LOGARITHMICALLY COMPLETE MONOTONICITY PROPERTIES AND CHARACTERIZATIONS OF THE GAMMA FUNCTION

AI-JUN LI AND CHAO-PING CHEN

Abstract. In this paper, the logarithmically complete monotonic properties of the functions \( \prod_{i=1}^{n} \frac{\Gamma(x - a_i)}{\Gamma(x - b_i)} \), \( \Gamma(x - \sum_{i=1}^{n} a_i) / \prod_{i=1}^{n} \Gamma(x - a_i) \), and \( x'(e/x)^x \Gamma(x) \) are obtained. Some characterizations of the gamma function are deduced.

1. Introduction

The classical gamma function is usually defined for \( \text{Re} z > 0 \) as

\[ \Gamma(z) = \int_{0}^{\infty} t^{z-1} e^{-t} dt. \]

It is one of the most important functions in analysis and its applications. The history and development of this function are described in detail [11].

The psi or digamma function, the logarithmic derivative of the gamma function, and the polygamma functions can be defined [20, p.16] as

\[ \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} = -\gamma + \int_{0}^{\infty} \frac{e^{-t} - e^{-xt}}{1 - e^{-t}} dt \] (1)

or

\[ \psi(x) = -\gamma + \sum_{n=0}^{\infty} \left( \frac{1}{1+n} - \frac{1}{x+n} \right), \] (2)

\[ \psi^{(k)}(x) = (-1)^{k+1} \int_{0}^{\infty} \frac{t^{k}}{1 - e^{-t}} e^{-xt} dt \] (3)

or

\[ \psi^{(k)}(x) = (-1)^{k+1} k! \sum_{i=0}^{\infty} \frac{1}{(x+i)^{k+1}} \] (4)

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for $x > 0$ and $k \in \mathbb{N}$, where $\gamma = 0.57721566490153286\ldots$ is the Euler-Mascheroni constant.

A function $f$ is said to be completely monotonic on an interval $I$ if $f$ has derivatives of all orders on $I$ which alternate successively in sign, that is

$$(-1)^n f^{(n)}(x) \geq 0 \quad (x \in I; n = 0, 1, 2, \cdots).$$

If the inequality is strict, then $f$ is said to be strictly completely monotonic on $I$.

"Completely monotonic functions have remarkable applications in different branches. For instance, they play a role in potential theory [8], probability theory [3, 15, 19], physics [12], numerical and asymptotic analysis [17, 28], and combinatorics [4]. A detailed collection of the most important properties of completely monotonic functions can be found in [27, Chapter IV], and in an abstract in [5]."

A positive function $f$ is said be logarithmically completely monotonic on an interval $I$ if its logarithm $\ln f$ satisfies

$$(-1)^n \ln[f(x)]^{(n)} \geq 0$$

for $x \in I$ and $n \in \mathbb{N} := 1, 2, \ldots$. If inequality is strict, then $f$ is said to be strictly logarithmically completely monotonic. The terminology "(strictly) logarithmically completely monotonic function" was introduced in [24]. It is also shown in this paper that a (strictly) logarithmically completely monotonic function is also (strictly) completely monotonic.

In the past many articles [1, 13, 18, 25] were published providing some different properties for the ratio $\Gamma(x+1)/\Gamma(x+s)$, where $x > 0$ and $s \in (0, 1)$. In 1986, J. Bustoz and M.E.H. Ismail [10] established the function

$$p(x; a, b) = \frac{\Gamma(x)\Gamma(x + a + b)}{\Gamma(x + a)\Gamma(x + b)} \quad (a, b > 0),$$

which can be represented in terms of Gauss’ hypergeometric series

$$\binom{2}{F_1}(a, b, c; z) = \sum_{n=0}^{\infty} \frac{(a)_{n}(b)_{n}}{(c)_{n}} \frac{z^n}{n!},$$

where $(a)_n = \Gamma(a+n)/\Gamma(a)$, namely,

$$\frac{\Gamma(x)\Gamma(x + a + b)}{\Gamma(x + a)\Gamma(x + b)} = \binom{2}{F_1}(-a, -b, x; 1), \quad (x > -a - b).$$

They showed that the function $p(x; a, b)$ is completely monotonic on $(0, \infty)$. This generalized a proposition of K. B. Stolarsky [24], who obtained that $p$ is decreasing in $x$.

