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On the Signed strong total Roman domination number of graphs

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Abstract. Let $G = (V, E)$ be a finite and simple graph of order n and maximum degree Δ. A signed strong total Roman dominating function on a graph G is a function $f : V(G) \to \{-1, 1, 2, \ldots, \lceil \frac{\Delta}{2} \rceil + 1\}$ satisfying the condition that (i) for every vertex v of G, $f(N(v)) = \sum_{u \in N(v)} f(u) \geq 1$, where $N(v)$ is the open neighborhood of v and (ii) every vertex v for which $f(v) = -1$ is adjacent to at least one vertex w for which $f(w) \geq 1 + \lceil \frac{1}{2} |N(w) \cap V_{-1}| \rceil$, where $V_{-1} = \{v \in V : f(v) = -1\}$. The minimum of the values $\omega(f) = \sum_{v \in V} f(v)$, taken over all signed strong total Roman dominating functions f of G, is called the signed strong total Roman domination number of G and is denoted by $\gamma_{ssTR}(G)$. In this paper, we initiate signed strong total Roman domination number of a graph and give several bounds for this parameter. Then, among other results, we determine the signed strong total Roman domination number of special classes of graphs.

Keywords. Signed total Roman dominating function, Signed total Roman domination number, Strong Roman dominating function, Signed strong Roman dominating function

1 Introduction

Let G be a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The *order* and *size* of a graph G are denoted by $n = n(G)$ and $m = m(G)$, respectively. For x, $y \in V(G)$ with $x \neq y$, $\mathrm{d(x,y)}$ denotes the length of a shortest path from x to y. If there is no such path, then we will make the convention $\mathrm{d(x,y)} = \infty$. A graph G is called *connected* if there is a path between each pair x and y in $V(G)$. The *diameter* of G is defined as $\mathrm{diam}(G) = \sup\{d(x,y)|\ x \text{ and } y \text{ are vertices of } G\}$. For every vertex $v \in V$, the *open neighborhood* $N_G(v)$ is the set $\{u \in V \mid uv \in E\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $d_G(v) = d(v) = |N_G(v)|$. The *minimum* and *maximum degree* of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. A graph G is *regular* if the degrees of all vertices of G are the same. We write K_n for the *complete graph*, P_n for a path and C_n for a *cycle* of order n. We also denote the *complete bipartite graph* with two parts of sizes r and s, by $K_{r,s}$. Let X and Y be two subsets of $V(G)$. We denote by $[X, Y]$ the set of edges of G with one end in X and the other end in Y.

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For a subset $S \subseteq V$ and $v \in V$, we denote by $G[S]$ the subgraph of G induced by vertices of S and by $d_S(v)$ the number of vertices in S that are adjacent to v. It is easy to see that $d_S(v) = d_{G[S]}(v)$ for every $v \in S$.

A subset S of vertices is called a 2-*packing* if $N[u] \cap N[v] = \emptyset$ for every pair of vertices $u, v \in S$. The 2-*packing number* $\rho := \rho_2(G)$ of a graph G is the maximum cardinality of a 2-packing in G.

A subset S of vertices of G is a *dominating set* if $N[S] = V$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. A dominating set of minimum cardinality of G is called a $\gamma(G)$-set. A subset D of vertices of a graph G is a *total dominating set* if each vertex in $V(G)$ is adjacent to some vertex in D. The cardinality of a smallest total dominating set in a graph G is called the *total domination number* of G and is denoted by $\gamma_t(G)$. We note that this parameter is only defined for graphs without isolated vertices.

The definition of a *Roman dominating function* was motivated by an article in Scientific American by Ian Stewart entitled Defend the Roman Empire (Stewart 1999) ([8]) and suggested even earlier by ReVelle (1997) ([6]). In this way, Emperor Constantine the Great can defend the Roman Empire, but it was so expensive to maintain a legion at a location, the Emperor would like to station as few legions as possible, while still defending the Roman Empire (See [1]). To solve this problem, other authors made some changes to the definition of the Roman function. In particular, the *signed Roman domination number* was introduced by Ahanghar et al. in [1] as below and has been studied in [7].

A *signed Roman dominating function* (SRDF) on a graph G is a function $f : V \to \{-1, 1, 2\}$ satisfying the conditions that (i) $f[v] = \sum_{x \in N[v]} f(x) \geq 1$ for each vertex $v \in V$, and (ii) every vertex u for which $f(u) = -1$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of an SRDF is the sum of its function values over all vertices and denoted by $\omega(f)$. The signed Roman domination number of G, denoted $\gamma_{sR}(G)$, is the minimum weight of an SRDF on G. In this way, the defensive strategy is based in the fact that every place in which there is established a Roman legion (a label 1) is able to protect itself under external attacks; and that every place with an auxiliary troop (a label -1) must have at least a stronger neighbor (a label 2). So, if an unsecured place (a label -1) is attacked, then a stronger neighbor could send one of its two legions in order to defend the weak neighbor vertex (label -1) from the attack.

Next, for more study, the *signed total Roman domination number* was introduced by Volkmann (2016) in [9]. A *signed total Roman dominating function* (STRDF) on a graph G is a function $f : V \to \{-1, 1, 2\}$ satisfying the conditions that (i) $f(N(v)) = \sum_{x \in N(v)} f(x) \geq 1$ for each vertex $v \in V$ (i.e., f is a total dominating function), and (ii) every vertex u for which $f(u) = -1$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of an STRDF is the sum of its function values over all vertices and denoted by $\omega(f)$. The signed total Roman domination number of G, denoted $\gamma_{stR}(G)$, is the minimum weight of an STRDF on G.

But still, there was a question. If several unsecured places which protected by one stronger place are attacked at the same time, the stronger place will be not able to defend all its neighbors. This persuaded researchers to define the concept *signed strong Roman domination number* as follows. In this definition, a strong place should be able to defend itself and, at least half of its weak neighbors.

Consider a graph G of order n and maximum degree Δ. A *signed strong Roman dominating function* (abbreviated SStRDF) on a graph G is a function $f : V(G) \to \{-1, 1, 2, \ldots, \lceil \frac{\Delta}{2} \rceil + 1\}$ satisfying the conditions that (i) for every vertex v of G, $f[v] = \sum_{u \in N[v]} f(u) \geq 1$ and (ii) every vertex v for which $f(v) = -1$ is adjacent to at least one vertex w for which $f(w) \geq 1 + \lceil \frac{1}{2} |N(w) \cap V_{-1}| \rceil$, where $V_{-1} = \{v \in V : f(v) = -1\}$. The minimum of the values $\omega(f) = \sum_{v \in V} f(v)$, taken over all signed strong Roman dominating functions f of G, is called the signed strong Roman domination number of G and is denoted by $\gamma_{ssR}(G)$. A signed strong Roman dominating function of weight $\gamma_{ssR}(G)$ is called $\gamma_{ssR}(G)$-function. This concept has been introduced in [2] and studied in [5].

With this motivation in mind and in graph theoretic terms, in this paper we generalize the concept of signed strong Roman dominating function and initiate the study of *signed strong total Roman dominating function* in graphs.

