A NOTE ON A CERTAIN CLASS OF FUNCTIONS RELATED TO HURWITZ ZETA FUNCTION AND LAMBERT TRANSFORM

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Abstract. In this paper we obtain multiple-series generating relations involving a class of function \( \theta_{(p_n)}(s, a; x_1, \ldots, x_n) \) which are connected to the Hurwitz zeta function. Also, a new generalization of Lambert transform is introduced, and its relationship with the above class of functions further depicted.

1. Introduction and Preliminaries

The generalized (Hurwitz’s) zeta function is defined by [3]

\[
\zeta(s, a) = \sum_{n=0}^{\infty} (a + n)^{-s}, \quad (\text{Re}(s) > 1; \ a \neq 0, -1, \ldots)
\] (1.1)

and when \( a = 1 \), we have

\[
\zeta(s, 1) = \sum_{n=1}^{\infty} n^{-s} = \zeta(s),
\] (1.2)

where \( \zeta(s) \) is the Riemann zeta function.

The function \( \phi(x, s, a) \) ([3, p.27]) extends (1.1) and is defined by

\[
\phi(x, s, a) = \sum_{n=0}^{\infty} (a + n)^{-s}x^n. \quad (\text{Re}(a) > 0; \ |x| < 1)
\] (1.3)

The integral representation of \( \phi(x, s, a) \) is of form

\[
\phi(x, s, a) = \frac{1}{\Gamma(s)} \int_{0}^{\infty} t^{s-1}a^{-s}(1 - xe^{-t})^{-1}dt,
\] (1.4)

provided that \( R(a) > 0 \) (and either \( |x| \leq 1 \), \( x \neq 1 \), and \( \text{Re}(s) > 0 \), or \( x = 1 \) and \( \text{Re}(s) > 1 \)).

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We introduce a multivariable function \( \theta^{(\mu_n)}_{(p_n)}(s, a; x_1, \ldots, x_n) \) which is defined by
\[
\theta^{(\mu_n)}_{(p_n)}(s, a; x_1, \ldots, x_n) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-at} \prod_{i=1}^n \left( 1 - x_i e^{-pt} \right)^{-\mu_i} dt,
\]
where \( \Omega = \sum_{i=1}^n p_i m_i, \mathrm{Re}(a) > 0, \mu_i \geq 1 \) (either \( |x_i| < 1, x_i \neq 1 \) or \( |x_i| = 1, \mathrm{Re}(s) > n, \forall i = 1, \ldots, n \)).

Equivalently, the integral representation of \( \theta^{(\mu_n)}_{(p_n)}(s, a; x_1, \ldots, x_n) \) is given by
\[
\theta^{(\mu_n)}_{(p_n)}(s, a; x_1, \ldots, x_n) = \sum_{m_1, \ldots, m_n = 0}^\infty (a + \Omega)^{-s} \prod_{i=1}^n \left( \frac{(\mu_i)_{m_i}}{m_i!} x_i^{m_i} \right),
\]
where \( \Omega = \sum_{i=1}^n p_i m_i, \mathrm{Re}(a) > 0, \mu_i \geq 1 \) (either \( |x_i| < 1, x_i \neq 1 \) or \( |x_i| = 1, \mathrm{Re}(s) > n, \forall i = 1, \ldots, n \)).

Special cases of (1.5)

(i) When \( n = p = 1 \), we have
\[
\phi^\mu_1(s, a; x) = \sum_{m=0}^\infty (a + m)^{-s} \frac{(\mu)_{m} x^{m}}{m!} = \phi^*_\mu(x, s, a).
\]

The function \( \phi^*_\mu(x, s, a) \) was studied recently by Goyal and Laddha [4].

