THE DUAL SPACES OF THE SETS OF Λ-STRONGLY
CONVERGENT AND BOUNDED SEQUENCES

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Abstract. In this paper we shall give the α–, β–, γ– and f–duals of the sets \( w^p_0(\Lambda) \), \( w^p_\infty(\Lambda) \), \( c^p_0(\Lambda) \), \( c^p(\Lambda) \) and \( c^p_\infty(\Lambda) \). Furthermore, we shall determine the continuous dual spaces of the sets \( w^p_0(\Lambda) \), \( c^p_0(\Lambda) \) and \( c^p(\Lambda) \).

1. Introduction

We write \( \omega \) for the set of all complex sequences \( x = (x_k)_{k=0}^{\infty} \), \( \phi \), \( l_\infty \), \( c \) and \( c_0 \) for the sets of all finite, bounded, convergent sequences and sequences convergent to naught, respectively, further \( cs \), \( bs \) and \( l_1 \) for the sets of all convergent, bounded and absolutely convergent series.

By \( e \) and \( e^{(n)} \) \( (n \in \mathbb{N}_0) \), we denote the sequences such that \( e_k = 1 \) for \( k = 0, 1, \ldots \), and \( e^{(n)}_n = 1 \) and \( e^{(n)}_k = 0 \) for \( k \neq n \). For any sequence \( x = (x_k)_{k=0}^{\infty} \), let \( x^{[n]} = \sum_{k=0}^{n} x_ke^{(k)} \) be its \( n \)–section.

Let \( X, Y \subset \omega \) and \( z \in \omega \). Then we write

\[
    z^{-1} \ast X = \{ x \in \omega : xz = (x_kz_k)_{k=0}^{\infty} \in X \}
\]

and

\[
    M(X, Y) = \bigcap_{x \in X} x^{-1} \ast Y = \{ a \in \omega : ax \in Y \text{ for all } x \in X \}
\]

for the multiplier space of \( X \) and \( Y \). The sets \( M(X, l_1) \), \( M(X, cs) \) and \( M(X, bs) \) are called the α–, β– and γ–duals of \( X \), respectively.

A Fréchet subspace \( X \) of \( \omega \) is called an FK space if it has continuous coordinates, that is if convergence in \( X \) implies coordinatewise convergence. An FK space \( X \supset \phi \) is said to have \( AK \) if, for every sequence \( x = (x_k)_{k=0}^{\infty} \in X \), \( x^{[n]} \rightarrow x \) \((n \rightarrow \infty)\); it is said to have \( AD \) if \( \phi \) is dense in \( X \). A BK space is an FK space which is a Banach space.

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If $X$ is a $p$-normed space then we write $X^*$ for the set of all continuous linear functionals on $X$, the so-called *continuous dual* of $X$, with its norm $\| \cdot \|$ given by

$$\|f\| = \sup\{|f(x)| : \|x\| = 1\} \text{ for all } f \in X^*.$$  

Let $X \supset \phi$ be an FK space. Then the set $X' = \{(f(e(n)))_{n=0}^\infty : f \in X^*\}$ is called the $f$–*dual* of $X$.

The sets $c_0(\Lambda)$, $c(\Lambda)$ and $c_{\infty}(\Lambda)$ of sequences that are $\Lambda$–strongly convergent to naught, $\Lambda$–strongly convergent and $\Lambda$–strongly bounded were introduced and studied by Móricz [12]. Their $\beta$– and continuous duals were determined in [10] and [11]. In this paper, we shall extend these results to $0 < p \leq 1$ where $p$ is an index. Furthermore, we shall give the $\alpha$–, $\gamma$– and $f$–duals of the spaces $w_0^p(\Lambda)$, $w_{\infty}^p(\Lambda)$, $c_0^p(\Lambda)$, $c^p(\Lambda)$ and $c_{\infty}^p(\Lambda)$.

### 2. Some Notations and Preliminary Results

We shall frequently apply the following inequality (cf. [8, p. 22])

$$(a + b)^p \leq a^p + b^p \ (0 < p \leq 1) \text{ for all } a, b \geq 0. \quad (2.1)$$

Given any infinite matrix $A = (a_{nk})_{n,k=0}^\infty$ of complex numbers and any sequence $x \in \omega$, we shall write $A_n(x) = \sum_{k=0}^\infty a_{nk}x_k \ (n = 0, 1, \ldots)$, $A(x) = (A_n(x))_{n=0}^\infty$, provided the series converge, and $X_A = \{x \in \omega : A(x) \in X\}$.

We define the matrix $\Delta$ by $\Delta_{nk} = 1$ for $k = n$, $\Delta_{nk} = -1$ for $k = n-1$ and $\Delta_{nk} = 0$ otherwise $(n = 0, 1, \ldots)$, and use the convention that any symbol with a negative subscript has the value 0.

Given any real $p > 0$ and any sequence $x$, we write $|x|^p = (|x_k|^p)_{k=0}^\infty$ and

$$M_n^p(x) = \frac{1}{\mu_n} \sum_{k=0}^n |(\Delta(\mu x))_k|^p \text{ for } n = 0, 1, \ldots.$$  

Let $0 < p < \infty$ and $\mu = (\mu_n)_{n=0}^\infty$ be a nondecreasing sequence of positive reals tending to infinity throughout. We shall consider the sets

$$w_0^p(\mu) = \left\{ x \in \omega : \lim_{n \to \infty} \left( \frac{1}{\mu_n} \sum_{k=0}^n |x_k|^p \right) = 0 \right\}, \quad c_0^p(\mu) = (\mu)^{-1} * (w_0^p(\mu))_\Delta,$$

$$w_{\infty}^p(\mu) = \left\{ x \in \omega : \sup_n \left( \frac{1}{\mu_n} \sum_{k=0}^n |x_k|^p \right) < \infty \right\}, \quad c_{\infty}^p(\mu) = (\mu)^{-1} * (w_{\infty}^p(\mu))_\Delta,$$

$$c^p(\mu) = \{ x \in \omega : x - le \in c_0^p(\mu) \text{ for some } l \in \mathbb{C} \}.$$  

If $p = 1$ then we omit the index $p$, that is we write $w_0(\mu) = w_0^1(\mu)$ etc.