In 1997, H. Alzer [2] proved that the function

$$\phi(x) = \prod_{i=1}^{n} \frac{\Gamma(x + a_i)}{\Gamma(x + b_i)}$$
is completely monotonic on \((0, \infty)\) with \(0 \leq a_1 \leq \cdots \leq a_n\), \(0 \leq b_1 \leq \cdots \leq b_n\), and \(\sum_{i=1}^k a_i \leq \sum_{i=1}^k b_i\) for \(k = 1, \ldots, n\). This extended Bustoz’s result.

Let us extend \(a_i, b_i\) to all real numbers. For \(a_i\) and \(b_i\) \((i = 1, \ldots, n)\) are positive real numbers, define

\[
\Phi(x) = \prod_{i=1}^n \frac{\Gamma(x-a_i)}{\Gamma(x-b_i)}.
\]

Then we obtain that \(\Phi(x)\) is logarithmically completely monotonic in the following theorems.

**Theorem 1.** Let \(a_i\) and \(b_i\) be positive real numbers, \(m = \max\{a_i, b_i\} (i = 1, \ldots, n)\) and \(a_i \geq b_i\). Then the function \(\Phi(x)\) is logarithmically completely monotonic on \((m, \infty)\).

**Theorem 2.** Let \(a_i\) and \(b_i\) be real numbers such that \(0 \leq a_1 \leq \cdots \leq a_n\), \(0 \leq b_1 \leq \cdots \leq b_n\), \(m = \max\{a_i, b_i\} (i = 1, \ldots, n)\) and \(\sum_{i=1}^k a_i \geq \sum_{i=1}^k b_i\) for \(k = 1, \ldots, n\). Then the function \(\Phi(x)\) is logarithmically completely monotonic on \((m, \infty)\).

**Corollary 1.** The function

\[
P(x; a, b) = 2F_1(a, b, x; 1) = \frac{\Gamma(x-a-b)}{\Gamma(x-a)\Gamma(x-b)} \quad (a, b > 0)
\]

is strictly logarithmically completely monotonic on \((a + b, \infty)\).

In 1995, L. Maligranda et al. [21] concluded that the function

\[
x \to \Gamma(x)^{n-1}\Gamma\left(x + \sum_{i=1}^n a_i\right)/\prod_{i=1}^n \Gamma(x + a_i)
\]

\((a_i > 0; i = 1, \ldots, n)\) is decreasing on \((0, \infty)\). Subsequently, Alzer extended this result in [2, p.385], and obtained the necessary and sufficient condition of the statement that the function is strictly completely monotonic. The following theorem provides a slight extension of Alzer’ result.

**Theorem 3.** Let \(\alpha\) be a real number, \(a_i\) \((i = 1, \ldots, n; n \geq 2)\) be positive real numbers and \(m = \sum_{i=1}^n a_i\). The function

\[
x \to \Gamma(x)^{\alpha-1}\Gamma\left(x - \sum_{i=1}^n a_i\right)/\prod_{i=1}^n \Gamma(x - a_i)
\]

is strictly logarithmically completely monotonic on \((m, \infty)\) if and only if \(\alpha = n - 1\).

In [2], Alzer proved that the function \(F_r(x) = x^r(e/x)^x\Gamma(x)\) is decreasing on \((0, \infty)\) if and only if \(r \leq 1/2\). Moreover, Alzer obtained that the function \(g(x)(f_1(x) - c)\) is strictly completely monotonic on \((0, \infty)\) if and only if \(c \leq 1/2\), where \(g(x)\) is a strictly completely monotonic function and \(f_1(x) = x(ln(x) - \psi(x))\). This extended a result of E. Muldoon.
who proved the complete monotonicity for the special case \( g(x) = 1/x \). In 2006, Alzer and Berg \[4\] established the completely monotonicity of function \([x^a(e/x)^b\Gamma(x)]^b\) for \(a, b \in \mathbb{R} \) and \(b \neq 0\).

Motivated by the results above, we establish several functions involving gamma and polygamma functions, and investigate their logarithmically completely monotonic properties in the following theorems.

**Lemma 1.** \([2, p.374]\) Let \( \alpha \) be a real number. The function
\[
f_\alpha(x) = x^\alpha(\ln(x) - \psi(x))
\]
is strictly completely monotonic on \((0, \infty)\) if and only if \(\alpha \leq 1\).

**Theorem 4.** Let \( r \) be real number. The function
\[
F_r(x) = x^r(e/x)^x\Gamma(x)
\]
is strictly logarithmically completely monotonic on \((0, \infty)\) if and only if \(r \leq \frac{1}{2}\); The function \((F_r(x))^{-1}\) is strictly logarithmically completely monotonic on \((0, \infty)\) for \(r \geq 1\).