(ii) For \( n = p = \mu = 1 \), we have
\[
\theta^1_1(s, a; 1) = \sum_{m=0}^\infty (a + m)^{-s} x^{m} = \phi(x, s, a).
\]

Evidently, the Hurwitz’s zeta function (1.1) is given by the relation
\[
\theta^1_1(s, a; 1) = \zeta(s, a).
\]

(iii) Corresponding to \( \mu_i = x_i = 1 \) (\( \forall i = 1, \ldots, n \)), we have
\[
\theta^{1, \ldots, 1}_{p_1, \ldots, p_n}(s, a; 1, \ldots, 1) = \sum_{m_1, \ldots, m_n = 0}^\infty (a + \Omega)^{-s} = \zeta_n(s, a; p_1, \ldots, p_n),
\]
where \( \Omega = \sum_{i=1}^n p_i m_i \). The class of functions \( \zeta_n(s, a; p_1, \ldots, p_n) \) is the \( n \)-tuple Hurwitz \( \zeta \)-function introduced by Barnes [1] (see also [9]).
(iv) For \( \mu_i = p_i = x_i = 1 \) (\( \forall i = 1, \ldots, n \)), we have

\[
\theta^{(1, \ldots, 1)}_{a, 1, \ldots, 1}(s, a; 1, \ldots, 1) = \sum_{m_1, \ldots, m_n = 0}^{\infty} (a + \Omega^*)^{-s} = \zeta_n(s, a), \tag{1.11}
\]

where \( \Omega^* = \sum_{i=1}^{n} m_i \). The function \( \zeta_n(s, a) \) is the multiple Hurwitz's zeta function (studied recently by Choi [2]).

In the present paper we first obtain certain multiple-series generating functions involving the multivariable function \( \theta^{(\mu_n)}_{(p_n)}(s, a; x_1, \ldots, x_n) \) defined by (1.5) above. A new generalization of Lambert transform is introduced, and its inversion formula, and relationship with the function \( \theta^{(\mu_n)}_{(p_n)}(s, a; x_1, \ldots, x_n) \) are also pointed out. The results presented provide extensions to some of the results in [4] and [6].

2. Generating Relations

Using (1.5) and the multinomial expansion ([7, p.329])

\[
\sum_{k_1, \ldots, k_r = 0}^{\infty} (\lambda)^{\sum_{i=1}^{r} k_i} \prod_{i=1}^{r} \frac{x_i}{k_i!} = (1 - \sum_{i=1}^{r} x_i)^{-\lambda}, \tag{2.1}
\]

provided that \( |\sum_{i=1}^{r} x_i| < 1 \), we easily obtain the generating function:

\[
\sum_{k_1, \ldots, k_r = 0}^{\infty} (\lambda)^{\sum_{i=1}^{r} k_i} \theta^{(\mu_n)}_{(p_n)}(\lambda + \sum_{i=1}^{r} k_i, a; x_1, \ldots, x_n) \prod_{i=1}^{r} \frac{x_i}{k_i!} = \theta^{(\mu_n)}_{(p_n)}(\lambda, a - \sum_{i=1}^{r} t_i; x_1, \ldots, x_n), \tag{2.2}
\]

provided that \( |\sum_{i=1}^{r} t_i| < |a|, \lambda \neq 1 \), and \( |x_i| < 1 \) (\( i = 1, \ldots, n \)).

The generating function (2.2) admits of a further extension. Indeed, in terms of the Lauricella’s multiple hypergeometric series \( F_D^{(r)} \) ([7, p.33]), which is defined by

\[
F_D^{(r)}[a, b_1, \ldots, b_r; c; x_1, \ldots, x_r] = \sum_{m_1, \ldots, m_r = 0}^{\infty} \frac{(a)_{m_1 + \cdots + m_r} (b_1)_{m_1} \cdots (b_r)_{m_r}}{(c)_{m_1 + \cdots + m_r}} \prod_{i=1}^{r} \frac{x_i^{m_i}}{m_i!}, \quad (\max\{||x_1|, \ldots, |x_r||\} < 1) \tag{2.3}
\]

it follows from (1.5) that

\[
\sum_{k_1, \ldots, k_r = 0}^{\infty} (\lambda)^{\sum_{i=1}^{r} k_i} \prod_{i=1}^{r} \frac{(\mu_i)_{k_i} k_i!}{k_i!} \theta^{(\mu_n)}_{(p_n)}(\lambda + \sum_{i=1}^{r} (\mu_i + k_i) - \nu, a; x_1, \ldots, x_n)
\]