The sets $w_0^p(\mu)$ and $w_{\infty}^p(\mu)$ are special cases of mixed normed spaces studied for instance in [1,2,5,6,9]. If $\frac{1}{\mu_n} = \frac{1}{n+1}$ for $n = 0, 1, \ldots$, then the sets $w_0^p(\mu)$ and $w_{\infty}^p(\mu)$ reduce to the sets $w_0^p$ and $w_{\infty}^p$ introduced and studied by Maddox [7], and the sets $c_0^p(\mu),$
$c^p(\mu)$ and $c^p_{\infty}(\mu)$ reduce to the sets $[c_0]_p$, $[c]_p$ and $[c_{\infty}]_p$ introduced and studied by Hyslop, Kuttner and Thorpe [3, 4]. For $p = 1$ the sets $c^p_0(\mu)$, $c^p(\mu)$ and $c^p_{\infty}(\mu)$ reduce to the sets $c_0(\mu)$, $c(\mu)$ and $c_{\infty}(\mu)$ introduced and studied by Móricz [12] and Malkowsky [10].

Obviously the sets $w^p_0(\mu)$, $w^p_{\infty}(\mu)$, $c^p_0(\mu)$, $c^p(\mu)$ and $c^p_{\infty}(\mu)$ are linear spaces and $w^p_0(\mu) \subset w^p_{\infty}(\mu)$, $c^p_0(\mu) \subset c^p(\mu)$ and $c^p_0(\mu) \subset c^p_{\infty}(\mu)$. Furthermore, we have

**Lemma 1.** (a) Let $0 < p < 1$. Then $c^p(\mu) \subset c^p_{\infty}(\mu)$ if and only if

$$\sup_n \frac{1}{\mu_n} \sum_{k=0}^n |(\Delta \mu)_k|^p < \infty \text{ or equivalently } e \in c^p_{\infty}(\mu). \quad (2.2)$$

(b) Let $1 \leq p < \infty$. Then $e \in c^p_{\infty}(\mu)$ and $c^p(\mu) \subset c^p_{\infty}(\mu)$.

(c) Let $0 < p < \infty$. If $x \in c^p(\mu)$, then $l \in C$ with $x - le \in c^p_0(\mu)$ is unique.

(d) Let $X^p(\mu)$ denote any of the spaces $w^p_0(\mu)$, $w^p_{\infty}(\mu)$, $c^p_0(\mu)$, $c^p(\mu)$ and $c^p_{\infty}(\mu)$. Then $X^p(\mu) \subset X^p(\mu)$ for $0 < p \leq \bar{p}$.

(e) If $0 < p \leq 1$, then $c^p_{\infty}(\mu) \subset c^p_{\infty}(\mu)$.

**Proof.** (a) First we assume that condition (2.2) holds. Let $x \in c^p(\mu)$ be given. Then there is $l \in C$ such that $x - le \in c^p_0(\mu)$, and so $x = x - le + le \in c^p_{\infty}(\mu)$, since $c^p_{\infty}(\mu)$ is a linear space.

Conversely, if condition (2.2) is not satisfied, then we can determine an increasing sequence $(n_m)_{m=0}^{\infty}$ of integers such that $M^p_{n_m}(e) > m$ $(m = 0, 1, \ldots)$. Then $x = e \in c^p(\mu) \setminus c^p_{\infty}(\mu)$, since

$$M_n(x - e) = 0 \quad (n = 0, 1, \ldots) \quad \text{and} \quad M^p_{n_m}(x) = M^p_{n_m}(e) > m \quad (m = 0, 1, \ldots).$$

(b) Now let $p \geq 1$. Since $1/p \leq 1$ and $\mu_n \geq \mu_{n-1}$ for all $n$, we have by (2.1)

$$(M^p_{n}(e))^{1/p} \leq M^1_{n}(e) = \frac{1}{\mu_n} \sum_{k=0}^n (\mu_k - \mu_{k-1}) = 1 \quad \text{for all } n = 0, 1, \ldots,$$

hence $e \in c^p_{\infty}(\mu)$. The inclusion $c^p(\mu) \subset c^p_{\infty}(\mu)$ now follows as in the first part of the proof of part (a).

(c) Let $x \in c^p(\mu)$ and $l, l' \in C$ such that $x - le \in c^p_0(\mu)$ and $x - l'e \in c^p_0(\mu)$. Given $\varepsilon > 0$, there is $n = n(\varepsilon) \in N_0$ such that $M^p_n(x - le), M^p_n(x - l'e) < \varepsilon$. Then, for $0 < p < 1$ by inequality (2.1)

$$|l - l'|^p \leq M^p_n((x - le) - (x - l'e)) \leq M^p_n(x - le) + M^p_n(x - l'e) < 2\varepsilon$$

and, for $p \geq 1$ by Minkowski’s inequality

$$|l - l'| \leq (M^p_n((x - le) - (x - l'e)))^{1/p} \leq (M^p_n(x - le))^{1/p} + (M^p_n(x - l'e))^{1/p} < 2\varepsilon^{1/p}.$$

Since $\varepsilon > 0$ was arbitrary, we have $l = l'$ in both cases.
(d) Since $p/\bar{p} \leq 1$, we have

$$\left( \frac{1}{\mu_n} \sum_{k=0}^{n} |x_k|^p \right)^{p/\bar{p}} \leq \frac{1}{\mu_n} \sum_{k=0}^{n} |x_k|^p \quad (n = 0, 1, \ldots).$$

From this, we obtain the inclusions $X^p(\mu) \subset X^{\bar{p}}(\mu)$ for $X^p(\mu) = w_0^p(\mu)$ and $X^{\bar{p}}(\mu) = w_0^{\bar{p}}(\mu)$.

