SOME REMARKS ON RECONSTRUCTION FROM LOCAL WEIGHTED AVERAGES

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Abstract. We solve the convolution equation of the type \( f \ast \mu = g \), where \( f \ast \mu \) is the convolution of \( f \) and \( \mu \) defined by \((f \ast \mu)(x) = \int_R f(x-y)d\mu(y)\), \( g \) is a given function and \( \mu \) is a finite linear combination of translates of an indicator function on an interval.

1. Introduction

We consider the convolution equation of the following type:

\[
f \ast \mu = g,
\]

where \( g \) is a known continuous function, \( \mu \) is a compactly supported measure and \( f \) is an unknown continuous function. Delsarte [3] was interested in solving the particular case of equation (1) which is of the type \( \frac{1}{\tau} \int_{x-\frac{\tau}{2}}^{x+\frac{\tau}{2}} f(t)dt = g(x) \). In the case when \( f \) is an integrable function with compact support van der Pol \([15, 16]\) has obtained reconstruction formula using two sided Laplace Transform. But such transform methods can not be used for the case of continuous functions on \( \mathbb{R} \). The special case of equation (1), namely \( g = 0 \), was analyzed by many authors citebag,ber1,deva,Ehr1,Ehr2,kah,schwartz,thangavelu,wei,sze on various groups. The solutions (1) for the particular case when \( g = 0 \) are called mean periodic functions. Laurent Schwartz \([18]\) gave an intrinsic characterization of such solutions. The corresponding non-homogeneous type equation is analysed in \([14]\) for the special case of when \( \mu \) is the indicator function on the interval \([-a, a]\). An explicit construction of a solution is given in \([17]\) for the same equation on the three dimensional Euclidean space when \( \mu \) is the indicator function of a ball in \( \mathbb{R}^3 \) using plane wave decomposition. When \( \mu \) is finitely supported, the equation (1) gets reduced to a non-homogeneous constant coefficient difference equation. Edgar and Rosenblatt \([6]\) have studied the homogeneous equation (ie, when \( g=0 \)). They have shown that a complex valued function \( f \) has linearly independent translates precisely when \( f \) does not

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satisfy a nontrivial homogeneous difference equation. An explicit construction of a solution is given in [4] on \( \mathbb{R} \) when \( \mu \) is an arbitrary finitely supported measure and \( g \) is a continuous function.

Malgrange [13], Ehrenpreis [9], John [10], and Hörmander [11] have studied the convolution equation of the type analogous to equation (1)

\[ P(D)u = f, \quad (2) \]

where \( P(D) \) is a constant coefficient partial differential operator and \( f \) is a given function. A criterion was given by Hörmander for the existence of solution \( u \in D'(\Omega) \) for an arbitrary \( f \in D'_c(\Omega) \) on an open set \( \Omega \subseteq \mathbb{R}^n \).

In general, no necessary and sufficient conditions for the existence of solutions of equation (1) are known. One can easily see the following:

(i) Equation (1) has no solution in \( C(\mathbb{R}) \) when \( g \) is a non smooth function and \( \mu \) is a compactly supported continuous function.

(ii) If \( f_0 \) is a particular solution of equation (1), then every other solution \( f \) can be written as \( f = f_0 + h \), where \( h \star \mu = 0 \).

(iii) If the Fourier-Laplace transform \( \hat{\mu}(\lambda) = 0 \) for some \( \lambda \in \mathbb{C} \) and if there exists a solution to equation (1), then there are infinitely many solutions to equation (1).

The methods of [14] can not be extended to the case when \( \mu \) is a sum of more than one indicator function. In this paper we analyze the case when \( \mu \) is a finite linear combination of the translates of an indicator function on an interval. A solution \( f \in C^r(\mathbb{R}) \) is constructed for every \( g \in C^{r+1}(\mathbb{R}) \).