**Corollary 2.** The function
\[
F_{r,\alpha}(x) = [x^r(e/x)^x\Gamma(x)]^\alpha
\]
is logarithmically completely monotonic on \((0, \infty)\) if and only if \(r \leq \frac{1}{2}\) and \(\alpha > 0\).

Finally, we study the problem of characterizing \(\Gamma(x)\) by means of the logarithmically completely monotonic functions related to \(\Gamma(x)\). From Theorem 4, we obtain the results as follows.

**Theorem 5.** If function \(x^{1/2}(e/x)^xf(x)\) is logarithmically completely monotonic on \((0, \infty)\) and that \(f(x_k) = \Gamma(x_k)\) for each point \(x_k\) in an increasing sequence \(\{x_k\} \subset (0, \infty)\) for which \(\sum(1/x_k)\) diverges, then \(f(x) = \Gamma(x), 0 < x < \infty\).

**Corollary 3.** If function \((f(x))^{-1}\) is logarithmically completely monotonic on \((0, \infty)\) for \(n = 2, 3, \ldots\) and that \(f(x_k) = \Gamma(x_k)\) for each point \(x_k\) in an increasing sequence \(\{x_k\} \subset (0, \infty)\) for which \(\sum(1/x_k)\) diverges, then \(f(x) = \Gamma(x), 0 < x < \infty\).

**Remark 1.** The function \((\Gamma(x))^{-1}\) is logarithmically completely monotonic on \((0, \infty)\) for \(n = 2, 3, \ldots\) ....

Moreover, the function \(\Gamma(x)\) can be characterized in the following way \[5, p.14\]: \(f(x) = \Gamma(x)\) \((0 < x < \infty)\) if and only if

1. \(f(1) = 1, f(x + 1) = xf(x), \quad 0 < x < \infty;\)
2. \(f(x)\) is defined and logarithmically convex for \(0 < x < \infty\).
Requirement (2) can be modified by logarithmically completely monotonicity properties. The result is as follows.

**Theorem 6.** Suppose that
1. \( f(1) = 1, f(x + 1) = xf(x), \ 0 < x < \infty; \)
2. \( f(x) \) is logarithmically completely monotonic on \((0, \infty)\).

Then \( f(x) = \Gamma(x), 0 < x < \infty. \)

### 2. Proofs of Theorems

**Proof of Theorem 1.** Taking logarithm and differentiation yields

\[
(\ln \Phi(x))' = \sum_{i=1}^{n} \psi(x - a_i) - \sum_{i=1}^{n} \psi(x - b_i)
\]

\[
= \int_{0}^{\infty} \sum_{i=1}^{n} \frac{e^{-(x-b_i)t}}{1-e^{-t}} - \sum_{i=1}^{n} \frac{e^{-(x-a_i)t}}{1-e^{-t}} dt
\]

\[
= \int_{0}^{\infty} \sum_{i=1}^{n} \frac{e^{b_it} - e^{a_it}}{1-e^{-t}} e^{-xt} dt
\]

(7)

Applying power series expansion of \( e^x \) to (7), we get

\[
(\ln \Phi(x))' = \int_{0}^{\infty} \sum_{i=1}^{n} \left( \sum_{k=1}^{\infty} \frac{(b_i - a_i) e^{kt}}{k!} \right) e^{-xt} dt \leq 0
\]

(8)

By (3), we have

\[
(-1)^m (\ln \Phi(x))^{(m)} = \left( \sum_{i=1}^{n} \psi^{(m-1)}(x - a_i) - \sum_{i=1}^{n} \psi^{(m-1)}(x - b_i) \right)
\]

\[
= \int_{0}^{\infty} \sum_{i=1}^{n} \frac{e^{a_it} - e^{b_it}}{1-e^{-t}} t^{m-1} e^{-xt} dt
\]

\[
= \int_{0}^{\infty} \sum_{i=1}^{n} \left( \sum_{k=1}^{\infty} \frac{(a_i - b_i) e^{kt}}{k!} \right) t^{m-1} e^{-xt} dt \geq 0
\]

(9)

The proof is complete.

In order to prove Theorem 2 we need the following lemma [22, p.10].