Since $x \in c_0^p(\mu)$ or $x \in c_0^{\bar{p}}(\mu)$ if and only if $\Delta(\mu x) \in w_0^p(\mu)$ or $\Delta(\mu x) \in w_0^{\bar{p}}(\mu)$, respectively, it follows that the inclusions also hold for $X^p(\mu) = c_0^p(\mu)$ or $X^{\bar{p}}(\mu) = c_0^{\bar{p}}(\mu)$.

Finally, the inclusion $c^p(\mu) \subset c^{\bar{p}}(\mu)$ holds, since $x \in c^p(\mu)$ if and only if $x - le \in c_0^p(\mu)$ for some $l \in C$.

(e) First

$$|x_n| = \frac{1}{\mu_n} \sum_{k=0}^{\infty} (\Delta(\mu x))_k \leq M_n^1(x) \quad (n = 0, 1, \ldots)$$

implies $c_\infty(\mu) \subset l_\infty$, and so $c_\infty^p(\mu) \subset l_\infty$ for $0 < p \leq 1$ by part (d).

Following the notations introduced in [10], we say that a nondecreasing sequence $\Lambda = (\lambda_n)_{n=0}^{\infty}$ of positive reals tending to infinity is \textit{exponentially bounded} if there are reals $s$ and $t$ with $0 < s \leq t < 1$ such that for some subsequence $(\lambda_{n(\nu)})_{\nu=0}^{\infty}$ of $\Lambda$, we have

$$s \leq \frac{\lambda_{n(\nu)}}{\lambda_{n(\nu)+1}} \leq t \quad \text{for all } \nu = 0, 1, \ldots; \quad (2.3)$$

such a subsequence $(\lambda_{n(\nu)})_{\nu=0}^{\infty}$ will be called an \textit{associated subsequence}.

If $(n(\nu))_{\nu=0}^{\infty}$ is a strictly increasing sequence of nonnegative integers then we shall write $K^{<\nu'}$ for the set of all integers $k$ with $n(\nu) \leq k \leq n(\nu' + 1) - 1$, and $\sum_\nu$ and $\max_\nu$ for the sum and maximum taken over all $k$ in $K^{<\nu'}$.

If $X$ is a $p$–normed sequence space and $a \in \omega$, then we write

$$\|a\|_X = \sup \left\{ \sum_{k=0}^{\infty} a_k x_k : \|x\| = 1 \right\}$$

provided the term on the right exists and is finite. This is the case whenever $X \supset \phi$ is a $p$–normed FK space and $a \in X^\beta$ by [13, Theorem 7.2.9, p. 107].

Let $\Lambda = (\lambda_n)_{n=0}^{\infty}$ be a nondecreasing exponentially bounded sequence of positive reals and $(\lambda_{n(\nu)})_{\nu=0}^{\infty}$ an associated subsequence throughout.

If $X^p(\Lambda)$ denotes any of the sets $w_0^p(\Lambda)$, $w_\infty^p(\Lambda)$, $c_0^p(\Lambda)$, $c^p(\Lambda)$ or $c_\infty^p(\Lambda)$ then we shall write $X^p(\Lambda)$ for the respective space with the sections $1/\lambda_n \sum_{k=0}^{n} \ldots$ replaced by the blocks $1/\lambda_n^{p} \sum_{k=0}^{n} \ldots$. Further, we define

$$\|x\|_{w_\infty^p(\Lambda)} = \begin{cases} \sup_n \left( \frac{1}{\lambda_n} \sum_{k=0}^{n} |x_k|^p \right)^{1/p} & (0 < p \leq 1) \\ \sup_n \left( \frac{1}{\lambda_n} \sum_{k=0}^{n} |x_k|^p \right)^{1/p} & (1 \leq p < \infty) \end{cases}$$
To show that c and 4.3.14, pp 63 and 46, except for the one that From this, all the assertions concerning holds for equivalent on and (sequence 1 space for < p < w ∥ · ∥ x W e have to show c sequence in w 0 (c) is a closed subspace of w p (c), c 0 (c) is a closed subspace of c p (c), w p (c) has AK for all p and c 0 (c) has AK for 0 < p ≤ 1. (b) We assume that condition (2.2) holds for 0 < p < 1. Then c p (µ) with ∥ · ∥ c p (µ) is a p–normed FK space for 0 < p < 1 and a BK space for 1 ≤ p < ∞, c p (µ) is a closed subspace of c p (µ), and if 0 < p ≤ 1, then every sequence x = (x k ) k=0 ∞ ∈ c p (µ) has a unique representation

\[
x = le + \sum_{k=0}^{∞} (x_k - l)e^{(k)} \text{ where } l \in C \text{ is such that } x - le \in c_p^L(Λ).
\]

(c) If X p (Λ) and X̃p(Λ) denote any of the sets w p (Λ), w p (c 0 (Λ)), c p (Λ), e p (Λ) and c p (Λ), w 0 (Λ), w 0 (c 0 (Λ)), c 0 (Λ), e 0 (Λ) and c 0 (Λ), respectively, then X p (Λ) = X̃p(Λ), ∥ · ∥ w p (Λ), and ∥ · ∥ c p (Λ) are equivalent on w p (Λ) and on w p (c 0 (Λ)), ∥ · ∥ w p (c 0 (Λ)), and ∥ · ∥ c p (Λ), in the case of c p (Λ) whenever condition (2.2) holds for 0 < p < 1.

Proof. (a) The assertions concerning the sets w p (µ) and w p (c 0 (µ)) were proved in [9]. From this, all the assertions concerning c p (µ) and c p (c 0 (µ)) follow from [13, Theorems 4.3.13 and 4.3.14, pp 63 and 46], except for the one that c p (µ) has AK for 0 < p ≤ 1.