2. Reconstruction Results

**Definition 1 ([4]).** We say a compactly supported Borel measure on \( \mathbb{R} \) is a discrete Borel measure, if there exists a finite set of distinct real numbers \( x_1, x_2, \ldots, x_n \) and nonzero complex constants \( c_1, c_2, \ldots, c_n \) such that \( \mu(E) = \sum_{i=1}^{n} c_i \delta_{x_i}(E) \) for every Borel set \( E \). The set of all compactly supported discrete Borel measures on \( \mathbb{R} \) is denoted by \( M_{cd}(\mathbb{R}) \).

For \( a, b \in \mathbb{R} \), the indicator function on the interval \( [a, b] \) is denoted by \( \chi_{[a,b]} \) and \( LST(\chi_{[a,b]}) \) denotes the linear span of the translates of \( \chi_{[a,b]} \). The set of all compactly supported regular Borel measures on \( \mathbb{R} \) is denoted by \( M_c(\mathbb{R}) \). We note that \( LST(\chi_{[a,b]}) \subseteq M_c(\mathbb{R}) \).

**Definition 2.** For \( f \in C(\mathbb{R}) \) and \( \mu \in M_c(\mathbb{R}) \), the convolution of \( f \) with \( \mu \) is defined as

\[ (f \star \mu)(x) = \int_{\mathbb{R}} f(x - y) d\mu(y). \]
When $\mu = \sum_{i=1}^{n} c_i \chi_{[a_i, b_i]}$, the convolution becomes

$$(f * \mu)(x) = \sum_{i=1}^{n} c_i \int_{a_i}^{b_i} f(x - y) dy.$$  

**Definition 3.** [4] For every real or complex valued function $f$ and discrete measure $\mu = \sum_{i=1}^{n} c_i \delta_{x_i} \in M_{cd}(\mathbb{R})$, the convolution of $f$ and $\mu$ is defined by

$$(f * \mu)(x) = \sum_{i=1}^{n} c_i f(x - x_i).$$

In [4] the special case $r = 0$ of the following lemma is proved. We extend the same for $r > 0$ along the lines of [4].

**Lemma 1.** For $\mu, \nu \in M_{cd}(\mathbb{R})$ and $g \in C^r(\mathbb{R})$, the following hold:

(i) If $\text{supp}(\mu) \subset (-\infty, -\alpha)$ for some $\alpha > 0$ and $\text{supp}(g) \subset (-\infty, \beta)$ for some $\beta \in \mathbb{R}$, then there exists $f \in C^r(\mathbb{R})$ such that $f \ast (\delta_0 + \mu) = g$.

(ii) If $\text{supp}(\nu) \subset (\alpha, \infty)$ for some $\alpha > 0$ and $\text{supp}(g) \subset (\beta, \infty)$ for some $\beta \in \mathbb{R}$, then there exists $f \in C^r(\mathbb{R})$ such that $f \ast (\delta_0 + \nu) = g$.

**Proof.** (i) We denote by $\mu^m$ the convolution of $\mu$ with itself m-times. As $\text{supp}(\mu) \subset (-\infty, -\alpha)$, we have $\text{supp}(\mu^n) \subset (-\infty, -n\alpha)$. Let $\mu^n = \sum_{i=1}^{l} c_i \delta_{y_i}$. Then

$$(g^{(j)} \ast \mu^n)(x) = \sum_{i=1}^{l} c_i g^{(j)}(x - y_i).$$

Since $\text{supp}(\mu^n) \subset (-\infty, -n\alpha)$, $y_i < -n\alpha$ and hence $x - y_i > x + n\alpha > \beta$ for sufficiently large $n$. Therefore for every $x$, $(g^{(j)} \ast \mu^n)(x) = 0$ for $n$ sufficiently large and for $0 \leq j \leq r$.

We define

$$f(x) := g(x) + \sum_{n=1}^{\infty} (-1)^n (g \ast \mu^n)(x).$$

Let us consider the following partial sums:

$$s_n(x) = g(x) + \sum_{k=1}^{n} (-1)^k (g \ast \mu^k)(x).$$

Then

$$s_n^{(j)}(x) = g^{(j)}(x) + \sum_{k=1}^{n} (-1)^k (g^{(j)} \ast \mu^k)(x).$$
We show that the above sequence converges uniformly on every compact set for $0 \leq j \leq r$. For, let $K$ be a compact subset of $\mathbb{R}$. Then $K \subset [a, b]$ for some real numbers $a$ and $b$. Choose $N$ such that $a + n\alpha > \beta$ and $b + n\alpha > \beta$, for $n \geq N$. For $x \geq a$, $x - y_i \geq a + n\alpha > \beta$. Now

$$s_{n}^{(j)}(x) - s_{m}^{(j)}(x) = \sum_{k=m+1}^{n} (-1)^{k}(g^{(j)} \ast \mu^{k})(x) = 0,$$

for $n \geq m \geq N$.

