CERTAIN CLASSES OF MEROMORPHICALLY MULTIVALENT
FUNCTIONS WITH FIXED ARGUMENT OF COEFFICIENTS

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Abstract. In this paper, we consider some classes of meromorphically multivalent functions with
fixed argument of coefficients. In those classes, we determine coefficient estimates, distortion
theorems and extreme points.

1. Introduction

Let $\Sigma_p$ denote the class of functions $f$ of the form

$$f(z) = \frac{1}{z^p} + \sum_{n=1}^{\infty} a_n z^n \quad (p \in N = \{1,2,\ldots\})$$

which are analytic in $U - \{0\}$, where $U = \{z : |z| < 1\}$. For analytic functions $f$ and
g, we say that $f$ is subordinate to $g$, written $f \prec g$, if there exists a Schwarz function $w$
such that $f(z) = g(w(z))$ for $z \in U$.

Definition 1.1. Let $\Sigma_{p}^{\theta}(A, B)$ denote the class of functions $f$ of the form (1.1)
such that

$$-z^{p+1}f'(z) \prec z^p \frac{1 + Az}{1 + Bz}$$

and $\text{arg}a_n = \theta$ for $n \in \mathbb{N}$, where $0 \leq B \leq 1$ and $-B \leq A < B$ $(B \neq 1$ or $\cos \theta > 0)$.

We note that every function $f$ belonging to the class $\Sigma_{p}^{\theta}(A, B)$ can be written as in
the form

$$f(z) = \frac{1}{z^p} + e^{i\theta} \sum_{n=1}^{\infty} |a_n| z^n \quad (z \in U - \{0\}).$$

Definition 1.2. Let $\tilde{\Sigma}_{p}^{\theta}(A, B)$ denote the class of functions $f$ of the form (1.3)
satisfying the following condition

$$\sum_{n=1}^{\infty} n|a_n| \leq \delta(\theta, A, B),$$

Received March 5, 1999.
1991 Mathematics Subject Classification. 30C45.
Key words and phrases. Meromorphically starlike of order $\alpha$, meromorphically convex of order
$\alpha$, fixed argument.
where
\[ \delta(\theta, A, B) = \frac{p(B - A)}{\sqrt{1 - B^2 \sin^2 \theta + B \cos \theta}}, \quad 0 \leq B \leq 1 \text{ and } -B \leq A < B \quad (B \neq 1 \text{ or } \cos \theta > 0). \] (1.5)

Meromorphic univalent functions have been extensively studied by Clunie [1], Libera [2], Mogra, Reddy and Juneja [4], Pommerenke [5] and others. In particular, the class \( \Sigma_p^0(A, B) \) was studied by Mogra [3].

The object of the present paper is to obtain coefficient estimates, distortion theorems and extreme points for the classes of functions defined above.

2. Coefficient Estimates

Theorem 2.1. If a function \( f \) of the form (1.3) belongs to the class \( \Sigma_p^0(A, B) \), then it satisfies the condition (1.4).

Proof. Let \( f \in \Sigma_p^0(A, B) \). By Definition (1.1), we obtain
\[ -z^{p+1}f'(z) = p\frac{1 + Aw(z)}{1 + Bw(z)}, \]
where \( w \) is an analytic function in \( U \) such that \( w(0) = 0 \) and \( |w(z)| < 1 \) for \( z \in U \). Thus we have
\[ \left| \frac{z^{p+1}f'(z) + p}{Ap + Bz^{p+1}f'(z)} \right| = |w(z)| < 1. \]

Then we have
\[ \left| \sum_{n=1}^{\infty} n|a_n|z^{n+p} \right| < |Ap + Bz^{p+1}f'(z)|. \]

Putting \( z = r(0 < r < 1) \), we obtain
\[ |w| < |(B - A)p - B e^{i\theta} w|, \] (2.1)
where
\[ w = \sum_{n=1}^{\infty} n|a_n|r^{n+p}. \]

Since \( w \) is a real number, by (2.1) we have
\[ (1 - B^2)w^2 + 2pB(B - A) \cos \theta w - p^2(B - A)^2 < 0. \]

Solving this inequality with respect to \( w \), we obtain
\[ \sum_{n=1}^{\infty} n|a_n|r^{n+p} < \delta(\theta, A, B), \]
where \( \delta(\theta, A, B) \) is defined by (1.5). Therefore, letting \( r \to 1^{-} \), we have (1.4).