To show that c p (µ) has AK for 0 < p ≤ 1, let x ∈ c p (µ) and ε > 0 be given. Then there is an integer m 0 ∈ N 0 such that M 0 (x) < ε/2 for all n ≥ m 0. Let m ≥ m 0. Then, since 0 < p ≤ 1, we conclude

\[
\|x^{[m]} - x\|_{c_p(µ)} = M_{p}(x^{[m]} - x) = \sup_{n \geq m+1} \frac{1}{µ^n} \left( |µ^{m+1}|p |x_{m+1}|^p + \sum_{k=m+2}^{n} |(∆(µx))_{k}|^p \right) < M_{p}(x) + ε/2 < ε/2 + ε/2 = ε.
\]

(b) First we show that c p (µ) is complete with ∥ · ∥ c p (µ).

By Lemma 1 (a) and (b), ∥ · ∥ c p (µ) is defined on c p (µ).

Let (x(m)) m=0 ∞ be a Cauchy sequence in c p (µ). For each m ∈ N 0, let l(m) ∈ C denote the number for which x(m) − l(m)e ∈ c p (µ). First we observe that (x(m)) m=0 ∞ is a Cauchy sequence in c p (µ), and so convergent by the completeness of c p (µ),

\[
\|x^{[m]} - x\|_{c_p(µ)} \rightarrow 0 \quad (m \rightarrow ∞), \text{ say.}
\]

We have to show x ∈ c p (µ).
First we show that the sequence \( (l^{(m)})_{m=0}^{\infty} \) converges.

Let \( \varepsilon > 0 \) be given. Since \( (x^{(m)})_{m=0}^{\infty} \) is a Cauchy sequence, we may choose \( M = M(\varepsilon) \in \mathbb{N}_0 \) such that \( \|x^{(m)} - x^{(j)}\|_{c^p(\mu)} < \varepsilon/3 \) for all \( m, j \geq M \). Let \( m, j \geq M \). Since \( x^{(m)} - l^{(m)} \in c^p(\mu), x^{(j)} - l^{(j)} \in c^p(\mu), \) there is \( n = n(m, j, \varepsilon) \in \mathbb{N}_0 \) such that \( M_n^{(p)}(x^{(m)} - l^{(m)})e, M_n^{(p)}(x^{(j)} - l^{(j)})e < \varepsilon/3 \). Then, for \( 0 < p < 1 \) by inequality (2.1)

\[
\|l^{(m)} - l^{(j)}\|^p \leq M_n^{(p)}(l^{(m)} - l^{(j)})e \leq M_n^{(p)}(x^{(m)} - l^{(m)})e + \|x^{(m)} - x^{(j)}\|_{c^p(\mu)}M_n^{(p)}(x^{(j)} - l^{(j)})e < \varepsilon/3 + \varepsilon/3 = \varepsilon,
\]

and, for \( 1 \leq p < \infty \) by Minkowski’s inequality

\[
\|l^{(m)} - l^{(j)}\| \leq \left( M_n^{(p)}(l^{(m)} - l^{(j)})e \right)^{1/p} \leq \left( M_n^{(p)}(x^{(m)} - l^{(m)})e \right)^{1/p} + \|x^{(m)} - x^{(j)}\|_{c^p(\mu)} + \left( M_n^{(p)}(x^{(j)} - l^{(j)})e \right)^{1/p} < 2(\varepsilon/3)^{1/p} + \varepsilon/3.
\]

Thus \( (l^{(m)})_{m=0}^{\infty} \) is a Cauchy sequence in \( C \), hence convergent,

\[
l = \lim_{m \to \infty} l^{(m)}, \quad \text{say.} \tag{2.6}
\]

Now we show \( x - le \in c^p(\mu) \).

Let \( \varepsilon > 0 \) be given. By (2.5) and (2.6), there is \( M \in \mathbb{N}_0 \) such that \( \|x^{(M)} - x\|_{c^p(\mu)} < \varepsilon/3 \), and, with \( C = \sup_n M_n^{(p)}(e) < \infty \) (for \( 0 < p < 1 \) by condition (2.2)),

\[
\|l - l^{(M)}\| < \left( \frac{\varepsilon}{3(C + 1)} \right)^{1/p}.
\]

Further, since \( x^{(M)} - l^{(M)} \in c^p(\mu) \), there is \( N \in \mathbb{N}_0 \) such that \( M_n^{(p)}(x^{(M)} - l^{(M)})e < \varepsilon/3 \). Let \( n \geq N \). Then, for \( 0 < p < 1 \) by inequality (2.1)

\[
M_n^{(p)}(x - le) \leq M_n^{(p)}(x^{(M)} - l^{(M)})e + \|x^{(m)} - x\|_{c^p(\mu)} + M_n^{(p)}(||l - l^{(M)}||e) < 2\varepsilon/3 + \|l - l^{(M)}\|^pM_n^{(p)}(e) < \frac{2\varepsilon}{3} + \frac{\varepsilon C}{3(C + 1)} \leq \varepsilon,
\]

and, for \( 1 \leq p < \infty \) by Minkowski’s inequality

\[
(M_n^{(p)}(x - le))^{1/p} < (\varepsilon/3)^{1/p} + \varepsilon/3 + \left( M_n^{(p)}(||l - l^{(M)}||e) \right)^{1/p} < (\varepsilon/3)^{1/p} + \varepsilon/3 + |l - l^{(M)}| (M_n^{(p)}(e))^{1/p} < 2(\varepsilon/3)^{1/p} + \varepsilon/3.
\]

This shows that \( c^p(\mu) \) is complete. Consequently \( c^p(\mu) \) is a \( p \)-normed FK space for \( 0 < p < 1 \) and a BK space for \( 1 \leq p < \infty \) by [13, Corollary 4.2.2, p. 56].

Finally, let \( 0 < p \leq 1 \) and \( x = (x_k)_{k=0}^{\infty} \in c^p(\mu) \). Then, by Lemma 1 (c) there is a uniquely determined \( l \in C \) such that \( x - le \in c^p(\mu) \). We put \( y = x - le \). Since \( c^p_0(\mu) \) has AK, \( y = \sum_{k=0}^{\infty} y_ke^{(k)} = \sum_{k=0}^{\infty} (x_k - le^{(k)}) \), and so the representation in (2.4) follows.
(c) Let $0 < p < \infty$.

