A HILBERT TYPE INEQUALITY

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Abstract. In this paper we obtain a new inequality of Hilbert type for a finite number of nonnegative sequences of real numbers from which we can recover as a special case an inequality due to Pachpatte. We also obtain an integral variant of the inequality.

1. Introduction

The well known Hilbert’s inequality [2, p.226] has been generalized in many directions by a number of mathematicians (see [1, 2, 3, 4, 5]). The purpose of the present paper is to derive a new inequality of Hilbert type, which will subsume, as a special case, a recent result of Pachpatte [7, Theorem 1].

Theorem 1. Let \( \{a_{i,m_i}\} \) (i = 1, \ldots, n) be n sequences of nonnegative real numbers defined for \( m_1 = 1, 2, \ldots, k_i \) with \( a_{1,0} = a_{2,0} = \cdots = a_{n,0} = 0 \) and let \( \{p_{i,m_i}\} \) be n sequences of positive real numbers defined for \( m_1 = 1, 2, \ldots, k_i \), where \( k_i \) (i = 1, 2, \ldots, n) are natural numbers. Set \( P_{i,m_i} = \sum_{s=1}^{m_i} p_{i,s} \) (i = 1, \ldots, n). Let \( \phi_i \) (i = 1, \ldots, n) be n real-valued nonnegative convex and submultiplicative functions defined on \( \mathbb{R}_+ = [0, \infty) \), let \( \alpha_i \in (0, 1) \), and set \( \alpha'_i = 1 - \alpha_i \) (i = 1, \ldots, n), \( \alpha = \sum_{i=1}^{n} \alpha_i \) and \( \alpha' = \sum_{i=1}^{n} \alpha'_i = n - \alpha \). Then

\[
\sum_{m_1=1}^{k_1} \cdots \sum_{m_n=1}^{k_n} \prod_{i=1}^{n} \frac{\phi_i(a_{i,m_i})}{(\sum_{s=1}^{m_i} \alpha'_s m_i) \alpha'} \leq M(k_1, \ldots, k_n) \prod_{i=1}^{n} \left( \sum_{m_i=1}^{k_i} (k_i - m_i + 1) \left[ p_{i,m_i} \phi_i \left( \frac{\nabla a_{i,m_i}}{p_{i,m_i}} \right) \right]^{1/\alpha_i} \right)^{\alpha_i},
\]

where

\[
M(k_1, \ldots, k_n) = \frac{1}{(\alpha')^{\alpha'}} \prod_{i=1}^{n} \left( \sum_{m_i=1}^{k_i} \left[ \phi_i(P_{i,m_i}) \right]^{1/\alpha'_i} \right)^{\alpha'_i}
\]

and

\[
\nabla a_{i,m_i} = a_{i,m_i} - a_{i,m_i-1} \quad (i = 1, \ldots, n).
\]

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Proof. From the hypotheses it is easy to observe that
\[ a_{i,m_i} = \sum_{s_i=1}^{m_i} \nabla a_{i,s_i} \quad (m_i = 1, 2, \ldots, k_i, \ i = 1, \ldots, n). \]

So we have
\[
\phi_i(a_{i,m_i}) = \phi_i \left\{ \frac{P_{i,m_i} \sum_{s_i=1}^{m_i} p_{i,s_i} \left( \nabla a_{i,s_i} / p_{i,s_i} \right)}{\sum_{s_i=1}^{m_i} p_{i,s_i}} \right\} \\
\leq \phi_i(P_{i,m_i}) \phi_i \left\{ \frac{\sum_{s_i=1}^{m_i} p_{i,s_i} \left( \nabla a_{i,s_i} / p_{i,s_i} \right)}{\sum_{s_i=1}^{m_i} p_{i,s_i}} \right\} \\
\leq \phi_i(P_{i,m_i}) \frac{\sum_{s_i=1}^{m_i} p_{i,s_i} \phi_i \left( \nabla a_{i,s_i} / p_{i,s_i} \right)}{P_{i,m_i}}
\]
for \( i = 1, \ldots, n. \)