From Theorem 2.1, we have

**Corollary 2.1.** \( \Sigma^\theta_p(A, B) \subset \tilde{\Sigma}^\infty_p(A, B) \).

**Corollary 2.2.** If a function \( f \) of the form (1.3) belongs to the class \( \tilde{\Sigma}^\theta_p(A, B) \), then

\[
|a_n| \leq \frac{\delta(\theta, A, B)}{n} \quad (n \in \mathbb{N}). \tag{2.2}
\]

The result is sharp for the extremal functions \( f_n \) of the form

\[ f_n(z) = \frac{1}{z^p} + e^{i\theta} \frac{\delta(\theta, A, B)}{n} z^n \quad (n \in \mathbb{N}). \]

By Corollary 2.1 and Corollary 2.2, we obtain

**Corollary 2.3.** If a function \( f \) of the form (1.3) belongs to the class \( \Sigma^\theta_p(A, B) \), then

\[
|a_n| \leq \frac{\delta(\theta, A, B)}{n} \quad (n \in \mathbb{N}),
\]

where \( \delta(\theta, A, B) \) is defined by (1.5). The result is sharp for \( \theta = 0 \). The extremal functions are functions \( f_n \) of the form

\[ f_n(z) = \frac{1}{z^p} + \frac{p(B - A)}{(1 + B)n} z^n \quad (n \in \mathbb{N}). \tag{2.3} \]

### 3. Distortion Theorems and Extreme Points

**Theorem 3.1.** If \( f \in \Sigma^\theta_p(A, B) \), then

\[
\frac{1}{|z|^p} - \delta(\theta, A, B)|z| \leq |f(z)| \leq \frac{1}{|z|^p} + \delta(\theta, A, B)|z| \tag{3.1}
\]

and

\[
\frac{p}{|z|^{p+1}} - \delta(\theta, A, B) \leq |f'(z)| \leq \frac{p}{|z|^{p+1}} + \delta(\theta, A, B), \tag{3.2}
\]

where \( \delta(\theta, A, B) \) is defined by (1.5). The result is sharp for \( \theta = 0 \). The extremal function is function \( f_1 \) of the form (2.3).

**Proof.** Let a function \( f \) of the form (1.3) belong to the class \( \Sigma^\theta_p(A, B) \). By Theorem 2.1, we obtain

\[
\sum_{n=1}^\infty |a_n| \leq \sum_{n=1}^\infty n|a_n| \leq \delta(\theta, A, B). \tag{3.3}
\]
Since
\[ |f(z)| = \left| \frac{1}{z^p} + e^{i\theta} \sum_{n=1}^{\infty} |a_n| z^n \right| \leq \frac{1}{|z|^p} + \sum_{n=1}^{\infty} |a_n||z|^n \leq \frac{1}{|z|^p} + |z| \sum_{n=1}^{\infty} |a_n| \leq \frac{1}{|z|^p} + |z| \delta(\theta, A, B), \]
and
\[ |f(z)| = \left| \frac{1}{z^p} + e^{i\theta} \sum_{n=1}^{\infty} |a_n| z^n \right| \geq \frac{1}{|z|^p} - \sum_{n=1}^{\infty} |a_n||z|^n \geq \frac{1}{|z|^p} - |z| \sum_{n=1}^{\infty} |a_n| \geq \frac{1}{|z|^p} - |z| \delta(\theta, A, B) \]
by (3.3), we obtain (3.1). Using (3.3), we prove the estimation (3.2) analogously.