This implies that the sequence of functions $\{s_{k}^{(j)}(x)\}$ is uniformly cauchy on every compact set and hence converges uniformly on every compact set for $0 \leq j \leq r$.

Therefore we get $s_{k}^{(j)}(x)$ converges uniformly to $f^{(j)}(x)$ on every compact set and

$$f^{(j)}(x) := g^{(j)}(x) + \sum_{n=1}^{\infty} (-1)^{n}(g^{(j)} \ast \mu^{n})(x)$$

for $0 \leq j \leq r$. Hence $f^{(r)}$ is continuous and hence $f^{(r)} \in C^{r}(\mathbb{R})$. It is very easy to check that $f \ast (\delta_0 + \mu) = g$.

(ii) Since $supp(\nu) \subset (a, \infty)$, we have $supp(\nu^{n}) \subset (n\alpha, \infty)$. Suppose the representation of $\nu^{n}$ is of the form: $\nu^{n} = \sum_{i=1}^{l} d_{i}\delta_{z_{i}}$. Then $(g^{(j)} \ast \nu^{n})(x) = \sum_{i=1}^{l} d_{i}g^{(j)}(x - z_{i})$. Since $supp(\nu^{n}) \subset (n\alpha, \infty)$, $z_{j} > n\alpha$ and hence $x - z_{i} < x - n\alpha < \beta$ for sufficiently large $n$. Therefore for every $x$,

$$(g^{(j)} \ast \nu^{n})(x) = 0 \text{ for } n \text{ sufficiently large.}$$

Hence $g^{(j)}(x) + \sum_{m=1}^{\infty} (-1)^{m}(g^{(j)} \ast \nu^{m})(x)$ is a finite sum for every $x$.

We define

$$f(x) := g(x) + \sum_{m=1}^{\infty} (-1)^{m}(g \ast \nu^{m})(x). \quad (3)$$

To show

$$f^{(r)}(x) := g^{(r)}(x) + \sum_{m=1}^{\infty} (-1)^{m}(g^{(r)} \ast \nu^{m})(x),$$

it is sufficient if we show that the partial sums of the series (3) and their derivatives converge uniformly on compact sets. For, let $K$ be a compact subset of $\mathbb{R}$. Then $K \subset [a, b]$ for some real numbers $a$ and $b$. Choose $N$ such that $b - n\alpha < \beta$, for $n \geq N$. Let us take the partial sums of the series as

$$t_{k}(x) = g(x) + \sum_{m=1}^{k} (-1)^{m}(g \ast \nu^{m})(x).$$

For $x \leq b$, $x - z_{i} < x - n\alpha \leq b - n\alpha$. Choose $N$ such that $b - n\alpha < \beta$ for $n \geq N$. Then

$$t_{n}^{(j)}(x) - t_{m}^{(j)}(x) = \sum_{k=m+1}^{n} (-1)^{k}(g^{(j)} \ast \nu^{k})(x) = 0,$$

for $n \geq m \geq N$ and $0 \leq j \leq r$. This implies that the sequence $\{t_{n}^{(j)}(x)\}$ is uniformly cauchy and hence converges uniformly on every compact set. Hence we get $t_{n}^{(j)}(x) \to f^{(j)}(x)$. Therefore $f \in C^{r}(\mathbb{R})$. One easily verifies $f \ast (\delta_0 + \nu) = g$. \(\blacksquare\)
Lemma 2. For $\mu = \chi_{[a,b]}$ and $g \in C^{r+1}(\mathbb{R})$, the following hold: If $\text{supp}(g) \subset (-\infty, \beta)$ or $\text{supp}(g) \subset (\beta, \infty)$ for some $\beta \in \mathbb{R}$, then there exists $f \in C^r(\mathbb{R})$ such that $f * \mu = g$.

