, 2, \ldots \).
By the definition of $D$, we have
\[
D(y(t), x_n(t)) \leq K \int_{t_0}^{t} D(y(s), x_{n-1}(s)) ds + K \int_{t_0}^{t} \int_{t_0}^{s} D(y(s), x_{n-1}(s)) ds du
\]
But $D(y(t), x_0(t)) \leq b$ on $|t - t_0| \leq \delta$, $y(t)$ being a solution of (12). It follows from (13) that
\[
D(y(t), x_1(t)) \leq K \int_{t_0}^{t} D(y(s), x_0(s)) ds + K \int_{t_0}^{t} \int_{t_0}^{s} D(y(s), x_0(s)) ds du
\]
\[
\leq K b \left[ |t - t_0| + \frac{|t - t_0|^2}{2!} \right]
\]
on $|t - t_0| \leq \delta$. Also,
\[
D(y(t), x_2(t)) \leq K \int_{t_0}^{t} D(y(s), x_1(s)) ds + K \int_{t_0}^{t} \int_{t_0}^{s} D(y(s), x_1(s)) ds du\]
\[
\leq K^2 b \left[ |t - t_0|^2 \frac{2!}{2!} + 2 \frac{|t - t_0|^3}{3!} + \frac{|t - t_0|^4}{4!} \right]
\]
on $|t - t_0| \leq \delta$. Now assume that
\[
D(y(t), x_n(t)) \leq K^n b \left[ \sum_{j=0}^{n} C_n^j \frac{|t - t_0|^j}{j!} \right]
\]
on the interval $|t - t_0| \leq \delta$. From
\[
D(y(t), x_{n+1}(t)) \leq K \int_{t_0}^{t} D(y(s), x_n(s)) ds + K \int_{t_0}^{t} \int_{t_0}^{s} D(y(s), x_n(s)) ds du
\]
and (14), one obtains
\[
D(y(t), x_{n+1}(t)) \leq K^{n+1} b \left[ \sum_{j=0}^{n} C_n^j \frac{|t - t_0|^j}{j!} + \sum_{j=n+1}^{n} C_n^j \frac{|t - t_0|^j}{j!} \right]
\]
Consequently, (14) holds for any $n$, which leads to the conclusion
\[
D(y(t), x_n(t)) = D(x(t), x_n(t)) \rightarrow 0
\]
on the interval $|t - t_0| \leq \delta$ as $n \rightarrow \infty$. Thus, there exists a unique solution on $[t_0, t_0 + L]$. Since $L$ is arbitrary in $(0, \beta)$. Therefore, there exists an unique solution on $[t_0, t_0 + \beta]$.

**Example.** Consider the fuzzy Volterra integral equation with infinite delay
\[
x'(t) = \left( \frac{5}{6} - \frac{6}{11e} \right) x(t) + \int_{-\infty}^{t} e^{(t-s-1)} x(s) ds,
\]
\[
\|x(t)\| = \|\phi(t)\| = e^{2t}[(1 + \alpha), (3 - \alpha)], \quad t \in (-\infty, 0], \alpha \in [0, 1].
\]
Since the conditions (A)-(C) hold, from Theorem 3.1 the above equation has a unique solution.

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References


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