GENERALIZATION ON SOME THEOREMS OF $L^1$-CONVERGENCE OF CERTAIN TRIGONOMETRIC SERIES

ŽIVORAD TOMOVSKI

Abstract. In this paper we study $L^1$-convergence of the $r$-th derivatives of Fourier series with complex-valued coefficients. Namely new necessary-sufficient conditions for $L^1$-convergence of the $r$-th derivatives of Fourier series are given. These results generalize corresponding theorems proved by several authors (see [7], [10], [13], [19]). Applying the Wang-Telyakovskii class $(BV)^{\sigma}$, $\sigma > 0$, $r = 0, 1, 2, \ldots$ we generalize also the theorem proved by Garrett, Rees and Stanojević in [5]. Finally, for $\sigma = 1$ some corollaries of this theorem are given.

1. Introduction

Let $\{c_k: \ k = 0, \pm 1, \pm 2, \ldots\}$ be a sequence of complex numbers and the partial sums of the complex trigonometric series
\[ \sum_{k=-\infty}^{\infty} c_k e^{ikt}, \]
be denoted by
\[ S_n(c) = S_n(c, t) = \sum_{k=-n}^{n} c_k e^{ikt}, \quad t \in (0, \pi]. \]

A sequence $\{c_k\}$ is of bounded variation of integer order $m \geq 1$, i.e. $\{c_k\} \in (BV)^m$ if
\[ \sum_{k=-\infty}^{\infty} |\Delta^m c_k| < \infty, \]
where
\[ \Delta^m c_k = \Delta(\Delta^{m-1} c_k) = \Delta^{m-1} c_k - \Delta^{m-1} c_{k+1}. \]

For $m = 1$, the class $(BV)^1$ is the class of complex sequences of bounded variation.

If the trigonometric series (1.1) is a Fourier series of some $f \in L^1$, we shall write $c_n = \hat{f}(n)$, for all $n$ and the partial sums of the corresponding Fourier series are denoted by $S_n(f) = S_n(f, t) = \sum_{k=-n}^{n} \hat{f}(k) e^{ikt}$.

A complex null sequence $\{c_n\}$ satisfying
\[ \sum_{n=1}^{\infty} |\Delta(c_n - c_{n-1})| \lg n < \infty \]
is called weakly even (see [10]). It is obvious that if $\{c_n\}$ is an even sequence then it is weakly even.

Received August 10, 2004; revised September 10, 2007.
2000 Mathematics Subject Classification. 26D15, 42A20.
Key words and phrases. $L^1$-convergence, trigonometric series, $r$-th derivatives of Fourier series, Wang-Telyakovskii class, Sheng class, Fomin class.
In the case of complex coefficients, i.e. if \( \{c_n\} \) is weakly even, the modified sums are defined as follows (see [2]):

\[
G_n(c, t) = \sum_{k=0}^{n} (\Delta c_k) D_k(t) + \sum_{k=0}^{n+1} (\Delta(c_{-k} - c_k))(E_{-k}(t) - \frac{1}{2}),
\]

where \( E_k(t) = \frac{1}{2} + \sum_{j=1}^{k} e^{ijt} \) and \( D_k \) is the Dirichlet kernel. In the case of real coefficients (see [4]) Garrett and Stanojević, defined the following class \( C \). A null sequence \( \{a_n\} \) of real numbers belongs to the class \( C \) if for every \( \varepsilon > 0 \), there exists a \( \delta > 0 \), independent of \( n \) and such that

\[
\int_{0}^{\delta} \left| \sum_{k=n+1}^{\infty} \Delta a_k D_k(t) \right| dt < \varepsilon \quad \text{for all } n.
\]

Let

\[
S_n(a) = S_n(a, t) = a_0 + \sum_{k=1}^{n} a_k \cos kt, \quad t \in (0, \pi],
\]

where \( \{a_n\} \) is a real null sequence of bounded variation of integer order \( m \geq 1 \), i.e. \( \{a_n\} \in (\text{BV})^m \). Then \( \lim_{n \to \infty} S_n(a, t) = f(t) \) exists in \( (0, \pi] \) (see [5]).

Garrett, Rees and Stanojević (see [5]) have given necessary and sufficient conditions for series (1.3) to be Fourier series of some \( f \in L^1(0, \pi) \). Namely they have proved the following theorem.