Proof. Case(i): Suppose that $\text{supp}(g) \subset (-\infty, \beta)$.

We can write

$$f * \chi_{[a,b]} = f * \chi_{[\frac{a-b}{2}, \frac{b-a}{2}]} * \delta_{\frac{a+b}{2}}.$$ 

Define

$$f_1(x) = -\sum_{n=0}^{\infty} \left( g' * \delta_{\frac{a+b}{2}}^{2n+1} \right)(x).$$

We show that the above series converges uniformly on compact sets. For $K$ a compact subset of $\mathbb{R}$. Then $K \subset [c, d]$ for some $c, d \in \mathbb{R}$. Let us take

$$s_n(x) = -\sum_{k=0}^{n} \left( g' * \delta_{\frac{a+b}{2}}^{2k+1} \right)(x).$$

Then

$$s_n^{(j)}(x) = -\sum_{k=0}^{n} \left( g^{(j+1)} * \delta_{\frac{a+b}{2}}^{2k+1} \right)(x).$$

Now

$$g' * \delta_{\frac{a+b}{2}}^{2k+1}(x) = g' \left( x + (2k+1)\left( \frac{b-a}{2} \right) \right).$$

Choose $N$ such that $c + (2k+1)(\frac{b-a}{2}) > \beta$ for $k \geq N$. Then $g^{(j+1)} \delta_{\frac{a+b}{2}}^{2k+1}(x) = 0$ for $k \geq N$ for all $x \in K$. Therefore $s_n^{(j)}(x) - s_m^{(j)}(x) = 0$ for all $n, m \geq N$, for all $x \in K$ and for $0 \leq j \leq r$. Hence $s_n^{(j)}(x) \to f_1^{(j)}(x)$ uniformly on $K$.

Therefore $f_1^{(j)}(x) = -\sum_{n=0}^{\infty} g^{(j+1)} * \delta_{\frac{a+b}{2}}^{2n+1}(x)$ and $f_1 \in C^r(\mathbb{R})$. Now

$$f_1 * \chi_{[\frac{a-b}{2}, \frac{b-a}{2}]} = -\sum_{n=0}^{\infty} \left( g' * \chi_{[\frac{a-b}{2}, \frac{b-a}{2}]} * \delta_{\frac{a+b}{2}}^{2n+1} \right)(x)$$

$$= -\sum_{n=0}^{\infty} \int_{\frac{a-b}{2}}^{\frac{b-a}{2}} g(x + (2n+1)(\frac{b-a}{2}) - y) \, dy$$

$$= \sum_{n=0}^{\infty} \left[ g \left( x + (2n+1)(\frac{b-a}{2}) - \frac{b-a}{2} \right) - g \left( x + (2n+1)(\frac{b-a}{2}) + \frac{b-a}{2} \right) \right]$$

$$= g(x).$$

Define $f(x) = f_1 * \delta_{-\frac{a+b}{2}}$. One easily verifies $f * \chi_{[a,b]} = g$.

Case(ii): Suppose that $\text{supp}(g) \subset (\beta, \infty)$. Now $f * \chi_{[a,b]} = f * \chi_{[\frac{a-b}{2}, \frac{b-a}{2}]} * \delta_{\frac{a+b}{2}}$. Define

$$f_1(x) = \sum_{n=0}^{\infty} g' * \delta_{\frac{a+b}{2}}^{2n+1}(x).$$

We show that the above series converges uniformly on compact sets. For, let $K$ be a compact subset of $\mathbb{R}$. Then $K \subset [c, d]$ for some $c, d \in \mathbb{R}$. Let $s_n(x) = \sum_{k=0}^{n} g' * \delta_{\frac{a+b}{2}}^{2k+1}(x)$. Then

$$s_n^{(j)}(x) = \sum_{k=0}^{n} g^{(j+1)} * \delta_{\frac{a+b}{2}}^{2k+1}(x).$$
We can write
\[ g \ast \delta^{2k+1}_{\frac{b-a}{2}}(x) = g(x - (2k+1)(\frac{b-a}{2})) \].