**Lemma 2.** Let \( a_i \) and \( b_i \) \((i = 1, \ldots, n)\) be real numbers such that \( a_1 \leq \cdots \leq a_n, \)

\( b_1 \leq \cdots \leq b_n, \) and \( \sum_{i=1}^{k} b_i \leq \sum_{i=1}^{k} a_i \) for \( k = 1, \ldots, n. \) If the function \( f \) is increasing and convex on \( \mathbb{R}, \) then

\[
\sum_{i=1}^{n} f(b_i) \leq \sum_{i=1}^{n} f(a_i).
\]
Proof of Theorem 2. Since the function \( x \to e^{xt}(t > 0) \) is increasing and convex on \( \mathbb{R} \), we conclude from Lemma 2 that \( \sum_{i=1}^{n}(e^{a_it} - e^{b_it}) \geq 0 \). Therefore \( (-1)^m(\ln \Phi(x))^{(m)} \geq 0 \) \( (m = 1, 2, \ldots) \)
\( \) for \( x > 0 \), and the function \( \Phi(x) \) is logarithmically completely monotonic on \( (0, \infty) \).

Proof of Corollary 1. With analogous proof method as Theorem 1, we get
\[
(-1)^m(\ln P(x; a, b))^{(m)} \\
= (-1)^m(\psi^{(n-1)}(x) + \psi^{(n-1)}(x - a - b) - \psi^{(n-1)}(x - a) - \psi^{(n-1)}(x - b)) \\
= \int_0^\infty \frac{t^{n-1}e^{-xt}}{1 - e^{-t}}(e^{(a+b)t} + 1 - e^{at} - e^{bt}) \mathrm{d}t \\
= \int_0^\infty \frac{t^{n-1}e^{-xt}}{1 - e^{-t}} \left( \sum_{k=2}^{\infty} ((a+b)^k - a^k - b^k) \frac{t^k}{k!} \right) \mathrm{d}t > 0 \quad (10)
\]

Now we provide another method to prove Corollary 1. For \( n \geq 0 \), we have that
\[
(\ln P(x; a, b))^{(n+1)} = \psi^{(n)}(x) + \psi^{(n)}(x - a - b) - \psi^{(n)}(x - a) - \psi^{(n)}(x - b) \\
= a \left( \frac{\psi^{(n)}(x) - \psi^{(n)}(x - a)}{a} - \frac{\psi^{(n)}(x - b) - \psi^{(n)}(x - b - a)}{a} \right). \quad (11)
\]
By (11), \( y \mapsto \psi^{(n)}(y) \) is strictly convex for odd \( n \), the ratio
\[
\frac{\psi^{(n)}(x) - \psi^{(n)}(x - a)}{a} \quad (12)
\]
is increasing with \( y \in (a, \infty) \). For even \( n \), the function \( \psi^{(n)}(x) \) is concave, and the ratio \( \frac{\psi^{(n)}(x) - \psi^{(n)}(x - a)}{a} \) is decreasing. Thus, by (12) we conclude that the sign of \( (\ln P(x; a, b))^{(n+1)} \) is \( (-1)^{n+1} \), for \( n \geq 0 \) and \( x \in (a + b, \infty) \).

Proof of Theorem 3. Let
\[
p_\alpha(x) = \Gamma(x)^\alpha \Gamma \left( x - \sum_{i=1}^{n} a_i \right) / \prod_{i=1}^{n} \Gamma(x - a_i).
\]
It is obvious that \( p_{n-1}(x) \) is strictly logarithmically completely monotonic on \( (m, \infty) \) from Theorem 2.

Next, we assume that \( p_\alpha(x) \) is strictly logarithmically completely monotonic on \( (m, \infty) \). Then, we get
\[
\frac{\partial}{\partial x} \ln p_\alpha(x) = \alpha \psi(x) + \psi(x - m) - \sum_{i=1}^{n} \psi(x - a_i) \leq 0. \quad (13)
\]
This implies for all sufficiently large $x$:

$$\alpha \leq \sum_{i=1}^{n} \frac{\psi(x - a_i)}{\psi(x)} - \frac{\psi(x - m)}{\psi(x)}.$$  \hfill (14)

Since $p_\alpha$ is completely monotonic on $(m, \infty)$, we obtain

$$0 \leq (p_\alpha(x))^{-2} \left[ p_\alpha(x) \frac{\partial^2 p_\alpha(x)}{\partial x^2} - \left( \frac{\partial p_\alpha(x)}{\partial x} \right)^2 \right]$$

$$= \alpha \psi'(x) + \psi'(x - m) - \sum_{i=1}^{n} \psi'(x - a_i).$$

Hence, we have for $x > m$:

$$\sum_{i=1}^{n} \frac{\psi'(x - a_i)}{\psi'(x)} - \frac{\psi'(x - m)}{\psi'(x)} \leq \alpha.$$  \hfill (15)

Since

$$\lim_{x \to \infty} \frac{\psi(x - A)}{\psi(x)} = \lim_{x \to \infty} \frac{\psi'(x - A)}{\psi'(x)} = 1 \quad (A > 0),$$

we conclude from (14) and (15) that $\alpha = n - 1$.