**Theorem A.** Let \( \{a_n\} \in (\text{BV})^m, m = 1, 2, 3, \ldots \) and \( a_n \log n = o(1), n \to \infty \). Then

\[
\|S_n - f\| = o(1), \quad n \to \infty \quad \text{if and only if } \quad \{a_n\} \in C.
\]

The difference of noninteger order \( \sigma \geq 0 \) of the sequence \( \{a_n\}_{n=0}^{\infty} \) is defined as follows:

\[
\Delta^\sigma a_n = \sum_{m=0}^{\infty} \binom{m-\sigma-1}{m} a_{n+m} \quad (n = 0, 1, 2, \ldots)
\]

where

\[
\binom{m+\alpha}{m} = \frac{(1+\alpha) \cdots (m+\alpha)}{m!}.
\]

Wang and Telyakovskii (see [20]) have considered the following class of real sequences \( \{a_n\} \).

Namely, a null-sequence \( \{a_n\} \) belongs to the class \( (\text{BV})^\sigma_{r}, r = 0, 1, 2, \ldots, \sigma \geq 0 \) if \( \sum_{k=1}^{\infty} k^r |\Delta^\sigma a_k| < \infty \).

**Theorem B.**([20]) Let \( \rho \geq 0, \sigma \geq 0 \). Then for all \( \gamma > \sigma \) the following embedding relation holds,

\[
(\text{BV})^\sigma_{\rho} \subset (\text{BV})^\gamma_{\rho}.
\]
In the same paper, Wang and Telyakovskii by considering the complex form of trigonometric series
\[ \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{inx}, \quad x \in (0, \pi) \]
have proved the following theorem.

**Theorem C.** If
\[ \{a_k\} \in (BV)_r^\sigma, \quad r = 0, 1, 2, \ldots, \quad \sigma \geq 0 \]
then cosine series (1.3) and sine series \( \sum_{n=1}^{\infty} a_n \sin nx \) have derivatives of \( r \)-th order on \((0, \pi]\).

The Wang-Telyakovskii class \((BV)_r^\sigma, r = 0, 1, 2, \ldots, \sigma \geq 0\), motivated us to consider a further class \((BV)_m^r, r = 0, 1, 2, \ldots, m = 1, 2, 3, \ldots\) (see [18]) of complex null-sequences \( \{c_n\} \) such that
\[ \sum_{k=-\infty}^{\infty} |k|^r |\Delta^m c_k| < \infty. \]

For \( r = 0 \), we have \((BV)_m^0 = (BV)_m^m\).

On the other hand in [14], we have defined the extension \( C_r, r = 1, 2, 3, \ldots \) of the Garret-Stanojević class \( C \) as follows:

A null real sequence \( \{a_n\} \) belongs to the class \( C_r, r = 1, 2, 3, \ldots \) if for every \( \varepsilon > 0 \), there exists \( \delta > 0 \), independent of \( n \) and such that
\[ \int_{0}^{\delta} \left| \sum_{k=n+1}^{\infty} \Delta a_k D_k^{(r)}(x) \right| dx < \varepsilon, \quad \text{for all} \ n. \]

V. B. Stanojević in [9] defined the class \( C^* \) of all weakly even complex sequences such that for every \( \varepsilon > 0 \), there exists \( \delta(\varepsilon) > 0 \), independent of \( n \), and
\[ \int_{|t| \leq \delta} \left| \sum_{k=n+1}^{\infty} (\Delta c_k) D_k(t) \right| dt < \varepsilon, \quad \text{for all} \ n. \]

In this paper, we shall consider complex null-sequences \( \{c_n\} \) such that
\[ \sum_{n=1}^{\infty} |\Delta(c_n - c_{-n})|^n n^r \ln n < \infty. \quad (1.4) \]

Let \( C_r^* \) denote the class of all complex null sequences \( \{c_n\} \) such that for every \( \varepsilon > 0 \) there exists \( \delta(\varepsilon) > 0 \), independent of \( n \) and such that
\[ \int_{|t| \leq \delta} \left| \sum_{k=n+1}^{\infty} (\Delta c_k) D_k^{(r)}(t) \right| dt < \varepsilon, \quad \text{for all} \ n. \]

Let
\[ \lim_{n \to \infty} S_n^{(r)}(f, t) = f_r(t), \quad r = 1, 2, 3, \ldots \]
If $f_r \in L^1$, then it is denoted by $f^{(r)}(t)$.

We note that the class $(BV)^m_r$, $r = 1, 2, \ldots, m = 1, 2, 3, \ldots$, for $m = 1$ is the class $(BV)_r$, defined in [13].