Choose \( N \) such that \( c - (2k+1)(\frac{b-a}{2}) < \beta \) for \( k \geq N \). Then \( g^{(j+1)} \ast \delta^{2k+1}_{\frac{b-a}{2}}(x) = 0 \) for \( k \geq N \) for all \( x \in K \). Therefore \( s_n^{(j)}(x) - s_m^{(j)}(x) = 0 \) for all \( n, m \geq N \), for all \( x \in K \) and for \( 0 \leq j \leq r \). Hence \( s_n^{(j)}(x) \rightarrow f_1^{(j)}(x) \) uniformly on \( K \).

Therefore \( f_1 \ast (x) = \sum_{n=0}^{\infty} g^{(j+1)} \ast \delta^{2n+1}_{\frac{b-a}{2}}(x) \) and \( f_1 \in C^r(\mathbb{R}) \).

Now
\[
\begin{align*}
\hat{f}_1 \ast \chi_{[\frac{a+b}{2}, \frac{b-a}{2}]} &= -\sum_{n=0}^{\infty} g' \ast \chi_{[\frac{a+b}{2}, \frac{b-a}{2}]} \ast \delta^{2n+1}_{\frac{b-a}{2}}(x) \\
&= -\sum_{n=0}^{\infty} \int_{\frac{a+b}{2}}^{\frac{b-a}{2}} g(x - (2n+1)(\frac{b-a}{2}) - y) \, dy \\
&= -\sum_{n=0}^{\infty} \left[ g \left( x - (2n+1)(\frac{b-a}{2}) - \frac{b-a}{2} \right) - g \left( x - (2n+1)(\frac{b-a}{2}) + \frac{b-a}{2} \right) \right] \\
&= g(x).
\end{align*}
\]

Define \( f(x) = f_1 \ast \delta_{-(\frac{a+b}{2})} \). It is easy to check that \( f \ast \chi_{[a,b]} = g \).

Lemma 3. Let \( \mu = \sum_{i=1}^{n} c_i \chi_{[a_i,b_i]} \) be a finite linear combination of indicator functions on intervals. If there exists \( r \in \mathbb{R} \) such that \( \frac{b_i-a_i}{r} \in \mathbb{Z} \), then the following hold:

(i) There exists \( g \in LST(\chi_{[a,b]}) \) such that \( \mu = g \) almost everywhere for some \( a, b \in \mathbb{R} \).

(ii) There exists \( \mu \in M_{cad}(\mathbb{R}) \), such that \( \mu = \chi_{[a,b]} \ast \nu \) a.e and \( f \ast \mu = f \ast \chi_{[a,b]} \ast \nu \) for all \( f \in C(\mathbb{R}) \).

Proof. (i) Let \( \frac{b_i-a_i}{r} = m_i \). Then \( \chi_{[a_i,b_i]} = \sum_{j=1}^{m_i} \chi_{[a_i+(j-1)r,a_i+jr]} \) a.e. As the indicator functions \( \chi_{[a_i+(j-1)r,a_i+jr]} \) are translates of the indicator function on \([0,r]\), we have \( \chi_{[a_i+(j-1)r,a_i+jr]} \in LST(\chi_{[0,r]}) \). Hence \( g_i = \sum_{j=1}^{m_i} \chi_{[a_i+(j-1)r,a_i+jr]} \in LST(\chi_{[0,r]}) \). Therefore \( g = \sum_{i=1}^{n} c_i g_i \in LST(\chi_{[0,r]}) \) and hence \( \mu = \sum_{i=1}^{n} c_i \chi_{[a_i,b_i]} = g \) a.e.