**Theorem 3.2.** Let \( \delta(\theta, A, B) \) be defined by (1.5) and let
\[ f_0(z) = \frac{1}{z^p} \]  
(3.4)
and
\[ f_n(z) = \frac{1}{z^p} + e^{i\theta} \frac{\delta(\theta, A, B)}{n} z^n \]  
(3.5)
Then a function \( f \) belongs to the class \( \tilde{\Sigma}_p(A, B) \) if and only if it is of the form
\[ f(z) = \sum_{n=0}^{\infty} \gamma_n f_n(z) \quad (z \in U - \{0\}), \]  
(3.6)
where \( \sum_{n=0}^{\infty} \gamma_n = 1 \) and \( \gamma_n \geq 0 \) \( (n \in N \cup \{0\}) \).

**Proof.** Let a function \( f \) of the form (1.3) belong to the class \( \tilde{\Sigma}_p(A, B) \). Put
\[ \gamma_n = \frac{n}{\delta(\theta, A, B)} |a_n| \quad (n \in N) \]
and
\[ \gamma_0 = 1 - \sum_{n=1}^{\infty} \gamma_n. \]
By the assumption and Definition 1.2, we have \( \gamma_n \geq 0 \) \( (n \in N) \) and \( \gamma_0 \geq 0 \). Thus
\[ \sum_{n=0}^{\infty} \gamma_n f_n(z) = \gamma_0 f_0(z) + \sum_{n=1}^{\infty} \gamma_n f_n(z) \]
\[ = \left( 1 - \sum_{n=1}^{\infty} \gamma_n \right) \frac{1}{z^p} + \sum_{n=1}^{\infty} \frac{n}{\delta(\theta, A, B)} |a_n| \left( \frac{1}{z^p} + e^{i\theta} \frac{\delta(\theta, A, B)}{n} z^n \right) \]
\[ = \frac{1}{z^p} - \sum_{n=1}^{\infty} \frac{n}{\delta(\theta, A, B)} |a_n| \frac{1}{z^p} + \sum_{n=1}^{\infty} \frac{n}{\delta(\theta, A, B)} |a_n| \frac{1}{z^p} + e^{i\theta} \sum_{n=1}^{\infty} |a_n| z^n \]
\[ = f(z) \]
and the condition (3.6) follows. Conversely, let the function $f$ satisfy (3.6). Since

$$f(z) = \sum_{n=0}^{\infty} \gamma_n f_n(z)$$

$$= \gamma_0 f_0(z) + \sum_{n=1}^{\infty} \gamma_n f_n(z)$$

$$= \left(1 - \sum_{n=1}^{\infty} \gamma_n\right) \frac{1}{z^p} + \sum_{n=1}^{\infty} \left(\frac{1}{z^p} + e^{i\theta} \frac{\delta(\theta, A, B)}{n} z^n\right) \gamma_n$$

$$= \frac{1}{z^p} + e^{i\theta} \sum_{n=1}^{\infty} \frac{\delta(\theta, A, B)}{n} \gamma_n z^n,$$

we can write the function $f$ in the form (1.3), where

$$|a_n| = \frac{\delta(\theta, A, B)}{n} \gamma_n.$$

Moreover,

$$\sum_{n=1}^{\infty} n|a_n| = \sum_{n=1}^{\infty} \gamma_n \delta(\theta, A, B) = \delta(\theta, A, B) (1 - \gamma_0) \leq \delta(\theta, A, B).$$

Thus we have $f \in \tilde{\Sigma}_p^\theta(A, B)$, which completes the proof of our result.

By using the same method as in the proof of Theorem 3.2, we can prove the following.

**Theorem 3.3.** Let $f_0(z) = \frac{1}{z^p}$ and let $f_n (n \in N)$ be defined by (2.3). Then a function $f$ belongs to the class $\Sigma_p^n$ if and only if it is of the form (3.6).

From Theorem 3.2 and Theorem 3.3, we obtain the following two corollaries.

**Corollary 3.1.** $\Sigma_p^n(A, B) = \tilde{\Sigma}_p^n(A, B)$.

**Corollary 3.2.** The class $\Sigma_p^n(A, B)$ is convex.

**Acknowledgement**

This work was partially supported by Pukyong National University (1998) and the Korea Research Foundation (Project No.: 1998-015-D00039).

**References**


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