In [13], we have proved the following theorem.

**Theorem D.** Let $\{c_n\}$ is a complex sequence such that (1.4) holds. If

$$\{c_n\} \in (BV)_r \cap C^*_r,$$

then

$$\|S_n^{(r)} - f^{(r)}\| = o(1), \quad n \to \infty$$

if and only if

$$n^r |c_n| \lg n = o(1), \quad n \to \infty.$$  

In this paper, we shall extend the Theorem D, by considering the class $(BV)^m_r$, $r = 1, 2, \ldots, m = 1, 2, 3, \ldots$ instead of $(BV)_r$.

In addition we shall give the extension of the Theorem A, by considering the Wang-Telyakovskii class $(BV)^\sigma_r$, $r = 0, 1, 2, \ldots, \sigma > 0$ instead of $(BV)^m_r$, $m = 1, 2, 3, \ldots$.

2. Lemmas

For the proofs of the our main results, we need the following Lemmas:

**Lemma 1.**([7]) For each $r = 0, 1, 2, \ldots$ the following inequality holds

$$\|E^{(r)}_{-n}(t)\| = O(n^r \lg n).$$

**Lemma 2.**([7]) For each nonnegative integer $n$, there holds

$$\|c_nE^{(r)}_{-n}(t) + c_{-n}E^{(r)}_{-n}(t)\| = o(1), \quad n \to \infty$$

if and only if

$$n^r |c_n| \lg n = o(1), \quad n \to \infty.$$  

We note that this lemma for $r = 0$, was proved by W. Bray and Ć. V. Stanovević in [1].

**Lemma 3.** For all $p \geq 1, r = 0, 1, 2, \ldots$ and $\delta > 0$ the following estimate holds

$$\int_{|t|>\delta} \left| \frac{d^r}{dt^r} \left( \frac{e^{it}}{\alpha^{it} - 1} \right) \right|^p dt = O_{p, r, \delta}(1), \quad t \in (0, \pi),$$

where $O_{p, r, \delta}$ depends on $p, r$ and $\delta$.

**Proof.** See the proof of Lemma 1 in [18].
Lemma 4. Let \( \{c_k\} \in (BV)^m, \ m = 1, 2, 3, \ldots, r = 1, 2, 3, \ldots \) and \( n'|c_n| \log n = o(1), \ \ n \to \infty. \) Then for \( \delta > 0 \) the following limit holds
\[
\int_{|t| > \delta} \left| \sum_{k=n+1}^{\infty} \Delta c_k D_k^{(r)}(t) \right| \, dt = o(1), \ \ n \to \infty,
\]
where \( t \in (0, \pi]. \)

Proof. It is easy to prove that
\[
\sum_{k=n+1}^{\infty} (\Delta c_k) E_k(t) = \sum_{k=n+1}^{\infty} c_k e^{i k t} + c_{n+1} E_n(t).
\]
Now we consider the identity, obtained by V. B. Stanovević in [8].

\[
\omega^m \sum_{j=M+n}^{N+n} c_j e^{i j t} = \sum_{j=M}^{N} (\Delta^m c_j) e^{i j t} - \sum_{k=0}^{m-1} \omega^k \left[ (\Delta^{m-k-1} c_{M+k}) e^{i(M+k)t} - (\Delta^{m-k-1} c_{N+k+1}) e^{i(N+k+1)t} \right],
\]
where \( \omega = 1 - e^{-i t}. \)

Setting \( M = n \) and letting \( N \to \infty, \) for \( t \neq 0, \) we obtain:

\[
\omega^m \sum_{j=n+1}^{\infty} c_j e^{i j t} = \omega^m \sum_{j=n}^{\infty} (\Delta^m c_j) e^{i j t} - \sum_{k=0}^{m-1} \omega^k \left[ (\Delta^{m-k-1} c_{n+k}) e^{i(n+k)t} \right],
\]
i.e.