\[
= \sum_{m_1, \ldots, m_n = 0}^{\infty} (a + \Omega)^{-\lambda - \nu} \prod_{i=1}^{r} \frac{(\mu_i)_{m_i} x_i^{m_i}}{m_i!} F_D^{(r)}[\lambda, \mu_1, \ldots, \mu_r; \nu, t_1, \ldots, t_r; a + \Omega, \ldots, a + \Omega], \tag{2.4}
\]
The function in several variables defined as follows (see [7, p.38]):

lead to the following multiple generating relation:

\[ \sum_{k_1, \ldots, k_r = 0}^{\infty} \nabla(k_1, \ldots, k_r) \theta_{(p_0)}(\mu) (\sigma + \sum_{i=1}^{r} k_i, a; x_1, \ldots, x_n) \prod_{i=1}^{r} \left\{ \frac{k_i}{k_i!} \right\} = \sum_{m_1, \ldots, m_n = 0}^{\infty} (a + \Omega)^{-\sigma} \prod_{i=1}^{n} \left\{ \frac{(\mu_m) m_i x_i^{m_i}}{m_i!} \right\} F_{s_1, \ldots, s_r}^{P, Q_1, \ldots, Q_r} \left[ \frac{t_1}{a + \Omega}, \ldots, \frac{t_r}{a + \Omega} \right], \tag{2.5} \]

provided that

(i) \( 1 + \sum_{i=1}^{r} (s_i - Q_i) + l - P \geq 0 \) and either \( P > l \) and \( \sum_{i=1}^{r} \frac{l_i}{r_i} \) or \( P \leq l \) and \( \max \{\frac{l_i}{r_i} \} < 1 \) \( (i = 1, \ldots, r) \)

(ii) \( \mu_i \geq 1, \; \Re(a) > 0 \)

where

\[ \nabla(k_1, \ldots, k_r) = \prod_{i=1}^{r} (a_i) r_{-1} \prod_{i=1}^{Q_i} (b_i^1)_{k_i} \cdots \prod_{i=1}^{Q_i} (b_i^r)_{k_i}, \tag{2.6} \]

The function \( F_{s_1, \ldots, s_r}^{P, Q_1, \ldots, Q_r} [x_1, \ldots, x_r] \) occurring in (2.5) is the generalized Lauricella series in several variables defined as follows (see [7, p.38]):

\[ F_{s_1, \ldots, s_r}^{P, Q_1, \ldots, Q_r} [x_1, \ldots, x_r] = F_{s_1, \ldots, s_r}^{P, Q_1, \ldots, Q_r} \left[ (a_1) : (b_1^1, \ldots, b_1^r); (a_2) : (b_2^1, \ldots, b_2^r); \ldots; (a_r) : (b_r^1, \ldots, b_r^r); x_1, \ldots, x_r \right] = \sum_{k_1, \ldots, k_r = 0}^{\infty} \nabla(k_1, \ldots, k_r) \prod_{i=1}^{r} \left\{ \frac{x_i}{k_i!} \right\}, \tag{2.7} \]

where, \( \nabla(k_1, \ldots, k_r) \) is defined above by (2.6).

Next, consider a set of polynomials \( \{S_{m}^q(x)\}_{m=0}^{\infty} \) defined by [5, p.1, Eq. (1)]:

\[ S_{m}^q(x) = \sum_{j=0}^{m/q} (-m)_{qj} j! C(j) x^j, \quad (q \in \mathbb{N}; m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}) \tag{2.8} \]

where \( C(j) \) are arbitrary constants (real or complex).