From
$$\frac{1}{\lambda_n^{(\nu+1)}} \sum_{k=0}^{n} |x_k|^p \leq \frac{1}{\lambda_n^{(\nu+1)}} \sum_{k=0}^{n} |x_k|^p \quad (\nu = 0, 1, \ldots),$$
we conclude $\mathcal{X}^p(\Lambda) \subset \tilde{\mathcal{X}}^p(\Lambda)$.

Conversely, let $x \in \tilde{w}^p_0(\Lambda)$ and $\varepsilon > 0$ be given. Then there is an integer $\nu_0 \in \mathbb{N}_0$ such that
$$\frac{1}{\lambda_n^{(\nu+1)}} \sum_{k=0}^{\nu} |x_k|^p < \varepsilon \quad \text{for all } \nu \geq \nu_0.$$

Since $\lambda_n^{(\nu)} \to \infty \quad (\nu \to \infty)$, we can choose an integer $\nu_1 > \nu_0$ such that
$$\frac{1}{\lambda_n^{(\nu_1)}} \sum_{k=0}^{\nu_0} |x_k|^p \quad \text{for all } \nu \geq \nu_1.$$

Let $m \geq n(\nu_1)$. Then there is an integer $\nu(m) \geq \nu_1$ such that $m \in K^{<\nu(m)>}$ and, using (2.3), we obtain
$$\frac{1}{\lambda_m^{(\nu)}} \sum_{k=0}^{m} |x_k|^p \leq \frac{1}{\lambda_n^{(\nu(m))}} \left( \sum_{k=0}^{n(\nu_0)-1} |x_k|^p + \sum_{\nu=\nu_0}^{\nu(m)} \sum_{k=0}^{\nu} |x_k|^p \right)
< \varepsilon + \left( \frac{\lambda_n^{(\nu(m)+1)}}{\lambda_n^{(\nu(m))}} \right)^p \frac{1}{\lambda_n^{(\nu(m)+1)}} \sum_{\nu=\nu_0}^{\nu(m)} \frac{1}{\lambda_n^{(\nu+1)}} \sum_{k=0}^{\nu} |x_k|^p
< \varepsilon + \left( 1 + \frac{1}{s^p \frac{1}{1-t^p}} \right).$$

This shows $w^p_0(\Lambda) \subset w^p_0(\Lambda)$. The inclusion $\tilde{w}^p_0(\Lambda) \subset w^p_\infty(\Lambda)$ is shown in exactly the same way. Now the identities $c_0^p(\Lambda) = \tilde{c}_0^p(\Lambda)$, $c^p(\Lambda) = \tilde{c}^p(\Lambda)$ and $c^p_\infty(\Lambda) = \tilde{c}^p_\infty(\Lambda)$ are obvious.

3. The Duals of The Sets $w^p_0(\Lambda)$ and $w^p_\infty(\Lambda)$

In this section, we shall give the duals of the sets $w^p_0(\Lambda)$ and $w^p_\infty(\Lambda)$ for $0 < p < \infty$.

Let $\Lambda = (\lambda_n)_{n=0}^{\infty}$ be a nondecreasing exponentially bounded sequence of positive reals throughout and $(\lambda_n(\nu))_{\nu=0}^{\infty}$ an associated subsequence. We put
$$\mathcal{W}^p(\Lambda) = \begin{cases} \{ a \in \omega : \sum_{\nu=0}^{\infty} \lambda_n(\nu+1) \max_{\nu} |a_k| < \infty \} & (0 < p \leq 1) \\
\{ a \in \omega : \sum_{\nu=0}^{\infty} \lambda_n(\nu+1) (\sum_{\nu} |a_k|^p)^{1/p} < \infty \} & (1 < p < \infty, q = \frac{p}{p-1}) \end{cases}$$
and on $W^p(\Lambda)$

$$
\|a\|_{W^p(\Lambda)} = \begin{cases} 
\sum_{\nu=0}^{\infty} \lambda_{\nu+1} \max_{\nu} |a_k| & (0 < p \leq 1) \\
\sum_{\nu=0}^{\infty} \lambda_{\nu+1} (\sum_{\nu} |a_k|^p)^{1/p} & (1 < p < \infty, q = \frac{p}{p-1}).
\end{cases}
$$

**Theorem 2.** Let $X^p(\Lambda) = w_0^p(\Lambda)$ or $X^\gamma(\Lambda) = w_\infty^p(\Lambda)$ and $\dagger$ stand for $\alpha$, $\beta$, $\gamma$ or $f$.

Then $(X^p(\Lambda))^\dagger = W^p(\Lambda)$. The continuous dual $(w_0^p(\Lambda))^* \subset W^p(\Lambda)$ is norm isomorphic to $W^p(\Lambda)$ when $w_0^p(\Lambda)$ has the norm $\| \cdot \|_{w_\infty^p(\Lambda)}$. This means, $g \in (w_0^p(\Lambda))^*$ if and only if there is a sequence $b = (b_n)_{n=0}^{\infty} \in W^p(\Lambda)$ such that

$$
g(y) = \sum_{n=0}^{\infty} b_n y_n \quad \text{for all } y \in w_0^p(\Lambda) \quad \text{and} \quad \|g\| = \|b\|_{W^p(\Lambda)}.
$$

Furthermore, $\|a\|_{w_\infty^p(\Lambda)} = \|a\|_{W^p(\Lambda)}$ on $(w_\infty^p(\Lambda))^\beta$.

**Proof.** The statements of the theorem with the exception of those concerning the $\gamma$ and $f$–duals are well known [9, Theorems 2.4,5 and 6].