\[
\sum_{j=n+1}^{\infty} c_j e^{i j t} = \omega^{-m} \sum_{j=n}^{\infty} (\Delta^m c_j) e^{i j t} - \omega^{-m} \sum_{k=0}^{m-1} \omega^k \left[ (\Delta^{m-k-1} c_{n+k}) e^{i(n+k)t} \right] + \sum_{j=n+1}^{m+n-1} c_j e^{i j t}.
\]
Hence,

\[
\sum_{k=n+1}^{\infty} (\Delta c_k) E_k(t) = \left( \frac{e^{i t}}{e^{i t} - 1} \right)^m \sum_{j=n}^{\infty} (\Delta^m c_j) e^{i j t} - \left( \frac{e^{i t}}{e^{i t} - 1} \right)^m \sum_{k=0}^{m-1} \sum_{q=0}^{k} (-1)^q \binom{k}{q} e^{i(q+k-q)(\Delta^{m-k-1} c_{n+k})}
\]
\[
+ \sum_{j=n+1}^{m+n-1} c_j e^{i j t} + c_{n+1} E_n(t).
\]
Applying the inequality (see [17])

$$|E_j^{(r)}(t)| \leq \frac{4\pi j^r}{|t|}, \quad 0 < |t| \leq \pi$$

we obtain that the series $\sum_{j=n+1}^{\infty} (\Delta c_j)E_j^{(r)}(t)$ is uniformly convergent on any compact subset of $(0, \pi]$. Also, by $\{c_n\} \in (BV)^{m}$, we obtain that the series $\sum_{j=n+1}^{\infty} j^r(\Delta^m c_j)e^{ijt}$, $m=1, 2, 3, \ldots$ is uniformly convergent on any compact subset of $(0, \pi]$.

Hence,

$$\sum_{k=n+1}^{\infty} (\Delta c_k)E_k^{(r)}(t) = \sum_{j=n+1}^{\infty} \left( r \int_0^t \frac{d^r}{dt^r} \left( \frac{e^{it}}{e^{it} - 1} \right)^m i^{r-v} \sum_{j=n}^{\infty} j^{r-v}(\Delta^m c_j)e^{ijt} \right)$$

$$- \sum_{j=n+1}^{\infty} j^r(\Delta^m c_j)e^{ijt} + \sum_{j=n+1}^{m+n-1} j^r c_j e^{ijt} + c_{n+1}E_n^{(r)}(t).$$

Then,

$$\left| \sum_{k=n+1}^{\infty} (\Delta c_k)E_k^{(r)}(t) \right| \leq \sum_{j=n+1}^{\infty} \left( r \int_0^t \frac{d^r}{dt^r} \left( \frac{e^{it}}{e^{it} - 1} \right)^m i^{r-v} \sum_{j=n}^{\infty} j^{r-v}(\Delta^m c_j) \right)$$

$$+ \sum_{j=n+1}^{\infty} j^r |c_j| + |c_{n+1}| |E_n^{(r)}(t)|.$$
\[
\leq 2^{m-1} \sum_{j=n}^{m+n-1} j|\Delta^{m+n-j-1} c_j|
\leq 2^{m-1} \sum_{j=n}^{m+n-1} \sum_{k=0}^{m+n-j-1} \left( m + n - j - 1 \right) j^r |c_{j+k}|
\]

and \( n^r |c_n| \log n = o(1), n \to \infty \), the second sum on the right-hand side of the inequality (2.2) is finite sum of \( o(1) \)-terms as \( n \to \infty \).

Hence,

\[
\int_{|t|>\delta} \left| \sum_{k=n+1}^{\infty} (\Delta c_k) E_k^{(r)}(t) \right| dt \leq O_{m,r,\delta}(1) \sum_{j=n+1}^{\infty} j^r |\Delta^m c_j| = o(1), \quad n \to \infty. \quad (2.3)
\]

Since \( E_k^{(r)}(t) = E_k^{(r)}(-t) \), by (2.3), we obtain

\[
\int_{|t|>\delta} \left| \sum_{k=n+1}^{\infty} (\Delta c_k) E_k^{(r)}(t) \right| dt = o(1), \quad n \to \infty.
\]

Using the equality

\[
D_n(t) = \frac{1}{2} (E_n(t) + E_n(-t))
\]

we obtain

\[
\int_{|t|>\delta} \left| \sum_{k=n+1}^{\infty} (\Delta c_k) D_k^{(r)}(t) \right| dt \leq \frac{1}{2} \int_{|t|>\delta} \left| \sum_{k=n+1}^{\infty} (\Delta c_k) E_k^{(r)}(t) \right| dt
\]
\[
+ \frac{1}{2} \int_{|t|>\delta} \left| \sum_{k=n+1}^{\infty} (\Delta c_k) E_{-k}^{(r)}(t) \right| dt
\]
\[
= o(1), \quad n \to \infty.
\]