Then, by simple series rearrangement method, and applying (2.2) in the process, we are lead to the following multiple generating relation:

\[ \sum_{k_1, \ldots, k_r = 0}^{\infty} (\lambda)^r \sum_{i=1}^{r} \theta_{(p_0)}(\mu) (\lambda + \sum_{i=1}^{r} k_i, a; x_1, \ldots, x_n) \prod_{i=1}^{r} S_{k_i}^q(y_i) \frac{k_i!}{k_i!} \]

\[ = \sum_{k_1, \ldots, k_r = 0}^{\infty} (\lambda)^r \sum_{i=1}^{r} \theta_{(p_0)}(\mu) (\lambda + \sum_{i=1}^{r} k_i q_i, a - \sum_{i=1}^{r} j_i; x_1, \ldots, x_n) \prod_{i=1}^{r} \left\{ \frac{C(k_i) y_i (t_i)^{\mu_i}}{k_i!} \right\}, \tag{2.9} \]
By specializing the sequence $C(k_i)$ as follows:

$$C(k_i) = \frac{\prod_{j=1}^{P} (a_j^{(i)})_{k_i}}{q_i^{k_i} \prod_{j=1}^{Q} (b_j^{(i)})_{k_i}},$$  \hspace{1cm} (2.10)

we then find from (2.9) that

$$\sum_{k_1, \ldots, k_r = 0}^{\infty} (\lambda) \sum_{i=1}^{r} \theta^{(\mu_n)}(\lambda + \sum_{i=1}^{r} k_i, a; x_1, \ldots, x_n) \prod_{i=1}^{r} \left\{ \frac{\Delta(q_i) \cdot \theta^{(\mu_n)}(b_q)}{y_i} \right\}$$

$$= \sum_{k_1, \ldots, k_r = 0}^{\infty} (\lambda) \sum_{i=1}^{r} k_i \theta^{(\mu_n)}(\lambda + \sum_{i=1}^{r} k_i, a - \sum_{i=1}^{r} t_i; x_1, \ldots, x_n) \prod_{i=1}^{r} \left\{ \frac{\Delta(q_i) \cdot \theta^{(\mu_n)}(b_q)}{y_i} \right\}$$

$$= \sum_{k_1, \ldots, k_r = 0}^{\infty} (\lambda) \sum_{i=1}^{r} k_i \theta^{(\mu_n)}(\lambda + \sum_{i=1}^{r} k_i, a - \sum_{i=1}^{r} t_i; x_1, \ldots, x_n) \prod_{i=1}^{r} \left\{ \frac{\Delta(q_i) \cdot \theta^{(\mu_n)}(b_q)}{y_i} \right\} \cdot \prod_{i=1}^{r} \frac{\prod_{j=1}^{P} (a_j^{(i)})_{k_i}}{k_i! \prod_{j=1}^{Q} (b_j^{(i)})_{k_i}} \left\{ y_i \left( \frac{-t_i}{q_i} \right) \right\} \left\{ \frac{y_i}{k_i!} \right\} \left\{ 1 + \frac{\alpha_i}{k_i} \right\} \left\{ 1 + \frac{\alpha_i}{k_i} \right\} \left\{ 1 + \frac{\alpha_i}{k_i} \right\} \left\{ \frac{y_i}{k_i!} \right\}$$

$$= \sum_{k_1, \ldots, k_r = 0}^{\infty} (\lambda) \sum_{i=1}^{r} k_i \theta^{(\mu_n)}(\lambda + \sum_{i=1}^{r} k_i, a - \sum_{i=1}^{r} t_i; x_1, \ldots, x_n) \prod_{i=1}^{r} \left\{ \frac{\Delta(q_i) \cdot \theta^{(\mu_n)}(b_q)}{y_i} \right\} \cdot \prod_{i=1}^{r} \frac{\prod_{j=1}^{P} (a_j^{(i)})_{k_i}}{k_i! \prod_{j=1}^{Q} (b_j^{(i)})_{k_i}} \left\{ y_i \left( \frac{-t_i}{q_i} \right) \right\} \left\{ \frac{y_i}{k_i!} \right\} \left\{ 1 + \frac{\alpha_i}{k_i} \right\} \left\{ 1 + \frac{\alpha_i}{k_i} \right\} \left\{ 1 + \frac{\alpha_i}{k_i} \right\} \left\{ \frac{y_i}{k_i!} \right\}$$

where $\Delta(m, \lambda)$ denotes the array of $m$-parameters

$$\frac{\lambda}{m}, \frac{\lambda + 1}{m}, \ldots, \frac{\lambda + m - 1}{m} \quad (m \in \mathbb{N}).$$