Consider a graph G of order n and maximum degree Δ. A signed strong total Roman dominating function (abbreviated SSTRDF) on a graph G is a function $f : V(G) \to \{-1, 1, 2, \ldots, \lceil \frac{\Delta}{2} \rceil + 1\}$ satisfying the condition that (i) for every vertex v of G, $f(N(v)) = \sum_{u \in N(v)} f(u) \geq 1$ (i.e., f is a total dominating function) and (ii) every vertex v for which $f(v) = -1$ is adjacent to at least one vertex w for which $f(w) \geq 1 + \lceil \frac{1}{2} |N(w) \cap V_{-1}| \rceil$, where $V_{-1} = \{v \in V : f(v) = -1\}$. The minimum of the values $\omega(f) = \sum_{v \in V} f(v)$, taken over all signed strong total Roman dominating functions f of G, is called the signed strong total Roman domination number of G and is denoted by $\gamma_{ssTR}(G)$. A signed strong total Roman dominating function of weight $\gamma_{ssTR}(G)$ is called $\gamma_{ssTR}(G)$-function.

A signed strong total Roman dominating function $f : V(G) \to \{-1, 1, 2, \ldots, \lceil \frac{\Delta}{2} \rceil + 1\}$ can be represented by the ordered partition $(V_{-1}, V_1, \ldots, V_{\lceil \frac{\Delta}{2} \rceil + 1})$, where $V_j = \{v \in V : f(v) = j\}$ for $j = -1, 1, 2, \ldots, \lceil \frac{\Delta}{2} \rceil + 1$. Let $|V_j| = n_j$ for $j = -1, 1, 2, \ldots, \lceil \frac{\Delta}{2} \rceil + 1$. In the course of paper, for simplicity, we set $B = \bigcup_{i=2}^{1+\lceil \frac{\Delta}{2} \rceil} V_i$, $|B| = n_B$, $V_{1B} = V_1 \bigcup B$, $|V_{1B}| = n_{1B}$. We also denote the size of $G[V_{1B}]$, $G[B]$ and $G[V_j]$ for $j = -1, 1, 2, \ldots, \lceil \frac{\Delta}{2} \rceil + 1$ by m_{1B}, m_B and m_j for $j = -1, 1, 2, \ldots, \lceil \frac{\Delta}{2} \rceil + 1$, respectively. Let $f(X) = \sum_{v \in X} f(v)$, where X is a subset of $V(G)$.

In this paper, we present some bounds on the signed strong total Roman domination number. Among other results, we prove that $\gamma_{ssTR}(G) \geq \frac{n(\delta+1)}{\Delta} - n$ and $\gamma_{ssTR}(G) \geq \frac{17+11\lceil \frac{\Delta}{2} \rceil}{1+\lceil \frac{\Delta}{2} \rceil} n - 12m$. In Section 3, the signed strong total Roman domination number is determined for some classes of graphs. Finally, in section 5, we indicate some possible directions of future research.

2 Preliminary results and examples

In this section we present basic properties of the signed strong Roman dominating function.

We make use of the following results in this paper.

Observation 2.1. *Let $f = (V_{-1}, V_1, \ldots, V_{1+\lceil \frac{\Delta}{2} \rceil})$ be an SSTRDF for a graph G without isolated vertices of order n. Then the following results hold:*

(a) $|B| + |V_1| + |V_{-1}| = n$.

(b) $V_1 \cup B$ *is a total dominating set in* G.

(c) $\omega(f) = \sum_{i=1}^{1+\lceil \frac{\Delta}{2} \rceil} i|V_i| - |V_{-1}|$.

Proposition 2.2. *Let $f = (V_{-1}, V_1, \ldots, V_{1+\lceil \frac{\Delta}{2} \rceil})$ be an SSTRDF on a graph G of order n. Let $\delta = \delta(G)$ and $\Delta = \Delta(G)$. Then the following holds:*

(i) $\sum_{i=1}^{1+\lceil \frac{\Delta}{2} \rceil} (i\Delta - 1)|V_i| \geq (\delta + 1)|V_{-1}|$.

(ii) $\sum_{i=1}^{1+\lceil \frac{\Delta}{2} \rceil} (i\Delta + \delta)|V_i| \geq (\delta + 1)n$.

Proof. (i) We have

$$\begin{aligned} n &\le \sum_{v\in V} f(N(v)) = \sum_{v\in V} d(v)f(v) \\ &= \sum_{i=1}^{1+\lceil \frac{\Delta}{2}\rceil} \sum_{v\in V_i} id(v) - \sum_{v\in V_{-1}} d(v) \\ &\le \sum_{i=1}^{1+\lceil \frac{\Delta}{2}\rceil} i\Delta|V_i| - \delta|V_{-1}| \end{aligned}$$

by Observation 2.1 (a), the desired result follows.

(ii) It follows immediately from Part (i) by substituting $|V_{-1}| = n - \sum_{i=1}^{1+\lceil \frac{\Delta}{2}\rceil} |V_i|$. □

Observation 2.3. *For any connected graph G with $\Delta \le 2$, $\gamma_{ssTR}(G) = \gamma_{stR}(G)$.*

We make use of the following results which have been proved in [9].

Proposition A. [9] *For $n \ge 3$,*

$$\gamma_{stR}(C_n) = \begin{cases} \frac{n}{2} & n \equiv 0 \pmod 4 \\ \frac{n+3}{2} & n \equiv 1,3 \pmod 4 \\ \frac{n+6}{2} & n \equiv 2 \pmod 4 \end{cases}$$

and

$$\gamma_{stR}(P_n) = \begin{cases} \frac{n}{2} & n \equiv 0 \pmod 4 \\ \lceil\frac{n+3}{2}\rceil & \text{otherwise.} \end{cases}$$

By Observation 2.3 and Proposition A, we have the following corollary.

Corollary 2.4. For $n \ge 3$,

$$\gamma_{ssTR}(C_n) = \begin{cases} \frac{n}{2} & n \equiv 0 \pmod 4 \\ \frac{n+3}{2} & n \equiv 1,3 \pmod 4 \\ \frac{n+6}{2} & n \equiv 2 \pmod 4 \end{cases}$$

and

$$\gamma_{ssTR}(P_n) = \begin{cases} \frac{n}{2} & n \equiv 0 \pmod 4 \\ \lceil\frac{n+3}{2}\rceil & \text{otherwise.} \end{cases}$$

The signed strong total Roman domination number is well-defined for all graphs G without isolated vertices. Thus we assume throughout this paper that $\delta(G) \ge 1$.