3. Main result

**Theorem 1.** Let \( \{c_n\} \) be a sequence of complex numbers such that (1.4) holds. If

\[
\{c_n\} \in (BV)^m_r \cap C^*_r, \quad m = 1, 2, 3, \ldots, \quad r = 1, 2, 3, \ldots
\]

then

\[
\| S_n^{(r)} - f^{(r)} \| = o(1), \quad n \to \infty
\]

if and only if

\[
n^r |c_n| \log n = o(1), \quad n \to \infty.
\]

**Proof.** Sufficiency: Assume that

\[
\{c_n\} \in (BV)^m_r \cap C^*_r, \quad n^r |c_n| \log n = o(1), \quad n \to \infty
\]
and (1.4) holds. Then (see [18])

$$\lim_{n \to \infty} S_n^{(r)}(c, t) = f_r(t).$$  \hfill (1) \hfill (*)

Let

$$G_{n,r}(c, t) = S_n^{(r)}(c, t) - (c_n E_n^{(r)}(t) + c_{-n} E_{-n}^{(r)}(t))$$

$$= \sum_{k=1}^{n} (\Delta c_k) D_k^{(r)}(t) + \sum_{k=1}^{n} \Delta(c_{-k} - c_k)(E_{-k}^{(r)}(t)).$$

For \( t \neq 0 \) it follows from (1) that

$$f_r(t) - G_{n,r}(c, t) = \sum_{k=n+1}^{\infty} (\Delta c_k) D_k^{(r)}(t) + \sum_{k=n+1}^{\infty} \Delta(c_{-k} - c_k)(E_{-k}^{(r)}(t)).$$

From Lemma 1, (1.4) and Lemma 4, we obtain

$$\|f_r(t) - G_{n,r}(c, t)\| \leq \left\| \sum_{k=n+1}^{\infty} (\Delta c_k) D_k^{(r)}(t) \right\| + \alpha \sum_{k=n+1}^{\infty} |\Delta(c_{-k} - c_k)| k^\gamma \log k$$

$$= \left( \int_{|t| \leq \delta} + \int_{|t| > \delta} \right) \left\| \sum_{k=n+1}^{\infty} (\Delta c_k) D_k^{(r)}(t) \right\| dt + o(1), \quad n \to \infty,$$

i.e.

$$\|f_r(t) - G_{n,r}(c, t)\| = o(1), \quad n \to \infty.$$

Applying Lemma 2, we obtain

$$\|G_{n,r}(c, t) - S_n^{(r)}(c, t)\| = \|c_n E_n^{(r)}(t) + c_{-n} E_{-n}^{(r)}(t)\| = o(1), \quad n \to \infty.$$ 

Hence,

$$\|S_n^{(r)}(c, t) - f_r(t)\| \leq \|f_r(t) - G_{n,r}(c, t)\| + \|G_{n,r}(c, t) - S_n^{(r)}(c, t)\| = o(1), \quad n \to \infty,$$

i.e. \( f_r \in L^1 \). Thus

$$\lim_{n \to \infty} S_n^{(r)}(c, t) = f^{(r)}(t).$$

Necessity: Let \( \{c_n\} \in (BV)^{\infty} \cap C_c^{\infty} \), (1.4) holds and \( \|S_n^{(r)}(c, t) - f^{(r)}(t)\| = o(1), \quad n \to \infty.$$

Applying Lemma 2, it suffices to show that

$$\|\hat{f}(n) E_n^{(r)}(t) + \hat{f}(-n) E_{-n}^{(r)}(t)\| = o(1), \quad n \to \infty.$$ 

Indeed,

$$\|\hat{f}(n) E_n^{(r)}(t) + \hat{f}(-n) E_{-n}^{(r)}(t)\| = \|G_{n,r}(c, t) - S_n^{(r)}(c, t)\|$$

$$\leq \|G_{n,r}(c, t) - f_r(t)\| + \|f_r(t) - S_n^{(r)}(c, t)\| = o(1), \quad n \to \infty,$$

i.e.

$$n^{\gamma} \hat{f}(n) \log n = o(1), \quad n \to \infty.$$
Theorem 2. Let \( \{a_n\} \in (BV)^\sigma_r \cap C_r \), where \( \sigma > 0 \) and \( r = 0, 1, 2, \ldots \). Then the \( r \)-th derivative of the series (1.3) is a Fourier series of some \( f^{(r)} \in L^1(0, \pi) \) and

\[
\|S^{(r)}_n - f^{(r)}\| = o(1), \quad n \to \infty
\]

if and only if

\[
n' a_n \log n = O(1), \quad n \to \infty.
\]

Proof. Let \( m \) be the least integer such that \( m \geq \sigma \). Then by Theorem B, we obtain \( \{a_n\} \in (BV)^m_r \cap C_r \), and by Theorem C, the point-wise limit \( f^{(r)} \) of the \( r \)-th derivative of the sum \( S_n(a) \) exists in \( (0, \pi] \). Applying the same technique for series (1.3), as in the proof of Theorem 1, the proof of this theorem is obvious.