**Example.** By involving the Laguerre polynomials (which occurs when $P_i = 0, Q_i = 1 = q_i, b_j^{(i)} = 1 + \alpha_i (i = 1, \ldots, r)$), (2.11) yields

$$\sum_{k_1, \ldots, k_r = 0}^{\infty} (\lambda) \sum_{i=1}^{r} k_i \theta^{(\mu_n)}(\lambda + \sum_{i=1}^{r} k_i, a; x_1, \ldots, x_n) \prod_{i=1}^{r} \left\{ \frac{\Delta(q_i) \cdot \theta^{(\mu_n)}(b_q)}{y_i} \right\} \cdot \prod_{i=1}^{r} \frac{\prod_{j=1}^{P} (a_j^{(i)})_{k_i}}{k_i! \prod_{j=1}^{Q} (b_j^{(i)})_{k_i}} \left\{ y_i \left( \frac{-t_i}{q_i} \right) \right\} \left\{ \frac{y_i}{k_i!} \right\} \left\{ 1 + \frac{\alpha_i}{k_i} \right\} \left\{ 1 + \frac{\alpha_i}{k_i} \right\} \left\{ 1 + \frac{\alpha_i}{k_i} \right\} \left\{ \frac{y_i}{k_i!} \right\}$$

Several other examples similar to (2.12) can be obtained from (2.11) by suitably specializing the sequence $C(k_i)$. We omit further details.

3. **An Integral Transform**

Let $f(t) (t \geq 0)$ be a continuous function, and

$$f(t) = O(e^{kt}) (t \to \infty).$$  \hspace{1cm} (3.1)
Then, the Lambert transform of \( f(t) \) is defined by

\[
F(s) = LM \{f(t)\} = \int_0^\infty \frac{st}{e^{st} - 1} f(t) \, dt. \quad (\text{Re}(s) > 0)
\] (3.2)

We introduce a generalization of the Lambert transform (3.2) in the following form:

\[
H^* \{f(t)\} = H^*(\alpha_\mu_\nu)(x_1, \ldots, x_n; s) = H^*(\mu_1, \ldots, \mu_n)(x_1, \ldots, x_n; s)
\]

\[
= \int_0^\infty \frac{st}{\prod_{i=1}^n (e^{pi s} - x_i)\mu_i} f(t) \, dt,
\] (3.3)

provided that \( \text{Re}(s) > 0, p_i > 0, |\mu_i| \geq 1, \max |x_i| \leq 1 \, (\forall i = 1, \ldots, n) \), \( f(t) \in A \) and \( \text{Re}(\gamma) > -2 \), where \( A \) denotes the class of functions \( f(t) \) which are continuous for \( t > 0 \) and satisfy the order estimates:

\[
f(t) = \begin{cases} O(t^\gamma) & (t \to 0^+), \\ O(t^\delta) & (t \to \infty). 
\end{cases}
\] (3.4)

The parameter \( \delta \) is unrestricted, in general, since \( \text{Re}(s) > 0, p_i > 0 \) \((i = 1, \ldots, n)\). We note that on putting \( n = p = 1 \), and setting \( f = i^{k-1} g \) in (3.3), we have

\[
H^*_{1}(x_i; s) = \int_0^\infty \frac{st}{(e^{s x_i} - t)^\mu} g(t) \, dt,
\] (3.5)

which was recently studied by Goyal and Laddha [4]. Evidently,

\[
H^*_{1}(x; s) = \int_0^\infty \frac{st}{e^{s x} - t} f(t) \, dt,
\] (3.6)

the transform investigated by Raina and Srivastava [6]. It readily follows from (3.3) and (1.6) that

\[
H^* \{t^{\alpha-1} e^{-\nu st}\} = \frac{\Gamma(\alpha + 1)}{s^\alpha} \theta_{(\mu_\nu)}(\alpha + 1, \nu + \sum_{i=1}^n p_i \mu_i; x_1, \ldots, x_n),
\] (3.7)

provided that \( \text{Re}(s) > 0, \text{Re}(\alpha) > -1, \text{Re}(\nu) > 0, \mu_i \geq 1, p_i > 0 \), and \( |x_i| \leq 1 \, (\forall i = 1, \ldots, n) \).