Observation 2.5. *Let G be a graph of order n. Then $\gamma_{ssTR}(G) \le n$, and this bound is tight.*

Proposition 2.6. *Let G be a graph of order n. Then $\gamma_{ssTR}(G) \ge 2\gamma_t(G) - n$.*

Proof. Suppose that f is a $\gamma_{ssTR}(G)$-function on G. Then by Observation 2.1, one has

$$\begin{aligned}\gamma_{ssTR}(G) &= \sum_{i=1}^{1+\lceil\frac{\Delta}{2}\rceil} i|V_i| - |V_{-1}| \\ &= \sum_{i=1}^{1+\lceil\frac{\Delta}{2}\rceil} (i+1)|V_i| - n \\ &\geq 2\sum_{i=1}^{1+\lceil\frac{\Delta}{2}\rceil} |V_i| - n \\ &\geq 2\gamma_t(G) - n.\end{aligned}$$

□

3 Bounds on the signed strong total Roman domination number

In this section, we present some sharp bounds for the signed strong total Roman domination number of graphs in terms of several parameters.

The next theorem gives a simple lower bound for the signed strong total Roman domination number using order of a graph, maximum and minimum degree.

Theorem 3.1. Let G be a graph of order n with maximum degree Δ and minimum degree δ. Then $\gamma_{ssTR}(G) \geq \frac{n(\delta+1)}{\Delta} - n$. Moreover, this bound is sharp.

Proof. Suppose that f is a signed strong total Roman dominating function for G. Define $g : V(G) \to \{0, 2, 3, \dots, \lceil\frac{\Delta}{2}\rceil + 2\}$ by $g(v) = f(v) + 1$. So we have

$$\begin{aligned}\sum_{x\in V(G)} g(N(x)) &= \sum_{x\in V(G)} (d(x) + f(N(x)) \\ &\geq \sum_{x\in V(G)} (\delta+1) = n(\delta+1).\end{aligned}$$

One also has

$$\begin{aligned}\sum_{x\in V(G)} g(N(x)) &= \sum_{x\in V(G)} d(x)g(x) \\ &\leq \sum_{x\in V(G)} \Delta g(x) \\ &=\Delta \sum_{x\in V(G)} g(x) = \Delta g(V).\end{aligned}$$

Since $f(V) = g(V) - n$, we obtain

$$f(V) \geq \frac{\sum_{x\in V(G)} g(N(x))}{\Delta} - n \geq \frac{n(\delta+1)}{\Delta} - n.$$

This implies that $\gamma_{ssTR}(G) \geq \frac{n(\delta+1)}{\Delta} - n$. This Bound occurs for C_n, $n \equiv 0 \pmod 4$, by Corollary 2.4. □

As an immediate consequence of Theorem 3.1, we obtain a lower bound on the signed strong Roman domination number of a regular graph.

Corollary 3.2. If G is an r-regular graph of order n, then $\gamma_{ssTR}(G) \geq \frac{n}{r}$.

We propose a so called Nordhaus-Gaddum type inequality for the signed strong total Roman domination number of regular graphs. The proof of next result is similar to the proof of [3, Theorem 6] and therefore it is omitted.

Theorem 3.3. *Let G be an r-regular graph of order n. Then*

$$\gamma_{ssTR}(G) + \gamma_{ssTR}(\overline{G}) \geq \begin{cases} \frac{4n}{n-1} & \text{if } n \text{ is odd} \\ \frac{4(n-1)}{n-2} & \text{if } n \text{ is even.} \end{cases}$$

Proposition 3.4. If G is a graph of order n with maximum degree Δ, then $\gamma_{ssTR}(G) \geq 1 + \Delta - n$.

Proof. Let f be a $\gamma_{ssTR}(G)$-function and v be a vertex of degree Δ. Since $f(N(v)) \geq 1$, we have

$$\gamma_{ssTR}(G) = \omega(f) \geq f(N(v)) - 1 - (n - \Delta - 1) \geq 1 + \Delta - n.$$

□

The next results give a lower and upper bound for the signed strong total Roman domination number using 2-packing number.

Proposition 3.5. If G is a graph of order n with $\delta \geq 1$, then $\gamma_{ssTR}(G) \geq \rho(G)(\delta + 1) - n$.

Proof. Let $\{v_1, v_2, \ldots, v_{\rho(G)}\}$ be a 2-packing of G, and let f be a $\gamma_{ssTR}(G)$-function. Assume that $A = \bigcup_{i=1}^{\rho(G)} N(v_i)$. Since $\{v_1, v_2, \ldots, v_{\rho(G)}\}$ is a 2-packing, one has

$$|A| = \sum_{i=1}^{\rho(G)} d(v_i) \geq \rho(G)\delta.$$

So we have

$$\begin{aligned}\gamma_{ssTR}(G) &= \sum_{x \in V(G)} f(x) = \sum_{i=1}^{\rho(G)} f(N(v_i)) + \sum_{x \in V(G)-A} f(x) \\ &\geq \rho(G) + \sum_{x \in V(G)-A} f(x) \geq \rho(G) - (n - |A|) \\ &\geq \rho(G) - n + \rho(G)\delta \\ &= \rho(G)(\delta + 1) - n.\end{aligned}$$

□

Proposition 3.6. *Let G be a graph of order n with minimum degree $\delta \geq 3$. Then $\gamma_{ssTR}(G) \leq n - \rho$.*

Proof. Suppose that $S = \{v_1, v_2, \ldots, v_\rho\}$ is a 2-packing set of G. For $1 \le i \le \rho$, choose an element $u_i \in N_G(v_i)$ and define f as follows:

$$f(x) = \begin{cases} -1 & x \in S \\ 2 & x = u_i \text{ and } \quad i = 1, \ldots, \rho \\ +1 & \text{otherwise.} \end{cases}$$

We have $\delta \ge 3$, so clearly, f is a signed strong total Roman dominating function on G. Since S is a 2-packing set, one has

$$\gamma_{ssTR}(G) \le \omega(f) = -\rho + 2\rho + n - 2\rho = n - \rho$$

as desired. □

The following corollary is an immediate consequence of Proposition 3.6.

Corollary 3.7. Let G be a cubic graph of order n different from the Peterson graph. Then $\gamma_{ssTR}(G) \le \lceil \frac{7n}{8} \rceil$.

Proof. By [4, Lemma], G contains a 2-packing set of at least $\frac{n}{8}$ vertices which in turn implies that $\gamma_{ssTR}(G) \le n - \lfloor \frac{n}{8} \rfloor = \lceil \frac{7n}{8} \rceil$, by Proposition 3.6. □

We next present a lower bound for signed strong total Roman domination number with regard to the diameter.

Proposition 3.8. *Let G be a graph of order n with $\delta \ge 1$. Then*

$$\gamma_{ssTR}(G) \ge (\delta + 1)(1 + \lfloor \frac{\text{diam}(G)}{3} \rfloor) - n.$$

Proof. Suppose that $v_0, \ldots, v_{\text{diam}(G)}$ is a diametral path, $\text{diam}(G) = 3t + r$ with integers $t \ge 0$ and $0 < r \le 2$. It is easy to see that $A = \{v_0, v_3, \ldots, v_{3t}\}$ is a 2-packing set of G such that $|A| = 1 + \lfloor \frac{\text{diam}(G)}{3} \rfloor$. Then we have $\rho \ge |A|$. So by Proposition 3.5, one has

$$\gamma_{ssTR}(G) \ge (\delta + 1)\rho - n \ge (\delta + 1)|A| - n = (\delta + 1)(1 + \lfloor \frac{\text{diam}(G)}{3} \rfloor) - n.$$

□

The following proposition bounds the signed strong total Roman domination number in terms of the maximum degree, when $\gamma(G) = 1$.