Further, by Hölder’s inequality (see [6, p. 99]) we have
\[
\prod_{i=1}^{n} \phi_i(a_{i,m_i}) \leq \prod_{i=1}^{n} \left( \frac{\phi_i(P_{i,m_i}) \sum_{s_i=1}^{m_i} p_{i,s_i} \left( \nabla a_{i,s_i} / p_{i,s_i} \right)}{\sum_{s_i=1}^{m_i} p_{i,s_i}} \right)^{\alpha'_i} \\
\leq \prod_{i=1}^{n} \left\{ \left[ \frac{\phi_i(P_{i,m_i}) \sum_{s_i=1}^{m_i} p_{i,s_i} \left( \nabla a_{i,s_i} / p_{i,s_i} \right)}{\sum_{s_i=1}^{m_i} p_{i,s_i}} \right]^{\alpha'_i} \right\}. \tag{2}
\]

Let us note that
\[
\left( \prod_{i=1}^{n} (m_i)^{\alpha'_i} \right)^{1/\alpha'} \leq \frac{1}{\alpha'} \sum_{i=1}^{n} \alpha'_i m_i,
\]
so we have
\[
\prod_{i=1}^{n} (m_i)^{\alpha'_i} \leq \left( \frac{1}{\alpha'} \right)^{\alpha'} \left( \sum_{i=1}^{n} \alpha'_i m_i \right)^{\alpha'},
\]
and (2) becomes
\[
\prod_{i=1}^{n} \phi_i(a_{i,m_i}) \leq \left( \frac{\sum_{i=1}^{n} \alpha'_i m_i}{(\alpha')^{\alpha'}} \right) \prod_{i=1}^{n} \left\{ \left[ \frac{\phi_i(P_{i,m_i}) \sum_{s_i=1}^{m_i} p_{i,s_i} \left( \nabla a_{i,s_i} / p_{i,s_i} \right)}{\sum_{s_i=1}^{m_i} p_{i,s_i}} \right]^{1/\alpha_i} \right\}^{\alpha_i}. \tag{3}
\]
Dividing both sides of (3) by \((\sum_{i=1}^{n} \alpha'_i m_i)^{\alpha'}\) and taking the sum over \(m_i\) \((i = 1, \ldots, n)\) from 1 to \(k_i\), then using Hölder’s inequality (see [6, p. 99]) and interchanging the order of summation, we observe that
\[
\sum_{m_1=1}^{k_1} \cdots \sum_{m_n=1}^{k_n} \frac{\prod_{i=1}^{n} \phi_i(a_{i,m_i})}{(\sum_{i=1}^{n} \alpha'_i m_i)^{\alpha'}}
\]
where

\[ \int p_i(s) \phi_i(\nabla a_{i,s}/p_i(s_i))^{1/\alpha_i} \]

which is equivalent to (1).

**Remark 2.** For \( \alpha_1 = \cdots = \alpha_n = (n-1)/n \), (1) becomes

\[
\sum_{m_1=1}^{k_1} \cdots \sum_{m_n=1}^{k_n} \prod_{i=1}^{m_i} \frac{\phi_i(a_{i,m_i})}{P_i(m_i)}^{\alpha_i} \leq \bar{M}(k_1, \ldots, k_n) \prod_{i=1}^{n} \left\{ \sum_{m_i=1}^{k_i} (k_i - m_i + 1) \left[ p_i, m_i, \phi_i \frac{\nabla a_{i,m_i}}{P_i(m_i)} \right]^{n/(n-1)} \right\}^{(n-1)/n},
\]

where

\[ \bar{M}(k_1, \ldots, k_n) = \frac{1}{n} \prod_{i=1}^{n} \left\{ \sum_{m_i=1}^{k_i} \left[ \frac{\phi_i(P_i(m_i))}{P_i(m_i)} \right] \right\}^{1/n}. \]

For \( n = 2 \), this is Pachpatte’s result [7, Theorem 1].