(ii) In the above proof,
\[
\begin{align*}
g_i &= \sum_{j=1}^{m_i} \chi_{[a_i+(j-1)r,a_i+jr]} \\
&= \sum_{j=1}^{m_i} \chi_{[0,r]} \ast \delta_{a_i+(j-1)r} \\
&= \chi_{[0,r]} \ast (\sum_{j=1}^{m_i} \delta_{a_i+(j-1)r}) \\
&= \chi_{[0,r]} \ast \nu_i,
\end{align*}
\]
where \( v_i = \sum_{j=1}^{m_j} \delta_{d_{i+1}(j-1)r} \in M_{cd}(\mathbb{R}) \). But \( g = \sum_{i=1}^{n} c_i g_i \) a.e. Therefore

\[
g = \sum_{i=1}^{n} c_i g_i
= \sum_{i=1}^{n} \chi_{[0,r]} \ast c_i v_i
= \chi_{[0,r]} \ast (\sum_{i=1}^{n} c_i v_i)
= \chi_{[0,r]} \ast v,
\]

where \( v = \sum_{i=1}^{n} c_i v_i \in M_{cd}(\mathbb{R}) \). Since \( \mu = g \) a.e, we have \( \mu = \chi_{[0,r]} \ast v \) a.e.

Also we have

\[
f \ast \chi_{(a,b)}(x) = \int_{a_i}^{b_i} f(x-y)dy = \sum_{j=1}^{m_i} \int_{a_i+(j-1)r}^{a_i+jr} f(x-y)dy = (f \ast g_i)(x).
\]

Therefore \( f \ast \mu = f \ast g = f \ast \chi_{[0,r]} \ast v. \)

The first part of the following theorem is an extension of [4] and the second part is a simple extension of [14].

**Theorem 2.1.** (i) For every \( g \in C^r(\mathbb{R}) \) and every \( v \in M_{cd}(\mathbb{R}) \) with \( v \neq 0 \) there exists \( f \in C^r(\mathbb{R}) \) such that \( f \ast v = g \).

(ii) For every \( g \in C^{r+1}(\mathbb{R}) \), there exists \( f \in C^r(\mathbb{R}) \) such that \( f \ast \chi_{[a,b]} = g \).

**Proof.** For \( \epsilon > 0 \), choose \( \phi_{\epsilon} \in C^{r+1}(\mathbb{R}) \) such that \( supp(\phi_{\epsilon}) \subset (-\epsilon, \epsilon) \) and \( \int_{-\epsilon}^{\epsilon} \phi_{\epsilon}(x)dx = 1 \).

Define \( \eta_1, \eta_2 \in C(\mathbb{R}) \) by

\[
\eta_1(x) := \begin{cases} 
0 & \text{if } x \leq -k \\
\frac{k-x}{2k} & \text{if } -k \leq x \leq k \\
1 & \text{if } x \geq k
\end{cases}
\]

\[
\eta_2(x) := \begin{cases} 
1 & \text{if } x \leq -k \\
\frac{k-x}{2k} & \text{if } -k \leq x \leq k \\
0 & \text{if } x \geq k
\end{cases}
\]

It is simple to check that \( \eta_1(x) + \eta_2(x) = 1 \) for all \( x \in \mathbb{R} \). Convolving both sides with \( \phi_{\epsilon} \), we get

\( (\eta_1 \ast \phi_{\epsilon})(x) + (\eta_2 \ast \phi_{\epsilon})(x) = 1 \) for all \( x \in \mathbb{R} \). Define

\[g_1(x) = g(x)(\eta_1 \ast \phi_{\epsilon})(x) \quad \text{and} \quad g_2(x) = g(x)(\eta_2 \ast \phi_{\epsilon})(x).\]

Then \( g_1, g_2 \in C^{r+1}(\mathbb{R}) \) and \( supp(g_1) \subset [-k-2\epsilon, \infty) \) and \( supp(g_2) \subset (-\infty, k+2\epsilon] \). Also we have \( g_1 + g_2 = g \).
(i) We show that \( f_1 \ast \mu = g_1 \) and \( f_2 \ast \mu = g_2 \) have solutions \( f_1 \) and \( f_2 \) respectively in \( C^r(\mathbb{R}) \). These \( f_1 \) and \( f_2 \) are then used to construct a solution of the equation \( f \ast \mu = g \).