Proposition 3.9. *Let G be a graph and $\gamma(G) = 1$. Then $0 \le \gamma_{ssTR}(G) \le 2 + \lceil \frac{\Delta}{2} \rceil$. Moreover, these bounds are sharp.*

Proof. Suppose that $\{v\}$ is an arbitrary $\gamma(G)$-set. Hence $d(v) = \Delta = n - 1$. Let $V(G) = \{v, v_1, v_2, \ldots, v_\Delta\}$. Suppose first that Δ is even and define $f : V(G) \to \{-1, 1, 2, \ldots, \lceil \frac{\Delta}{2} \rceil + 1\}$ by $f(v) = 1 + \lceil \frac{\Delta}{2} \rceil$, $f(v_i) = (-1)^i$ for $1 \le i \le n-2$ and $f(v_{n-1}) = 2$. Assume now that Δ is odd and define $f : V(G) \to \{-1, 1, 2, \ldots, \lceil \frac{\Delta}{2} \rceil + 1\}$ by $f(v) = 1 + \lceil \frac{\Delta}{2} \rceil$ and $f(v_i) = (-1)^i$ for $1 \le i \le n-2$ and $f(v_{n-1}) = +1$. Clearly in each cases f is a signed strong total Roman dominating function for G and one has

$$\gamma_{ssTR}(G) \le \omega(f) = 2 + \lceil \frac{\Delta}{2} \rceil.$$

To prove the left inequality, assume that g is a $\gamma_{ssTR}(G)$-function. Then one has

$$\gamma_{ssTR}(G) = \omega(g) = g(v) + g(N(v)) \geq 0.$$

□

We next present a trivial necessary and sufficient condition for a graph G with $\gamma(G) = 1$ such that $\gamma_{ssTR}(G) = 2 + \lceil \frac{\Delta(G)}{2} \rceil$.

Proposition 3.10. Let G be a graph of order n and $\gamma(G) = 1$. Then $\gamma_{ssTR}(G) = 2+\lceil \frac{\Delta(G)}{2} \rceil$ if and only if there exists a $\gamma_{ssTR}(G)$-function $f = (V_{-1}, V_1, V_2, \dots, V_{1+\lceil \frac{\Delta(G)}{2} \rceil})$ such that $\bigcup_{i=2}^{\lceil \frac{\Delta(G)}{2} \rceil} V_i = \emptyset$, $|V_{-1}| = |V_1| - 1$ when n is even and $\bigcup_{i=3}^{\lceil \frac{\Delta(G)}{2} \rceil} V_i = \emptyset$ and $|V_{-1}| = |V_1| + 1$ when n is odd.

Proof. Suppose that n is even and there exists a $\gamma_{ssTR}(G)$-function $f = (V_{-1}, V_1, \dots, V_{\lceil \frac{\Delta(G)}{2} \rceil+1})$ such that $\bigcup_{i=2}^{\lceil \frac{\Delta(G)}{2} \rceil} V_i = \emptyset$, $|V_{-1}| = |V_1| - 1$. Hence one has

$$\gamma_{ssTR}(G) = |V_1| - |V_{-1}| + (1 + \lceil \frac{\Delta(G)}{2} \rceil)|V_{1+\lceil \frac{\Delta(G)}{2} \rceil}| \geq 2 + \lceil \frac{\Delta(G)}{2} \rceil.$$

By Proposition 3.9, we have $\gamma_{ssTR}(G) = 2 + \lceil \frac{\Delta(G)}{2} \rceil$. Assume now that n is odd and there exists a $\gamma_{ssTR}(G)$-function $f = (V_{-1}, V_1, \dots, V_{\lceil \frac{\Delta(G)}{2} \rceil+1})$ such that $\bigcup_{i=3}^{\lceil \frac{\Delta(G)}{2} \rceil} V_i = \emptyset$ and $|V_{-1}| = |V_1| + 1$. So

$$\gamma_{ssTR}(G) = |V_1| - |V_{-1}| + 2|V_2| + (1 + \lceil \frac{\Delta(G)}{2} \rceil)|V_{1+\lceil \frac{\Delta}{2} \rceil}| \geq 2 + \lceil \frac{\Delta(G)}{2} \rceil.$$

By Proposition 3.9, this yields $\gamma_{sstR}(G) = 2 + \lceil \frac{\Delta(G)}{2} \rceil$. Conversely is clear according to the proof of the Proposition 3.9. □

Proposition 3.11. If T is a tree of order n and maximum degree $\Delta(T) \geq 2$, then

$$\gamma_{ssTR}(G) \geq \Delta(T) + 4 - n.$$

Moreover this bound is sharp for graphs $K_{1,3}$ and $K_{1,4}$.

Proof. Let $f = (V_{-1}, V_1, \dots, V_{1+\lceil \frac{\Delta(T)}{2} \rceil})$ be a $\gamma_{ssTR}(T)$-function, v a vertex of maximum degree $\Delta(T)$ and $\Delta_i = V_i \bigcap N(v)$ for $i = \{-1, 1, \dots, 1 + \lceil \frac{\Delta(T)}{2} \rceil\}$. Suppose first that $f(v) = 1$. Each vertex x in Δ_{-1} must have a neighbor x' with label at least two. Note that if $y \neq x \in \Delta_{-1}$, then $y' \neq x'$. Thus we have

$$\begin{aligned}
\gamma_{ssTR}(G) &\geq f(v) + \sum_{i=2}^{1+\lceil \frac{\Delta(T)}{2} \rceil} i\Delta_i - \Delta_{-1} + \Delta_1 + 2\Delta_{-1} - (n - (\Delta(T) + 1 + \Delta_{-1})) \\
&\geq 1 + 2 \sum_{i=2}^{1+\lceil \frac{\Delta(T)}{2} \rceil} \Delta_i + 2\Delta_{-1} + \Delta_1 - n + \Delta(T) + 1 \\
&\geq 1 + \Delta(T) - n + \Delta(T) + 1 \\
&\geq \Delta(T) - n + 4.
\end{aligned}$$

Assume that $f(v) \geq 2$. Then we have

$$\begin{aligned}\gamma_{ssTR}(G) &\geq f(v) + \sum_{i=2}^{1+\lceil\frac{\Delta(T)}{2}\rceil} i\Delta_i - \Delta_{-1} + \Delta_1 - (n - (\Delta(T)+1)) \\ &\geq 2 + 1 - n + \Delta(T) + 1 \\ &\geq \Delta(T) - n + 4.\end{aligned}$$

Finally, suppose that $f(v) = -1$ and note that each vertex x in $N(v)$ have a neighbor x' with label at least two. Note that if $y \neq x \in \Delta_{-1}$, then $y' \neq x'$. Thus we have

$$\begin{aligned}\gamma_{ssTR}(G) &\geq f(v) + \sum_{i=2}^{1+\lceil\frac{\Delta(T)}{2}\rceil} i\Delta_i - \Delta_{-1} + \Delta_1 + 2\Delta(T) - (n - (2\Delta(T)+1)) \\ &\geq 2\Delta(T) - n + 2\Delta(T) + 1 \\ &= 4\Delta(T) - n + 1 \\ &> \Delta(T) - n + 4.\end{aligned}$$

as desired.