**Proof of Theorem 4.** Using Binet’s formula [14, p.18, (22)], we get

$$\left( \ln F_r(x) \right)' = \frac{r}{x} - \ln x + \psi(x)$$

$$= \frac{r}{x} - \ln x + \ln x - \frac{1}{2x} + \int_{0}^{\infty} \left( \frac{1}{2} + \frac{1}{t} - \frac{1}{1 - e^{-t}} \right) e^{-xt} dt$$

$$= \frac{r - 1/2}{x} + \int_{0}^{\infty} \left( \frac{1}{2} + \frac{1}{t} - \frac{1}{1 - e^{-t}} \right) e^{-xt} dt$$

$$< 0.$$  \hfill (16)

(16) follows from $r \leq \frac{1}{2}$ and the inequality

$$\frac{1}{2} + \frac{1}{t} - \frac{1}{1 - e^{-t}} = \frac{2 - t - (2 + t)e^{-t}}{2t(1 - e^{-t})} < 0 \quad (0 < t < \infty).$$  \hfill (17)

Indeed, let $g(t) = 2 - t - (2 + t)e^{-t}$, we can get $g'(t) = -te^{-t} < 0$ and $\lim_{t \to 0} g'(t) = 0$. This implies that $g(t)$ is decreasing on $(0, \infty)$. Since $\lim_{t \to 0} g(t) = 0$, (17) holds.
Taking the $n$th derivative of $\ln F_r(x)$, we obtain
\[
(-1)^n \left( \ln F_r(x) \right)^{(n)} = (-1)^n \left[ \left( \frac{r - 1/2}{x} \right)^{(n-1)} + \left( \int_0^\infty \left( \frac{1}{2} + \frac{1}{t} - \frac{1}{1 - e^{-t}} \right) e^{-xt} \, dt \right)^{(n-1)} \right]
\]
\[
= -\frac{(r - 1/2)(n-1)!}{x^n} - \int_0^\infty \left( \frac{1}{2} + \frac{1}{t} - \frac{1}{1 - e^{-t}} \right) t^{n-1} e^{-xt} \, dt > 0 \quad (r \leq \frac{1}{2}).
\]
(18)

Next, it is clear from (16) that
\[
r < x(\ln x - \psi(x)).
\]
(19)

By Lemma 1, we obtain that $f_1(x) = x(\ln x - \psi(x))$ is strictly decreasing on $(0, \infty)$. Moreover,
\[
\lim_{x \to \infty} f_1(x) = \frac{1}{2}.
\]

It follows from the representations
\[
f_1(x) = x \ln x - x\psi(x + 1) + 1
\]
and
\[
f_1(x) = \frac{1}{2} + \frac{1}{12x} - \frac{\theta}{120x^3} \quad (0 < \theta < 1);
\]
see [16, p.824]. Therefore, we conclude $r \leq \frac{1}{2}$.

To prove the second part, a simple calculation shows that
\[
\left( \ln(F_r(x))^{-1} \right)' = \int_0^\infty \left( \frac{1}{1 - e^{-t}} - \frac{1}{t - \alpha} \right) e^{-xt} \, dt < 0
\]
and
\[
(-1)^n \left( \ln(F_r(x))^{-1} \right)^{(n)} = -\int_0^\infty \left( \frac{1}{1 - e^{-t}} - \frac{1}{t - \alpha} \right) t^{n-1} e^{-xt} \, dt > 0
\]
for $r \geq 1$. Since we write
\[
f(t) = \frac{1}{1 - e^{-t}} - \frac{1}{t},
\]
we have $f(0+) = \frac{1}{2}$, $f(\infty) = 1$.

This completes the whole proof.

**Proof of Theorem 5.** It is shown by W. Feller [15, p.671] that two functions which are completely monotonic on $(0, \infty)$ must be identical if they coincide at the points of an increasing unbounded sequence $\{x_k\}$ where $\sum (1/x_k)$ diverges.

Since the functions $x^{1/2}(e/x)^{\theta} f(x)$ and $x^{1/2}(e/x)^{\theta} \Gamma(x)$ are both (logarithmically) completely monotonic on $(0, \infty)$, we get $f(x) = \Gamma(x)$ easily.
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