Further, in view of (2.8) and (3.7) we obtain

\[
H^* \left\{ t^{\alpha-1} e^{-\nu st} \prod_{i=1}^r (S_{m_i}^{y_i} (y_i t)) \right\}
\]

\[
= \frac{1}{s^\alpha} \sum_{j_1=0}^{[m_1/q_1]} \cdots \sum_{j_r=0}^{[m_r/q_r]} \Gamma(1 + \alpha + \sum_{i=1}^r j_i) \prod_{i=1}^r \left\{ \frac{(-m_i)_{j_i}}{j_i!} C(j_i) \left( \frac{y_i}{s} \right)^{j_i} \right\}
\]

\[
\theta_{(\mu_\nu)}(1 + \alpha + \sum_{i=1}^r j_i, \nu + \sum_{i=1}^n p_i \mu_i; x_1, \ldots, x_n)
\] (3.8)
provided that Re(s) > 0, Re(α) > -1, Re(ν) > 0 and μ_i ≥ 1, p_i > 0, |x_i| ≤ 1 (∀i = 1, . . . , n) and q_i ∈ N, m_i ∈ N_0 (∀i = 1, . . . , r). In particular, when C(j_i) = \frac{(1 + α_i)_m}{(t + α_i)_m}, q_i = 1 (∀i = 1, . . . , r), then (3.8) in terms of Laguerre polynomials gives

\begin{equation}
H^s \{ \ell^{α-1}e^{-νst}L_{m_j}(x_1 t) \cdots L_{m_r}(x_r t) \} = \prod_{i=1}^{r} \left\{ \frac{Γ(m_i + α_i + 1)}{m_i!} \right\} s^{m_i - 1} t \sum_{j=0}^{m_i} \sum_{j=0}^{m_r} Γ(1 + α + \sum_{i=1}^{r} j_i)
\end{equation}

\begin{equation}
θ_{(p)}^{(μ)}(1 + α + \sum_{i=1}^{r} j_i, ν + \sum_{i=1}^{n} p_i μ_i; x_1, . . . , x_n) \prod_{i=1}^{r} \left\{ \frac{(-m_i)_j, (x_i^{j_i})}{Γ(1 + α_i + j_i)!} \right\}. \tag{3.9}
\end{equation}

### Inversion formula for the transform (3.3)

Applying the Mellin transform [8, p.46], (3.3) then gives

\begin{align*}
Ψ(k) &= \int_{0}^{∞} s^{-k-1} H^{(μ)}_{(p)}(x_1, . . . , x_n; s) ds \\
&= \int_{0}^{∞} s^{-k-1} \left\{ \int_{0}^{∞} \prod_{i=1}^{n} (e^{p_i s t} - x_i)^{-μ_i} f(t) dt \right\} ds \\
&= \int_{0}^{∞} t f(t) \left\{ \int_{0}^{∞} s^{-k} \prod_{i=1}^{n} (e^{p_i s t} - x_i)^{-μ_i} ds \right\} dt, \tag{3.10}
\end{align*}

provided that, in addition to the existence and convergence conditions stated with (3.3), we also require that Re(k) < 1, for the convergence of the inner s-integral in (3.10) above.

By the Mellin inversion theorem [8, p.46], we obtain the following inversion formula for the integral transform (3.3):

\begin{align*}
\frac{1}{2} [f(t + 0) + f(t - 0)] \\
= \frac{1}{2πi} \int_{C}^{σ+∞} Γ(1 - k) \left\{ \prod_{i=1}^{n} p_i μ_i; x_1, . . . , x_n \right\}^{-1} t^{-k} Ψ(k) dk, \tag{3.12}
\end{align*}

provided that σ > 1/2, Re(k) < 1, t^k f(t) ∈ L(0, ∞), f(t) is of bounded variation in the neighbourhood of the point t, Ψ(k) is given by (3.10), and μ_i ≥ 1, p_i > 0, max{|x_i|} ≤ 1 (∀i = 1, . . . , n).
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