□

In the following theorem, a lower bound is presented for signed strong total Roman domination number in terms of order and size of a graph.

Theorem 3.12. Let G be a graph with minimum degree $\delta \geq 1$. Then

$$\gamma_{ssTR}(G) \geq \frac{17 + 11\lceil\frac{\Delta}{2}\rceil}{1 + \lceil\frac{\Delta}{2}\rceil} n - 12m.$$

Where n and m are order and size of graph, respectively.

Proof. Suppose that $f = (V_{-1}, V_1, V_2, \ldots, V_{1+\lceil\frac{\Delta}{2}\rceil})$ is an $\gamma_{ssTR}(G)$-function. Every vertex in V_{-1} is adjacent to at least one vertex in $B = \bigcup_{i=2}^{1+\lceil\frac{\Delta}{2}\rceil} V_i$. So one has

$$n_{-1} = |V_{-1}| \leq |[V_{-1}, B]| = \sum_{v \in G[B]} d_{V_{-1}}(v). \tag{3.1}$$

On the other hand, for every vertex v in $G[B]$ we have

$$1 \leq f(N(v)) \leq (1 + \lceil\frac{\Delta}{2}\rceil)d_B(v) + d_{V_1}(v) - d_{V_{-1}}(v). \tag{3.2}$$

Remember that m_B and n_B are size and order of $G[B]$, respectively. As mentioned in the introduction, $V_{1B} = V_1 \cup B$. Note that $m_1 + m_B + |[V_1, B]| = m_{1B}$ and $n_{1B} = n_1 + n_B$. Hence from (3.1) and (3.2), we obtain the following inequality

$$\begin{aligned}n_{-1} &\leq \sum_{v \in G[B]} d_{V_{-1}}(v) \\ &\leq \sum_{v \in G[B]} ((1 + \lceil\frac{\Delta}{2}\rceil)d_B(v) + d_{V_1}(v) - 1)\end{aligned}$$

$$= \sum_{v\in G[B]}(1+\lceil\frac{\Delta}{2}\rceil)d_B(v) + \sum_{v\in G[B]} d_{V_1}(v) + \sum_{v\in G[B]}(-1)$$
$$\leq 2(1+\lceil\frac{\Delta}{2}\rceil)m_B + |[V_1,B]| - n_B$$
$$= 2(1+\lceil\frac{\Delta}{2}\rceil)m_{1B} - 2(1+\lceil\frac{\Delta}{2}\rceil)m_1 + (-2(1+\lceil\frac{\Delta}{2}\rceil)+1)|[V_1,B]| - n_B.$$

It follows that

$$m_{1B} \geq \frac{n_{-1}+n_B+2(1+\lceil\frac{\Delta}{2}\rceil)m_1 - (-2(1+\lceil\frac{\Delta}{2}\rceil)+1)|[V_1,B]|}{2(1+\lceil\frac{\Delta}{2}\rceil)}.$$

On the other hand, one has $m = m_{1B} + |[V_{-1}, V_{1B}]| + m_{-1}$ and $n = n_{-1} + n_{1B}$. Hence we obtain the following inequality

$$m \geq m_{1B} + |[V_{-1}, V_{1B}]|$$
$$\geq \frac{1}{2(1+\lceil\frac{\Delta}{2}\rceil)}(n_{-1}+n_B+2(1+\lceil\frac{\Delta}{2}\rceil)m_1 - (-2(1+\lceil\frac{\Delta}{2}\rceil)+1)|[V_1,B]|) + n_{-1}$$
$$= \frac{1}{2(1+\lceil\frac{\Delta}{2}\rceil)}((2(1+\lceil\frac{\Delta}{2}\rceil)+1)n_{-1}+n_{1B}-n_1+2(1+\lceil\frac{\Delta}{2}\rceil)m_1 - (-2(1+\lceil\frac{\Delta}{2}\rceil)+1)|[V_1,B]|)$$
$$= \frac{1}{2(1+\lceil\frac{\Delta}{2}\rceil)}((2(1+\lceil\frac{\Delta}{2}\rceil)+1)(n-n_{1B})+n_{1B}-n_1+2(1+\lceil\frac{\Delta}{2}\rceil)m_1$$
$$+ (2(1+\lceil\frac{\Delta}{2}\rceil)-1)|[V_1,B]|)$$
$$\geq \frac{1}{2(1+\lceil\frac{\Delta}{2}\rceil)}((2(1+\lceil\frac{\Delta}{2}\rceil)+1)n-n_1+2(1+\lceil\frac{\Delta}{2}\rceil)m_1 + (2(1+\lceil\frac{\Delta}{2}\rceil)-1)|[V_1,B]|) - \frac{1}{4}n_{1B}.$$

This means that

$$n_{1B} \geq \frac{4}{2(1+\lceil\frac{\Delta}{2}\rceil)}(2(1+\lceil\frac{\Delta}{2}\rceil)+1)n-n_1+2(1+\lceil\frac{\Delta}{2}\rceil)m_1+(2(1+\lceil\frac{\Delta}{2}\rceil)-1)|[V_1,B]|-(2(1+\lceil\frac{\Delta}{2}\rceil))m).$$

It yields that

$$\gamma_{ssR}(G) \geq 2n_B + n_1 - n_{-1}$$
$$= 3n_{1B} - n - n_1$$
$$\geq 3\frac{4}{2(1+\lceil\frac{\Delta}{2}\rceil)}((2(1+\lceil\frac{\Delta}{2}\rceil)+1)n - n_1$$
$$+ 2(1+\lceil\frac{\Delta}{2}\rceil)m_1 + (2(1+\lceil\frac{\Delta}{2}\rceil)-1)|[V_1,B]| - (2(1+\lceil\frac{\Delta}{2}\rceil))m) - n - n_1$$
$$= \frac{17+11\lceil\frac{\Delta}{2}\rceil}{1+\lceil\frac{\Delta}{2}\rceil}n - 12m + \frac{-7-\lceil\frac{\Delta}{2}\rceil}{1+\lceil\frac{\Delta}{2}\rceil}n_1 + 12m_1 + \frac{6+12\lceil\frac{\Delta}{2}\rceil}{1+\lceil\frac{\Delta}{2}\rceil}|[V_1,B]|.$$