(a) Since $w_0^p(\Lambda)$ has AK, we have $(w_0^p(\Lambda))^\beta = (w_0^p(\Lambda))^f$ by [13, Theorem 7.2.7 (ii), p. 106], and so $(w_0^p(\Lambda))^f = W^p(\Lambda)$. Further, since an AK space obviously has AD, we also have $(w_0^p(\Lambda))^\gamma = (w_0^p(\Lambda))^f$ by [13, Theorem 7.2.7 (iii), p. 106], and so $(w_0^p(\Lambda))^\gamma = W^p(\Lambda)$. Since $w_0^p(\Lambda)$ is a closed subspace of $w_\infty^p(\Lambda)$, it follows that $(w_\infty^p(\Lambda))^f = (w_0^p(\Lambda))^f$ by [13, Theorem 7.2.6, p.106], and so

$$(w_\infty^p(\Lambda))^f = W^p(\Lambda).$$

Finally, by [13, Theorem 7.2.7 (i), p. 106], $(w_\infty^p(\Lambda))^\gamma \subset (w_\infty^p(\Lambda))^f$, and so by (3.1)

$$W^p(\Lambda) \subset (w_\infty^p(\Lambda))^\gamma \subset (w_\infty^p(\Lambda))^f = W^p(\Lambda),$$

hence $(w_\infty^p(\Lambda))^\gamma = W^p(\Lambda)$.

**4. The Duals of The Sets** $c_0^p(\Lambda)$, $c^p(\Lambda)$ and $c_\infty^p(\Lambda)$

In this section, we shall determine the $\alpha$, $\beta$, $\gamma$– and $f$–duals of the sets $c_0^p(\Lambda)$, $c^p(\Lambda)$ and $c_\infty^p(\Lambda)$ and the continuous duals of $c_0^p(\Lambda)$ and $c^p(\Lambda)$ for $0 < p \leq 1$.

Let $\Lambda = (\lambda_n)_{n=0}^{\infty}$ be a nondecreasing exponentially bounded sequence of positive reals throughout.

We need the following lemma for the determination of the $\alpha$–duals of $c_0^{\infty}(\Lambda)$, $c^p(\Lambda)$ and $c_\infty^p(\Lambda)$.

**Lemma 2.** Let $X \subset l_\infty$ be a BK space such that $\sup_n \|e^{[n]}\|_{c_\infty^p(\Lambda)} < \infty$. Then $X^\alpha = l_1$. 

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Proof. First we observe that \( X \subset l_\infty \) implies \( l_\infty^o = l_1 \subset X^o \).

Conversely, let \( a \in X^o \). For each \( m \in \mathbb{N}_0 \), we define the map \( f_a^{(m)} : X \to \mathbb{R} \) by \( f_a^{(m)}(x) = \sum_{k=0}^m |a_k x_k| \) \((x \in X)\). Then \( (f_a^{(m)})_{m=0}^\infty \) is a sequence of seminorms on \( X \) which are continuous, since \( X \) is a BK space. Further \( f_a^{(m)}(x) \leq \sum_{k=0}^\infty |a_k x_k| = |M| < \infty \) for all \( m \in \mathbb{N}_0 \) and all \( x \in X \). By the uniform boundedness principle, there is a constant \( M_1 \) such that \( \|f^{(m)}\| \leq M_1 \) for all \( m \in \mathbb{N}_0 \). From this and \( \sup_m \|e^{[n]}\|_{c_0^\infty(\Lambda)} < \infty \), we conclude \( a \in l_1 \).

Theorem 3. Let \( X^p(\Lambda) \) denote any of the sets \( c_0^p(\Lambda) \), \( c^p(\Lambda) \) or \( c_\infty^p(\Lambda) \), then \( (X^p(\Lambda))^o \) \( = l_1 \) for \( 0 < p \leq 1 \).

Proof. Since obviously \( \sup_n \|e^{[n]}\|_{c_0^\infty(\Lambda)} \leq 2 \), and \( c_0(\Lambda) \subset c(\Lambda) \subset c_\infty(\Lambda) \subset l_\infty \) by Lemma 1 (b) and (e), we conclude from Lemma 2

\[
c_0^p(\Lambda) = c^p(\Lambda) = c_\infty^p(\Lambda) = l_1. \tag{4.1}
\]

Now we assume \( a \in (c_0^p(\Lambda))^o \). For each \( m \in \mathbb{N}_0 \), we define the map \( f^{(m)} : c_0^p(\Lambda) \to \mathbb{R} \) as in the proof of Lemma 2, and again there is a constant \( M > 0 \) such that

\[
\sum_{k=0}^\infty |a_k x_k| \leq M \quad \text{for all } x \in c_0^p(\Lambda) \text{ with } \|x\|_{c_\infty^p(\Lambda)} = 1. \tag{4.2}
\]

Since \( 1/\Lambda = (1/\lambda_n)_{n=0}^\infty \in c_0^p(\Lambda) \), we have

\[
R_n = R_n(|a|/\lambda) = \sum_{k=n}^\infty |a_k|/\lambda_k < \infty \quad \text{for all } n = 0, 1, \ldots
\]

Let \( \nu(m) \in \mathbb{N}_0 \) be given. We define the sequence \( x^{(\nu(m))} \) by

\[
x^{(\nu(m))}_n = \begin{cases}
\frac{1}{\lambda_n} \sum_{\nu(m)}^{n} \lambda_{n(\nu+1)} & (n \in N^{<\nu}; \nu = 0, 1, \ldots, \nu(m)) \\
\frac{1}{\lambda_n} \sum_{\nu(m)}^{\infty} \lambda_{n(\nu+1)} & (n \geq n(\nu(m) + 1)).
\end{cases}
\]