Let \( \mu = \sum_{i=1}^{n} c_i \delta_{x_i}, \ x_{i_0} = \min\{x_1, x_2, \ldots, x_n\} \) and \( x_{j_0} = \max\{x_1, x_2, \ldots, x_n\} \).

We can write

\[
\mu = \sum_{i=1}^{n} c_i \delta_{x_i}
\]

\[
= \sum_{i=1}^{n} c_i \delta_{x_i} \ast \delta_{-x_{i_0}} \ast \delta_{x_{i_0}}
\]

\[
= \delta_{x_{i_0}} \ast \sum_{i=1}^{n} c_i \delta_{x_i-x_{i_0}}
\]

\[
= c_{i_0} \delta_{x_{i_0}} \ast \left( \delta_{0} + \sum_{i=1, i \neq i_0}^{n} \frac{c_i}{c_{i_0}} \delta_{x_i-x_{i_0}} \right)
\]

\[
= c_{i_0} \delta_{x_{i_0}} \ast \left( \delta_{0} + \nu \right), \quad \text{where } \nu = \sum_{i=1, i \neq i_0}^{n} \frac{c_i}{c_{i_0}} \delta_{x_i-x_{i_0}}.
\]

Also we can write \( \mu \) as

\[
\mu = \sum_{j=1}^{n} c_j \delta_{x_j}
\]

\[
= \sum_{j=1}^{n} c_j \delta_{x_j} \ast \delta_{-x_{j_0}} \ast \delta_{x_{j_0}}
\]

\[
= \delta_{x_{j_0}} \ast \sum_{j=1}^{n} c_j \delta_{x_j-x_{j_0}}
\]

\[
= c_{j_0} \delta_{x_{j_0}} \ast \left( \delta_{0} + \sum_{j=1, j \neq j_0}^{n} \frac{c_j}{c_{j_0}} \delta_{x_j-x_{j_0}} \right)
\]

\[
= c_{j_0} \delta_{x_{j_0}} \ast \left( \delta_{0} + \psi \right), \quad \text{where } \psi = \sum_{j=1, j \neq j_0}^{n} \frac{c_j}{c_{j_0}} \delta_{x_j-x_{j_0}}.
\]

Define \( \alpha = \frac{1}{2} \min\{x_i - x_{i_0}/1 \leq i \leq n, i \neq i_0\} \) and \( \beta = \frac{1}{2} \min\{x_{j_0} - x_j/1 \leq j \leq n, j \neq j_0\} \).

Then \( x_i - x_{i_0} > \alpha, x_j - x_{j_0} < -\beta \). Hence \( supp(\nu) \subset (\alpha, \infty) \) and \( supp(\psi) \subset (-\infty, -\beta) \).

Using lemma 1, we get \( h_1, h_2 \in C^r(\mathbb{R}) \) such that

\[
h_1 \ast (\delta_{0} + \nu) = g_1 \tag{4}
\]

and

\[
h_2 \ast (\delta_{0} + \psi) = g_2 \tag{5}
\]

Convolving both sides of the equation (4) with \( c_{i_0} \delta_{x_{i_0}} \) and the equation (5) with \( c_{j_0} \delta_{x_{j_0}} \), we get

\[
h_1 \ast (\delta_{0} + \nu) \ast c_{i_0} \delta_{x_{i_0}} = c_{i_0} g_1 \ast \delta_{x_{i_0}} \]

and

\[
h_2 \ast (\delta_{0} + \psi) \ast c_{j_0} \delta_{x_{j_0}} = c_{j_0} g_2 \ast \delta_{x_{j_0}}.
\]

That is

\[
h_1 \ast \mu = c_{i_0} g_1 \ast \delta_{x_{i_0}} \tag{6}
\]
and
\[ h_2 \ast \mu = c_{j_0} g_2 \ast \delta_{x_{j_0}}. \]  
(7)

Equations (6) and (7) imply \((\frac{1}{c_{j_0}} h_1 \ast \delta_{-x_{j_0}}) \ast \mu = g_1\) and \((\frac{1}{c_{j_0}} h_2 \ast \delta_{-x_{j_0}}) \ast \mu = g_2\).