To complete the proof, we claim that $\frac{-7-\lceil\frac{\Delta}{2}\rceil}{1+\lceil\frac{\Delta}{2}\rceil}n_1 + 12m_1 + \frac{6+12\lceil\frac{\Delta}{2}\rceil}{1+\lceil\frac{\Delta}{2}\rceil}|[V_1,B]| \geq 0$. Suppose first that $n_1 = 0$. Then obviously the assertion holds. So assume that $n_1 \geq 1$. Since $\delta \geq 1$, if $v \in V_1$ and $d_{V_{1B}}(v) = 0$, then every neighbor of v belongs to V_{-1}. Hence $f(N(v)) \leq -1$ and this is a contradiction. Therefore $d_{V_{1B}}(v) > 0$, for every $v \in V_1$. So we have

$$\frac{-7-\lceil\frac{\Delta}{2}\rceil}{1+\lceil\frac{\Delta}{2}\rceil}n_1 + 12m_1 + \frac{6+12\lceil\frac{\Delta}{2}\rceil}{1+\lceil\frac{\Delta}{2}\rceil}|[V_1,B]|$$

$$
\begin{aligned}
&= 12m_1 + 6|[V_1, B]| + \frac{6\lceil\frac{\Delta}{2}\rceil}{1+\lceil\frac{\Delta}{2}\rceil}|[V_1, B]| \\
&- n_1 - \frac{6}{1+\lceil\frac{\Delta}{2}\rceil}n_1 \\
&\geq 6\sum_{v\in V_1} d_{V_1}(v) + 6\sum_{v\in V_1} d_B(v) + \frac{6\lceil\frac{\Delta}{2}\rceil}{1+\lceil\frac{\Delta}{2}\rceil}|[V_1, B]| - 4n_1 \\
&\geq 6\sum_{v\in V_1} d_{V_{1B}}(v) + \frac{6\lceil\frac{\Delta}{2}\rceil}{1+\lceil\frac{\Delta}{2}\rceil}|[V_1, B]| - 4n_1 \\
&\geq 2n_1 + \frac{6\lceil\frac{\Delta}{2}\rceil}{1+\lceil\frac{\Delta}{2}\rceil}|[V_1, B]| \\
&> 0.
\end{aligned}
$$

□

4 Special values of signed strong Roman domination number

In this section, we determine the signed strong total Roman domination number of special classes of graphs including star graphs and complete graphs.

Proposition 4.1. *For $n \geq 2$, $\gamma_{ssTR}(K_{1,n-1}) = 2 + \lceil\frac{n-2}{4}\rceil$.*

Proof. Let $G = K_{1,n-1}$ and $V(G) = \{w, v_1, v_2, \ldots, v_{n-1}\}$. Suppose that w is central vertex of the star graph. Consider the following two cases:
Case 1. Suppose first that n is odd and define $f : V(K_{1,n-1}) \longrightarrow \{-1, 1, 2, \ldots, 1 + \lceil\frac{n-1}{2}\rceil\}$ as follows:

$$
f(x) = \begin{cases} 1 + \lceil\frac{n-2}{4}\rceil & x = w \\ 2 & x = v_{n-1} \\ (-1)^{i+1} & x = v_i \quad \text{and} \quad 1 \leq i \leq n-2. \end{cases}
$$

Obviously, f is a signed strong total Roman dominating function for $K_{1,n-1}$ of weight $\omega(f) = 2 + \lceil\frac{n-1}{4}\rceil$.
Case 2. Assume now that n is even. Define $f : V(K_{1,n-1}) \longrightarrow \{-1, 1, 2, \ldots, 1 + \lceil\frac{n-1}{2}\rceil\}$ as follows:

$$
f(x) = \begin{cases} 1 + \lceil\frac{n-2}{4}\rceil & x = w \\ 1 & x = v_{n-1} \\ (-1)^{i+1} & x = v_i \quad \text{and} \quad 1 \leq i \leq n-2. \end{cases}
$$

One can see easily that f is a signed strong total Roman dominating function for $K_{1,n-1}$ of weight $\omega(f) = 2 + \lceil\frac{n-2}{4}\rceil$.

The proof is completed by showing that this inequality becomes an equality. To this, suppose that g is an $\gamma_{ssTR}(K_{1,n-1})$-function, for $n \geq 2$. If $|V_{-1}| = 0$, then one has $\omega(g) = n \geq 2 + \lceil\frac{n-2}{4}\rceil$ and we are done. So assume that $|V_{-1}| > 0$. Since g is a signed strong total Roman dominating function, it is clear that $g(w) \geq 1$. Let $|V_{-1} \cap \{v_1, \ldots, v_{n-1}\}| = t$. Then one has $g(w) \geq 1 + \lceil\frac{t}{2}\rceil$ and $\omega(g) = g(N(w)) + g(w) \geq 2 + \lceil\frac{t}{2}\rceil$. If $t \geq \lceil\frac{n-1}{2}\rceil$, then

$$
\omega(g) \geq 2 + \lceil\frac{t}{2}\rceil \geq 2 + \lceil\frac{n-1}{4}\rceil \geq 2 + \lceil\frac{n-2}{4}\rceil
$$

and we are done.

Suppose thus $t < \lceil\frac{n-1}{2}\rceil$. In this case

$$\omega(g) \geq 1 + \lceil\frac{t}{2}\rceil + (n-t-1) + (-t) \geq n - \lfloor\frac{3t}{2}\rfloor.$$

Consider the following two subcases.

Subcase 1. Assume that n is even. Hence $t \leq \frac{n}{2} - 1$, so $\lfloor\frac{3t}{2}\rfloor \leq \lfloor\frac{3n}{4}\rfloor - 2$. Then

$$\omega(g) \geq n - \lfloor\frac{3t}{2}\rfloor \geq \lceil\frac{n}{4}\rceil + 2 \geq \lceil\frac{n-2}{4}\rceil + 2.$$

Subcase 2. Assume now that n is odd. Hence $t \leq \frac{n-1}{2} - 1$, so $\lfloor\frac{3t}{2}\rfloor \leq \lfloor\frac{3n}{4}\rfloor - 3$. Then

$$\omega(g) \geq n - \lfloor\frac{3t}{2}\rfloor \geq \lceil\frac{n}{4}\rceil + 3 \geq \lceil\frac{n-2}{4}\rceil + 2$$

and the assertion holds. □

Proposition 4.2. *For $n \geq 4$, $\gamma_{ssTR}(K_n) = \begin{cases} 1 + \lceil\frac{n-2}{4}\rceil & n \ is \ even \\ 1 + \lceil\frac{n-3}{4}\rceil & n \ is \ odd. \end{cases}$*

Proof. Let $G = K_n$ and $V(G) = \{v_1, v_2, \dots, v_n\}$. Suppose first that n is even. Then the function $f : V(G) \longrightarrow \{-1, 1, 2, \dots, 1 + \lceil\frac{n-1}{2}\rceil\}$ define by $f(v_1) = 1 + \lceil\frac{n-2}{4}\rceil$, $f(v_n) = 1$ and $f(v_j) = (-1)^{j+1}$ for $2 \leq j \leq n-1$, is a signed strong total Roman dominating function on K_n of weight $1 + \lceil\frac{n-2}{4}\rceil$. Assume now that n is odd. Then the function $g : V(G) \longrightarrow \{-1, 1, 2, \dots, 1 + \lceil\frac{n-1}{2}\rceil\}$ define by $g(v_1) = 1 + \lceil\frac{n-3}{4}\rceil$, $g(v_2) = g(v_n) = 1$ and $g(v_j) = (-1)^j$ for $3 \leq j \leq n-1$, is a signed strong total Roman dominating function on K_n of weight $1 + \lceil\frac{n-3}{4}\rceil$. Hence For $n \geq 4$,

$$\gamma_{ssTR}(K_n) \leq \begin{cases} 1 + \lceil\frac{n-2}{4}\rceil & n \ is \ even \\ 1 + \lceil\frac{n-3}{4}\rceil & n \ is \ odd. \end{cases}$$