Then

\[
\left( \Delta(Ax^{(\nu(m))}) \right)_n = \begin{cases}
0 & (n \geq n(\nu(m) + 1) \text{ or } n \neq n(\nu); \nu = 0, 1, \ldots, \nu(m)) \\
\lambda_{n(\nu+1)} & (n = n(\nu); \nu = 0, 1, \ldots, \nu(m)).
\end{cases} \tag{4.3}
\]

and

\[
\sum_\nu \left| \Delta(Ax^{(\nu(m))}) \right|_n = \begin{cases}
\lambda_{n(\nu+1)} & (0 \leq \nu \leq \nu(m)) \\
0 & (\nu \geq \nu(m)).
\end{cases} \tag{4.4}
\]
Therefore \(x^{(\nu(m))} \in c_0^p(\Lambda)\) and \(\|x^{(\nu(m))}\|_{c_0^p(\Lambda)} = 1\). Further, by (4.3), (4.4) and (4.2), and since \(x_k^{(\nu(m))} \geq 0\) for all \(k\),

\[
\sum_{\nu=0}^{\nu(m)} \lambda_{n(\nu+1)} R_n(\nu) = \sum_{n=0}^{\infty} \left( \Delta (\Lambda x^{(\nu(m))}) \right) _n \sum_{k=n}^{\infty} \frac{|a_k|}{\lambda_k} = \sum_{k=0}^{\infty} \frac{|a_k|}{\lambda_k} \sum_{n=0}^{k} \left( \Delta (\Lambda x^{(\nu(m))}) \right) _n = \sum_{k=0}^{\infty} |a_k||x_k^{(\nu(m))}| \leq M.
\]

Since \(\nu(m) \in \mathbb{N}_0\) was arbitrary, we conclude \(\sum_{\nu=0}^{\infty} \lambda_{n(\nu+1)} R_n(\nu) < \infty\). Now

\[
\sum_{k=0}^{\infty} |a_k| = \sum_{\nu=0}^{\infty} \sum_{\nu=0}^{\infty} |a_k| \leq \sum_{\nu=0}^{\infty} \lambda_{n(\nu+1)} \sum_{\nu=0}^{\infty} \frac{|a_k|}{\lambda_k} \leq \sum_{\nu=0}^{\infty} \lambda_{n(\nu+1)} R_n(\nu) < \infty.
\]

Thus \((c_0^p(\Lambda))^\alpha \subset l_1\), and consequently, by (4.1)

\[(c_{\infty}^p(\Lambda))^\alpha \subset (c_0^p(\Lambda))^\alpha \subset l_1 = c_{\infty}^p(\Lambda) \subset (c_0^p(\Lambda))^\alpha \subset (c_0^p(\Lambda))^\alpha,
\]

hence \((c_{\infty}^p(\Lambda))^\alpha = (c_0^p(\Lambda))^\alpha = l_1\) for \(0 < p \leq 1\).

Finally \(c_0^p(\Lambda) \subset c^p(\Lambda)\) implies \((c^p(\Lambda))^\alpha \subset l_1\), and \(c(\Lambda) \subset c_{\infty}(\Lambda)\) implies \(l_1 = c_{\infty}(\Lambda) \subset c^p(\Lambda) \subset (c^p(\Lambda))^\alpha\), so \((c^p(\Lambda))^\alpha = l_1\).

Now we give the \(\beta\)-, \(\gamma\)- and \(f\)-duals of the sets \(c_0^p(\Lambda)\), \(c^p(\Lambda)\) and \(c_{\infty}^p(\Lambda)\) for \(0 < p \leq 1\), and the continuous duals of \(c_0^p(\Lambda)\) and \(c^p(\Lambda)\) in some cases.

If \(a \in cs\) then we shall write \(R(a)\) for the sequence with \(R_n(a) = \sum_{k=n}^{\infty} a_k\) \((n = 0, 1, \ldots)\). We shall frequently apply Abel’s summation by parts

\[
\sum_{n=0}^{m-1} a_n y_n = \sum_{n=0}^{m} R_n(a)(\Delta y)_n - R_m(a)y_m \text{ for all } m = 0, 1, \ldots \quad (4.5)
\]

If \(u\) is a sequence with \(u_k \neq 0\) for all \(k = 0, 1, \ldots\) then we shall write \(1/u\) for the sequence with \((1/u)_k = 1/u_k\) for all \(k\).

**Theorem 4.** Let \(0 < p \leq 1\). We put

\[
C_\beta(\Lambda) = \left\{ a \in \omega : \sum_{\nu=0}^{\infty} \lambda_{n(\nu+1)} \max_{\nu} \left| \sum_{k=n}^{\infty} \frac{a_k}{\lambda_k} \right| < \infty \right\}
\]

and

\[
\|a\|_{C_\beta(\Lambda)} = \sum_{\nu=0}^{\infty} \lambda_{n(\nu+1)} \max_{\nu} \left| \sum_{k=n}^{\infty} \frac{a_k}{\lambda_k} \right|.
\]

(a) If \(X^p(\Lambda)\) is any of the sets \(c_0^p(\Lambda)\) or \(c_{\infty}^p(\Lambda)\) and \(\dagger\) stands for any of the symbols \(\beta\), \(\gamma\) or \(f\), then

\[
X^p(\Lambda)\dagger = C_\beta(\Lambda).
\]
This also holds when $X^p(\Lambda) = c(\Lambda)$ or $X^p(\Lambda) = c^p(\Lambda)$ for $0 < p < 1$ whenever condition (2.2) is satisfied. Otherwise

$$(c^p(\Lambda))^{\beta} = C^\beta(\Lambda) \cap cs$$ and $$(c^p(\Lambda))^\gamma = C^\gamma(\Lambda) \cap bs.$$

(b) The continuous dual $(c^0_\infty(\Lambda))^*$ of $c^0_\infty(\Lambda)$ is norm isomorphic to $C^\beta(\Lambda)$ when $c^0_\infty(\Lambda)$ has the $p$–norm $\| \cdot \|$ of $c^\infty_\infty(\Lambda)$. Further

$$\|a\|_{c^\infty_\infty(\Lambda)} = \|a\|_{C^\beta(\Lambda)} \text{ on } c^\infty_\infty(\Lambda).$$

(4.6)

(c) We have $f \in c^*(\Lambda)$ if and only if

$$f(x) = l \chi_f + \sum_{n=0}^{\infty} a_n x_n \text{ for all } x \in c(\Lambda)$$

where $a \in C^\beta(\Lambda), l \in \mathbb{C}$ with $x - le \in c_0(\Lambda)$ and

$$\chi_f = f(e) - \sum_{n=0}^{\infty} a_n.$$

Further, $\|f\|$ is equivalent to

$$|\chi_f| + \|a\|_{C^\beta(\Lambda)}.$$

(4.8)

If condition (2.2) is satisfied, then this also holds for $c^p(\Lambda)$ ($0 < p < 1$).