Define \(f = \frac{1}{c_{j_0}} h_1 \ast \delta_{-x_{j_0}} + \frac{1}{c_{j_0}} h_2 \ast \delta_{-x_{j_0}}\). Then \(f \in C^r(\mathbb{R})\). Now
\[ f \ast \mu = \left(\frac{1}{c_{j_0}} h_1 \ast \delta_{-x_{j_0}}\right) \ast \mu + \left(\frac{1}{c_{j_0}} h_2 \ast \delta_{-x_{j_0}}\right) \ast \mu = g_1 + g_2 = g. \]

(ii) Using Lemma 2, we get \(f_1, f_2 \in C^r(\mathbb{R})\) such that \(f_1 \ast \chi_{[a, b]} = g_1\) and \(f_2 \ast \chi_{[a, b]} = g_2\). Then \(f = f_1 + f_2 \in C^r(\mathbb{R})\) will satisfy \(f \ast \chi_{[a, b]} = g. \)

**Theorem 2.2.** For \(g \in C^{r+1}(\mathbb{R})\), the following hold:

(i) For every \(\mu \in LST(\chi_{[a, b]})\) with \(\mu \neq 0\) a.e., there exists \(f \in C^r(\mathbb{R})\) such that \(f \ast \mu = g\).

(ii) If there exists \(r \in \mathbb{R}\) such that \(\frac{b - a}{r} \in \mathbb{Z}\) and \(\mu = \sum_{i=1}^{n} c_i \chi_{[a_i, b_i]} \neq 0\) a.e., then there exists \(f \in C^r(\mathbb{R})\) such that \(f \ast \mu = g\).

**Proof.** (i) By lemma 3, there exists \(v \in M_{cd}(\mathbb{R})\) such that \(\mu = \chi_{[a, b]} \ast v\). Applying Theorem 2.1, we get a \(h \in C^{r+1}(\mathbb{R})\) and \(f \in C^r(\mathbb{R})\) such that \(h \ast v = g\) and \(f \ast \chi_{[a, b]} = h\). It is simple to verify that \(f \ast \mu = g\).

(ii) Using Lemma 3, we can write \(\mu = \chi_{[a, b]} \ast v\) a.e for some \(v \in M_{cd}(\mathbb{R})\). As in previous part we obtain \(f \in C^r(\mathbb{R})\) such that \(f \ast \mu = g. \)

**Remark 1.** The operator \(T_\mu\) defined by \(T_\mu(f) = f \ast \mu\) is 1-1 if we restrict the domain of \(T_\mu\) to the space of integrable functions \(L_1(\mathbb{R})\). This can be seen as follows: Suppose \(f \ast \mu = 0\) and \(f \in L_1(\mathbb{R})\). Since \(f\) is integrable and \(\mu\) is compactly supported, the Fourier transforms of both \(f\) and \(\mu\) namely \(\hat{f}\) and \(\hat{\mu}\) are holomorphic on \(\mathbb{C}\). Hence the corresponding zero sets \(z(\hat{f})\) and \(z(\hat{\mu})\) are of measure zero. Therefore we get \(f = 0\) a.e.

**Remark 2.** When \(\mu \in LST(\chi_{[a, b]})\) or \(\mu = g\) a.e for some \(g \in LST(\chi_{[a, b]})\), the kernel of the operator \(T_\mu\) is a nontrivial subspace of \(C(\mathbb{R})\). For, since \(\mu\) can be written as \(\mu = \chi_{[0, r]} \ast v\) for some \(v \in M_{cd}(\mathbb{R})\). This implies that \(\lambda = \frac{2\pi n}{r} \in z(\hat{\mu})\) for \(n \in \mathbb{Z}\). Therefore \(e^{i\lambda x} \in Ker(T_\mu)\). Hence there are infinitely many solutions to the convolution equation \(f \ast \mu = g\).

**Remark 3.** Theorem 2.2 is possible even if \(g \in L_1(\mathbb{R})\) with \(\hat{g}(\lambda) \neq 0\) and the Fourier-Laplace transform \(\hat{\mu}(\lambda) = 0\) for some \(\lambda \in \mathbb{C}\).
References


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