The proof is completed by showing that this inequality becomes an equality. To this, suppose that h is an $\gamma_{ssTR}(K_n)$-function, for $n \geq 2$. If $|V_{-1}| = 0$, then one has

$$\omega(h) = n \geq 2 + \lceil\frac{n-2}{4}\rceil$$

and we are done. So assume that $|V_{-1}| = t > 0$. Without loss of generality, let $h(v_1) = -1$. So there exist a vertex v_j, $2 \leq j \leq n$, such that $h(v_j) \geq 1 + \lceil\frac{t}{2}\rceil$. One has

$$\omega(h) = h(N(v_j)) + h(v_j) \geq 2 + \lceil\frac{t}{2}\rceil.$$

If $t \geq \lceil\frac{n-1}{2}\rceil$, then

$$\omega(h) \geq 2 + \lceil\frac{t}{2}\rceil \geq 2 + \lceil\frac{n-1}{4}\rceil \geq 1 + \lceil\frac{n-2}{4}\rceil$$

and we are done.

Suppose thus $t < \lceil\frac{n-1}{2}\rceil$. In this case

$$\omega(h) \geq 1 + \lceil\frac{t}{2}\rceil + (n-t-1) + (-t) \geq n - \lfloor\frac{3t}{2}\rfloor.$$

Consider the following two cases.
Case 1. Assume that n is even. Hence $t \leq \frac{n}{2} - 1$, so $\lfloor \frac{3t}{2} \rfloor \leq \lfloor \frac{3n}{4} \rfloor - 2$. Then

$$\omega(h) \geq n - \lfloor \frac{3t}{2} \rfloor \geq \lceil \frac{n}{4} \rceil + 2 \geq 1 + \lceil \frac{n-2}{4} \rceil.$$

Case 2. Assume now that n is odd. Hence $t \leq \frac{n-1}{2} - 1$, so $\lfloor \frac{3t}{2} \rfloor \leq \lfloor \frac{3n}{4} \rfloor - 3$. Then

$$\omega(h) \geq n - \lfloor \frac{3t}{2} \rfloor \geq \lceil \frac{n}{4} \rceil + 3 \geq 1 + \lceil \frac{n-3}{4} \rceil$$

and the assertion holds. □

Proposition 4.3. *For* $n \geq 1$, $\gamma_{ssTR}(K_{n,n}) = 2$.

Proof. Let $X = \{u_1, u_2, \ldots, u_n\}$ and $Y = \{v_1, \ldots, v_n\}$ be the partite sets of $K_{n,n}$ for $n \geq 1$. Consider the following two cases:
Case 1. Suppose first that n is odd. Define $f : V(K_{n,n}) \longrightarrow \{-1, 1, 2, \ldots, 1 + \lceil \frac{\Delta}{2} \rceil\}$ by $f(u_i) = f(v_i) = -1$ for $1 \leq i \leq n-2$, $f(u_{n-1}) = f(v_{n-1}) = \frac{n-1}{2} + 1$ and $f(u_n) = f(v_n) = \frac{n-1}{2} - 1$.

Obviously, f is a signed strong total Roman dominating function on $K_{n,n}$ of weight $\omega(f) = 2$.
Case 2. Assume now that n is even. Define $f : V(K_{n,n}) \longrightarrow \{-1, 1, 2, \ldots, 1 + \lceil \frac{\Delta}{2} \rceil\}$ by $f(u_i) = f(v_i) = -1$ for $1 \leq i \leq n-2$, $f(u_{n-1}) = f(v_{n-1}) = \frac{n-2}{2} + 1$ and $f(u_n) = f(v_n) = \frac{n-2}{2}$.

One can see easily that f is a signed strong total Roman dominating function on $K_{n,n}$ of weight $\omega(f) = 2$. Therefore for $n \geq 1$, one has $\gamma_{ssTR}(K_{n,n}) \leq 2$.

The proof is completed by showing that this inequality becomes an equality. To this, suppose that g is a $\gamma_{ssTR}(K_{n,n})$-function, for $n \geq 1$. By definition, we have $g(N(u_1)) \geq 1$ and $g(N(v_1)) \geq 1$. Hence $\omega(g) \geq 2$ and the assertion holds. □

Proposition 4.4. *For* $2 \leq r \leq s$, $\gamma_{ssTR}(K_{r,s}) \leq \begin{cases} 2 & r - \frac{s}{2} \geq 2 \\ \lceil \frac{s-2}{2} \rceil - r + 3 & r - \frac{s}{2} < 2. \end{cases}$

Proof. Let $X = \{u_1, \ldots, u_s\}$ and $Y = \{v_1, \ldots, v_r\}$ be the partite sets of $K_{r,s}$. Consider the following two cases:
Case 1. Suppose that $r \geq \frac{s}{2} + 2$ and consider the following two subcases:

Subcase 1.1. Suppose first that s is even. We define $f : V(K_{r,s}) \longrightarrow \{-1, 1, 2, \ldots, 1 + \lceil \frac{s}{2} \rceil\}$ by $f(v_i) = (-1)$ for $1 \leq i \leq r-2$, $f(v_{r-1}) = \frac{s-2}{2} + 1$, $f(v_r) = r - 2 - (\frac{s-2}{2})$, $f(u_i) = -1$ for $1 \leq i \leq s-2$, $f(u_{s-1}) = \frac{s-2}{2}$ and $f(u_s) = \frac{s-2}{2} + 1$. Clearly f is a signed strong total Roman dominating function on $K_{r,s}$ and $\omega(f) = 2$.

Subcase 1.2. Assume now that s is odd. We define $f : V(K_{r,s}) \longrightarrow \{-1, 1, 2, \ldots, 1 + \lceil \frac{s}{2} \rceil\}$ by $f(v_i) = (-1)$ for $1 \leq i \leq r-2$, $f(v_{r-1}) = \frac{s-1}{2} + 1$, $f(v_r) = r - 2 - (\frac{s-1}{2})$, $f(u_i) = -1$ for $1 \leq i \leq s-2$, $f(u_{s-1}) = \frac{s-1}{2}$ and $f(u_s) = \frac{s-1}{2}$. It is easy to see that f is a signed strong total Roman dominating function on $K_{r,s}$ and $\omega(f) = 2$.