There is also an integral analogue of Theorem 1.

**Theorem 3.** Let \( f_i \in C^1([0, k_i], \mathbb{R}_+) \), \( i = 1, \ldots, n \), with \( f_i(0) = 0 \) (i = 1, \ldots, n), let \( p_i(\sigma_i) \) be \( n \) positive functions defined for \( \sigma_i \in [0, x_i] \) (i = 1, \ldots, n), and set \( P_i(s_i) = \int_0^{s_i} p_i(\sigma_i) d\sigma_i \) for \( s_i \in [0, x_i] \) where \( x_i \) are positive real numbers. Let \( \phi_i, \alpha_i, \alpha_i', \alpha \) and \( \alpha' \) be as in Theorem 1. Then

\[
\int_0^{x_1} \cdots \int_0^{x_n} \prod_{i=1}^{n} \frac{\phi_i(f_i(s_i))}{(\sum_{i=1}^n \alpha_i s_i)}^{\alpha_i} \leq L(x_1, \ldots, x_n) \prod_{i=1}^{n} \left\{ \int_0^{x_i} (x_i - s_i) [p_i(s_i) \phi_i(f_i'(s_i)) / p_i(s_i)]^{1/\alpha_i} ds_i \right\}^{\alpha_i}, \tag{4}
\]

where

\[ L(x_1, \ldots, x_n) = \frac{1}{(\alpha')^{\alpha'}} \prod_{i=1}^{n} \left\{ \int_0^{x_i} \left[ \frac{\phi_i(P_i(s_i))}{P_i(s_i)} \right]^{1/\alpha_i'} ds_i \right\}^{\alpha_i'}. \]
Proof. From the hypotheses we have
\[ f_i(s_i) = \int_0^{x_i} f'(\sigma_i) d\sigma_i, \quad s_i \in [0, x_i], \]
Using Jensen’s integral inequality (see [6, p. 6]), we obtain
\[ \phi_i(f_i(s_i)) = \phi_i \left( \frac{\int_0^{x_i} p_i(\sigma_i)(f'_i(\sigma_i)/p_i(\sigma_i)) d\sigma_i}{\int_0^{x_i} p_i(\sigma_i) d\sigma_i} \right) \]
\[ \leq \phi_i(P_i(s_i)) \phi_i \left( \frac{\int_0^{x_i} p_i(\sigma_i)(f'_i(\sigma_i)/p_i(\sigma_i)) d\sigma_i}{\int_0^{x_i} p_i(\sigma_i) d\sigma_i} \right) \]
\[ \leq \frac{\phi_i(P_i(s_i))}{P_i(s_i)} \int_0^{x_i} p_i(\sigma_i) \phi_i(f'_i(\sigma_i)/p_i(\sigma_i)) d\sigma_i, \quad i = 1, \ldots, n. \]
The rest of the proof is similar to that for Theorem 1.

Remark 4. For \( \alpha_1 = \cdots = \alpha_n = (n-1)/n \), (4) becomes
\[ \int_0^{x_1} \cdots \int_0^{x_n} \prod_{i=1}^{n} \frac{\phi_i(f_i(s_i))}{s_1 + \cdots + s_n} ds_1 \cdots ds_n \]
\[ \leq \mathcal{L}(x_1, \ldots, x_n) \prod_{i=1}^{n} \left( \int_0^{x_i} (x_i - s_i) \left[ p_i(s_i) \phi_i( f_i(s_i) ) \right]^{n/(n-1)} ds_i \right)^{(n-1)/n} \]
where
\[ \mathcal{L}(x_1, \ldots, x_n) = \frac{1}{n} \prod_{i=1}^{n} \left( \int_0^{x_i} \left[ \frac{\phi_i(P_i(s_i))}{P_i(s_i)} \right]^{n} ds_i \right)^{1/n}. \]
For \( n = 2 \) we recover Pachpatte’s result [7, Theorem 2].

References

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