Proof. In the case $p = 1$, the statements of the theorem concerning the $\beta$– and continuous duals can be found in [10, 11].

(a) Let $0 < p < 1$. First $c^\infty(\Lambda) \subset c_\infty(\Lambda)$ implies

$$(c_\infty(\Lambda))^{\beta} = C^\beta(\Lambda) \subset (c^\infty(\Lambda))^{\beta}.$$ Conversely, let $a \in (c^0_\infty(\Lambda))^{\beta}$. Since $c^0_\infty(\Lambda)$ is a $p$–normed FK space, the map $f_a : c^0_\infty(\Lambda) \to \mathbb{C}$ defined by $f_a(x) = \sum_{k=0}^{\infty} a_k x_k \ (x \in c^0_\infty(\Lambda))$ is an element of $(c^0_\infty(\Lambda))^*$. We define the matrix $\Delta(\Lambda)$ by

$$\Delta_{nk}(\Lambda) = \begin{cases} -\lambda_{n-1} & (k = n - 1) \\ \lambda_n & (k = n) \\ 0 & (\text{otherwise}) \end{cases} \ (n = 0, 1, \ldots).$$

By [13, Theorem 4.4.2, p. 66], there is $g \in (c^0_\infty(\Lambda))^*$ with

$$f = g \circ \Lambda(\Delta)$$

(4.9)

Since $w^p_0(\Lambda)$ is an FK space with AK, we have

$$b = (g(e^{\infty}_n))_{n=0}^{\infty} \in (w^p_0(\Lambda))^{\beta}$$

(4.10)
Let \( m \in \mathbb{N}_0 \) be given. Then, for the sequence \( x^{(m)} \) defined by
\[
x^{(m)}_n = \begin{cases} 0 & (n < m) \\ \frac{1}{\lambda_n} & (n \geq m), \end{cases}
\]
we have \( x^{(m)} \in c_0^p(\Lambda) \) and
\[
(\Delta_n(\Lambda))(x^{(m)}) = \begin{cases} 1 & (n = m) \\ 0 & (n \neq m) \end{cases} = e^{(m)} \in w_0^p(\Lambda),
\]
and so \( x^{(m)} \in c_0^p(\Lambda) \). From (4.9) and (4.10), we obtain
\[
b_m = g(e^{(m)}) = g\left((\Delta(\Lambda))(x^{(m)})\right) = f(x^{(m)}) = \sum_{n=0}^{\infty} a_n x^{(m)}_n = \sum_{n=m}^{\infty} a_n \frac{\lambda_n}{x^{(m)}_n} (m = 0, 1, \ldots),
\]
hence \( a \in C_0^p(\Lambda) \), since \((w_0^p(\Lambda))^\beta = W^p(\Lambda)\) by Theorem 2 (a). Therefore \((c_0^p(\Lambda))^\beta \subset C_0^p(\Lambda)\). Thus we have shown \((c_0^p(\Lambda))^\beta = (c_0^\infty(\Lambda))^\beta = C_0^\beta(\Lambda)\) for \(0 < p < 1\).

If condition (2.2) holds, then \( c_0^p(\Lambda) \subset c^p(\Lambda) \subset c^{\infty}(\Lambda)\), and so \((c^p(\Lambda))^\beta = C_0^\beta(\Lambda)\).

The assertions concerning the \(\gamma\)– and \(f\)–duals are proved in the same way as in Theorem 2 (a).

Now we consider the case where condition (2.2) does not hold. We assume \( a \in C_0^\beta(\Lambda) \cap cs \).

Let \( x \in c^p(\Lambda) \) be given. Then there is \( l \in \mathbb{C} \) such that \( x - le \in c^p(\Lambda) \), and so \( ax = a(x - le) + lae \in cs \), hence \( a \in (c^p(\Lambda))^\beta \). Conversely, let \( a \in (c^p(\Lambda))^\beta \). Then \( a \in (c_0^p(\Lambda))^\beta = C_0^\beta(\Lambda) \), since \( c_0^p(\Lambda) \subset c^p(\Lambda) \) implies \((c^p(\Lambda))^\beta \subset (c_0^p(\Lambda))^\beta \). Since \( e \in c^p(\Lambda) \), we also have \( a = ae \in cs \).

The identity \((c^p(\Lambda))^\gamma = C_0^\beta(\Lambda) \cap bs\) is proved in exactly the same way.

(b) Since \( c_0^p(\Lambda) \) is an FK space with AK, the representation of \((c_0^p(\Lambda))^\ast \) follows from [13, Theorem 7.2.9, p. 107].

(c) Let \( 0 < p < 1 \) and condition (2.2) hold.

We assume \( f \in (c^p(\Lambda))^\ast \). Then \( f_1 = f \big|_{c_0^p(\Lambda)} \in (c_0^p(\Lambda))^\ast \). Given \( x \in c^p(\Lambda) \), there is a sequence \( a \in C_0^\beta(\Lambda) \) such that \( f_1(x - le) = \sum_{k=0}^{\infty} a_k(x_k - l) \). Since \( a \in C_0^\beta(\Lambda) \), we have \( a \in cs \) for all \( x \in c^p(\Lambda) \), in particular, for \( x = e \in c^p(\Lambda) \), this implies \( ae = a \in cs \), and we may write
\[
f(x) = l \left( f(e) - \sum_{k=0}^{\infty} a_k \right) + \sum_{k=0}^{\infty} a_k x_k.
\]

Putting \( \chi f = f(e) - \sum_{k=0}^{\infty} a_k x_k \), we obtain the given representation.

Conversely, if \( f \) has the given representation, then \( f \in c^*(\Lambda) \), and so \( f \in (c^p(\Lambda))^\ast \).

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