Case 2. Let $r < \frac{s}{2} + 2$ and consider the following two subcases:

Subcase 2.1. Suppose first that s is even. We define $f : V(K_{r,s}) \longrightarrow \{-1, 1, 2, \ldots, 1 + \lceil \frac{s}{2} \rceil\}$ by $f(v_i) = (-1)$ for $1 \leq i \leq r-1$, $f(v_r) = \frac{s-2}{2} + 1$, $f(u_i) = -1$ for $1 \leq i \leq s-2$,

$f(u_{s-1}) = \frac{s-2}{2}$ and $f(u_s) = \frac{s-2}{2} + 1$. Then f is a signed strong total Roman dominating function and $\omega(f) = \lceil \frac{s-2}{2} \rceil - r + 3$.

Subcase 2.2. Assume that s is odd. We define $f : V(K_{r,s}) \longrightarrow \{-1, 1, 2, \ldots, 1 + \lceil \frac{s}{2} \rceil\}$ by $f(v_i) = (-1)$ for $1 \le i \le r-1$, $f(v_r) = \frac{s-1}{2} + 1$, $f(u_i) = -1$ for $1 \le i \le s-2$, $f(u_{s-1}) = \frac{s-1}{2}$ and $f(u_s) = \frac{s-1}{2}$. Clearly f is a signed strong total Roman dominating function and $\omega(f) = \lceil \frac{s-2}{2} \rceil - r + 3$. □

Proposition 4.5. *For* $1 \le r \le s$, $\gamma_{ssTR}(DS_{r,s}) \le \lceil \frac{s}{2} \rceil - r + 4$.

Proof. Suppose that u and v are non-leaf vertices. Let $u_1, \ldots, u_r$ and $v_1, \ldots, v_s$ are leaves adjacent to u and v, respectively. Consider the following three cases:

Case 1. Assume that $r = \lceil \frac{s}{2} \rceil + 1$ and consider the following two subcases:

Subcase 1.1. Suppose first that s is even. We define $f : V(DS_{r,s}) \longrightarrow \{-1, 1, 2, \ldots, 1 + \lceil \frac{s}{2} \rceil\}$ by $f(v_i) = -1$ for $1 \le i \le s-1$, $f(v_s) = \frac{s}{2}$, $f(v) = \frac{s}{2} + 2$, $f(u_i) = -1$ for $1 \le i \le r$, $f(u) = \frac{s}{2}$. Obviously f is a signed strong total Roman dominating function on $DS_{r,s}$ and $\omega(f) = 2$.

Subcase 1.2. Assume now that s is odd. We define $f : V(DS_{r,s}) \longrightarrow \{-1, 1, 2, \ldots, 1 + \lceil \frac{s}{2} \rceil\}$ by $f(v_i) = -1$ for $1 \le i \le s-1$, $f(v_s) = \frac{s+1}{2} - 1$, $f(v) = \frac{s+1}{2} + 2$, $f(u_i) = -1$ for $1 \le i \le r$ and $f(u) = \frac{s+1}{2}$. Clearly f is a signed strong total Roman dominating function on $DS_{r,s}$ and $\omega(f) = 2$.

Case 2. Assume that $r > \lceil \frac{s}{2} \rceil + 1$ and consider the following two subcases:

Subcase 2.1. Suppose first that s is even. We define $f : V(DS_{r,s}) \longrightarrow \{-1, 1, 2, \ldots, 1 + \lceil \frac{s}{2} \rceil\}$ by $f(v_i) = -1$ for $1 \le i \le s-1$, $f(v_s) = \frac{s}{2} - 1$, $f(v) = \frac{s}{2} + 1$ $f(u_i) = -1$ for $1 \le i \le r-1$, $f(u_r) = r - (\frac{s}{2} + 1)$ and $f(u) = \frac{s}{2} + 1$. It is easy to see that f is a signed strong total Roman dominating function on $DS_{r,s}$ and $\omega(f) = 2$.

Subcase 2.2. Assume that s is odd. We define $f : V(DS_{r,s}) \longrightarrow \{-1, 1, 2, \ldots, 1 + \lceil \frac{s}{2} \rceil\}$ by $f(v_i) = -1$ for $1 \le i \le s-1$, $f(v_s) = \frac{s+1}{2} - 1$ and $f(v) = \frac{s+1}{2} + 1$, $f(u_i) = -1$ for $1 \le i \le r-1$, $f(u_r) = r - (\frac{s+1}{2} + 1)$ and $f(u) = \frac{s+1}{2}$. Then f is a signed strong total Roman dominating function on $DS_{r,s}$ and $\omega(f) = 2$.

Case 3. Let $r < \lceil \frac{s}{2} \rceil + 1$ and consider the following two subcases:

Subcase 3.1. Assume that $\lceil \frac{r}{2} \rceil + 1 \le 4r - s + 1$. Then define $f : V(DS_{r,s}) \longrightarrow \{-1, 1, 2, \ldots, 1 + \lceil \frac{s}{2} \rceil\}$ by $f(v_i) = -1$ for $1 \le i \le 2r$, $f(v_i) = +1$ for $2r + 1 \le i \le s$, $f(v) = r + 1$, $f(u_i) = -1$ for $1 \le i \le r$ and $f(u) = 4r - s + 1$. Clearly f is a signed strong total Roman dominating function on $DS_{r,s}$ and $\omega(f) = 2$.

Subcase 3.2. Let $\lceil \frac{r}{2} \rceil + 1 > 4r - s + 1$ and define $f : V(DS_{r,s}) \longrightarrow \{-1, 1, 2, \ldots, 1 + \lceil \frac{s}{2} \rceil\}$ by $f(v_i) = -1$ for $1 \le i \le \lceil \frac{r}{2} \rceil$, $f(v_i) = (-1)^i$ for $\lceil \frac{r}{2} \rceil + 1 \le i \le s$, $f(v) = \lceil \frac{s + \lceil \frac{r}{2} \rceil}{4} \rceil + 1$, $f(u_i) = -1$ for $1 \le i \le r$ and $f(u) = \lceil \frac{r}{2} \rceil + 2$. Then f is a signed strong total Roman dominating function on $DS_{r,s}$ and $\omega(f) = -r + \lceil \frac{s + \lceil \frac{r}{2} \rceil}{4} \rceil + 4$. □

5 Concluding remarks

In this section we indicate some possible directions of future research. We present some questions and open problems about signed strong total Roman domination number which can be a new fields of research.

Let $k \leq |V(G)|$ be a positive integer. Does G have a signed strong total Roman dominating function of weight at most k? Since we can check in polynomial time that a function $f : V(G) \rightarrow \{-1, 1, 2, \ldots, \lceil \frac{\Delta}{2} \rceil + 1\}$ has weight at most k and is a signed strong Roman dominating function, so SSTRDF is a member of NP and we conjecture that:

Conjecture 5.1. *Problem* SSTRDF *is* NP*-complete for bipartite graphs.*

We gave an upper bound for the signed strong total Roman domination number of $K_{r,s}$, for $2 \leq r \leq s$ and double stars. So the following problem is another thing worth trying could be to see.

Problem 5.2. Give equality for Proposition 4.4 and Proposition 4.5.

We conclude the section and the paper by the following problem.

Problem 5.3. Characterize all connected graphs G of order n and size m attaining the bound of Theorem 3.12.

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