4. Some corollaries for \( \sigma = 1 \)

Firstly, we shall define some known classes of real sequences introduced in [12], [14], [15], [16], [17], [19].

A null-sequence \( \{a_k\} \) belongs to the class \( S_r, r = 0, 1, 2, \ldots \) (see [15], [17]) if there exists a monotonically decreasing sequence \( \{A_k\} \) such that

\[
\sum_{k=1}^{\infty} k^r A_k < \infty \quad \text{and} \quad \frac{1}{n} \sum_{k=1}^{n} |\Delta a_k|^q A_k^q = O(1).
\]

It is clear that \( S_r \subset S_q \).

On the other hand, in [16] we defined an equivalent form of the Sheng’s class \( S^\alpha_{q,r} \), \( q > 1 \), \( \alpha \geq 0 \), \( r = 0, 1, 2, \ldots \) (see [7]) as follows: a null sequence \( \{a_k\} \) belongs to the class \( S^\alpha_{q,r} \), \( q > 1 \), \( \alpha \geq 0 \), \( r = 0, 1, 2, \ldots \) if there exists a monotonically decreasing sequence \( \{A_k\} \) such that

\[
\sum_{k=1}^{\infty} k^r A_k < \infty \quad \text{and} \quad \frac{1}{n} \sum_{k=1}^{n} |\Delta a_k|^q A_k^q = O(1).
\]

The following embedding relation holds (see [16])

\[
S^\alpha_{q,r} \subset (BV)_r \cap C_r, \quad 1 < q \leq 2, \quad \alpha \geq 0, \ r \in [0, 1, 2, \ldots, [\alpha]].
\]

Corollary 4.1. Let \( \{a_n\} \in S^\alpha_{q,r}, 1 < q \leq 2, \alpha \geq 0, \ r \in [0, 1, 2, \ldots, [\alpha]]. \) Then the \( r \)-th derivative of the series (1.3) is a Fourier series of some \( f^{(r)} \in L^1(0, \pi) \) and

\[
\|S^{(r)}_n - f^{(r)}\| = o(1), \quad n \to \infty
\]
if and only if
\[ n' a_n \lg n = o(1), \quad n \to \infty. \]

We note that for \( \alpha = r \) we obtain analogical results for the classes \( S_{qr} \) and \( S_r \), and for \( r = 0 \) for the Sidon-Telyakovskii class \( S \).

Denote by \( I_m \) the diadic interval \( [2^{m-1}, 2^m) \), for \( m \geq 1 \).

A null sequence \( \{a_n\} \) belongs to the class \( F_{qr} \), \( q > 1 \), \( r = 0, 1, 2, \ldots \) if
\[
\sum_{m=1}^{\infty} 2^{m(1+r)} \left( \frac{1}{2^m} \sum_{k \in I_m} |\Delta a_k|^q \right)^{1/q} < \infty.
\]

It is obvious that for \( r = 0 \), we obtain the Fomin’s class \( F_q \), \( q > 1 \) (see [3]).

But in [12] we verified the embedding relation
\[ F_{qr} \subset (BV)_r \cap C_r, 1 < q \leq 2, r = 1, 2, \ldots. \]

**Corollary 4.2.** Let
\[ \{a_n\} \in F_{qr}, \quad 1 < q \leq 2, \quad r = 0, 1, 2, \ldots. \]

Then the \( r \)-th derivative of the series (1.3) is a Fourier series of some \( f^{(r)} \in L^1(0, \pi) \) and
\[ \|S_n^{(r)} - f^{(r)}\| = o(1), \quad n \to \infty \]
if and only if
\[ n^r a_n \lg n = o(1), \quad n \to \infty. \]

**References**


Faculty of Mathematical and Natural Sciences, Department of Mathematics, P.O. BOX 162, 1000 Skopje, MACEDONIA

E-mail: tomovski@iunona.pmf.ukim.edu.mk; zivoradt@yahoo.com