SUBCLASSES OF ANALYTIC FUNCTIONS WITH RESPECT TO SYMMETRIC AND CONJUGATE POINTS

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Abstract. In this paper, we introduce new subclasses of convex and starlike functions with respect to other points. The coefficient estimates for these classes are obtained.

1. Introduction

Let $U$ be the class of functions which are analytic and univalent in the open unit disc $D = \{ z : |z| < 1 \}$ given by

$$w(z) = z + \sum_{k=1}^{n} b_k z^k$$

and satisfying the conditions

$$w(0) = 0, \quad |w(z)| < 1, \quad z \in D.$$

Let $S$ denote the class of functions $f$ which are analytic and univalent in $D$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in D.$$ (1.1)

Also let $S^*_c$ be the subclass of $S$ consisting of functions given by (1.1) satisfying

$$\text{Re} \left\{ \frac{zf'(z)}{f(z) - f(-z)} \right\} > 0, \quad z \in D.$$

These functions are called starlike with respect to symmetric points and were introduced by Sakaguchi in 1959. Ashwah and Thomas in [2] introduced another class namely the class $S^*_c$ consisting of functions starlike with respect to conjugate points.
Let \( S_\alpha^* \) be the subclass of \( S \) consisting of functions given by (1.1) and satisfying the condition

\[
Re \left\{ \frac{zf''(z)}{f(z) + f'(z)} \right\} > 0, \quad z \in D.
\]

Motivated by \( S_\alpha^* \), many authors discussed the following class \( C_\alpha \) of function convex with respect to symmetric points and its subclasses.

Let \( C_\alpha \) be the subclass of \( S \) consisting of functions given by (1.1) and satisfying the condition

\[
Re \left\{ \frac{(zf''(z))'}{(f(z) - f(-z))'} \right\} > 0, \quad z \in D.
\]

In terms of subordination, Goel and Mehrok in 1982 introduced a subclass of \( S_\alpha^* \) denoted by \( S_\alpha^*(A,B) \).

Let \( S_\alpha^*(A,B) \) be the class of functions of the form (1.1) and satisfying the condition

\[
\frac{2zf'(z)}{f(z) - f(-z)} < \frac{1 + Az}{1 + Bz}, \quad -1 \leq B < A \leq 1, z \in D.
\]

Also let \( S_\alpha^*(A,B) \) be the class of functions of the form (1.1) and satisfying the condition

\[
\frac{2zf'(z)}{(f(z) + f'(z))} < \frac{1 + Az}{1 + Bz}, \quad -1 \leq B < A \leq 1, z \in D.
\]

Let \( C_\alpha(A,B) \) be the class of functions of the form (1.1) and satisfying the condition

\[
\frac{2(zf''(z))'}{(f(z) - f(-z))'} < \frac{1 + Az}{1 + Bz}, \quad -1 \leq B < A \leq 1, z \in D.
\]

Also let \( C_\alpha(A,B) \) be the class of functions of the form (1.1) and satisfying the condition

\[
\frac{2(zf''(z))'}{(f(z) + f'(z))'} < \frac{1 + Az}{1 + Bz}, \quad -1 \leq B < A \leq 1, z \in D.
\]

In this paper, we introduce the class \( M_\alpha(a, A, B) \) consisting of analytic functions \( f \) of the form (1.1) and satisfying

\[
\frac{2zf'(z) + 2az^2 f''(z)}{(1 - \alpha)(f(z) - f(-z)) + az(f(z) - f(-z))'} < \frac{1 + Az}{1 + Bz},
\]

\[-1 \leq B < A \leq 1, 0 \leq a \leq 1, z \in D.
\]

We note that \( M_\alpha(0, A, B) = S_\alpha^*(A,B) \) and \( M_\alpha(1, A, B) = C_\alpha(A,B) \). Also introduce the class \( M_\alpha(a, A, B) \) consisting of analytic functions \( f \) of the form (1.1) and satisfying

\[
\frac{2zf'(z) + 2az^2 f''(z)}{(1 - \alpha)(f(z) + f'(z)) + az(f(z) + f'(z))'} < \frac{1 + Az}{1 + Bz},
\]

\[-1 \leq B < A \leq 1, 0 \leq a \leq 1, z \in D.
\]
Note that $M_c(0, A, B) = S_c(A, B)$ and $M_c(1, A, B) = C_c(A, B)$.

By definition of subordination it follows that $f \in M_s(\alpha, A, B)$ if and only if
\[
\frac{2zf'(z) + 2az^2f''(z)}{(1 - \alpha)(f(z) - f(-z)) + az(f(z) - f(-z))'} = \frac{1 + Aw(z)}{1 + Bw(z)} = p(z), \ w \in U
\]
and that $f \in M_c(\alpha, A, B)$ if and only if
\[
\frac{2zf'(z) + 2az^2f''(z)}{(1 - \alpha)(f(z) + f(\bar{z})) + az(f(z) + f(\bar{z}))'} = \frac{1 + Aw(z)}{1 + Bw(z)} = p(z), \ w \in U
\]
where
\[
p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n
\]

We study the classes $M_s(\alpha, A, B)$ and $M_c(\alpha, A, B)$, the coefficient estimates for functions belonging to these classes are obtained.

2. Preliminary result

We need the following lemma for proving our results.

**Lemma 2.1.** ([3]) If $p(z)$ is given by (1.4) then
\[
|p_n| \leq A - B, \ n = 1, 2, 3, \ldots
\]

3. Main result

We give the coefficient inequalities for the classes $M_s(\alpha, A, B)$ and $M_c(\alpha, A, B)$.

**Theorem 3.1.** Let $f \in M_s(\alpha, A, B)$. Then for $n \geq 1, 0 \leq \alpha \leq 1$,
\[
|a_{2n}| \leq \frac{A - B}{2n! (1 + (2n - 1)\alpha)} \prod_{j=1}^{n-1} (A - B + 2j), \quad (3.1)
\]
\[
|a_{2n+1}| \leq \frac{A - B}{2n! (1 + 2n\alpha)} \prod_{j=1}^{n-1} (A - B + 2j). \quad (3.2)
\]

**Proof.** From (1.2) and (1.4), we have
\[
\begin{align*}
(z + 2a_2z^2 + 3a_3z^3 + 4a_4z^4 + 5a_5z^5 + \cdots + 2na_{2n}z^{2n} + \cdots) \\
+ \alpha (2a_2z^2 + 6a_3z^3 + 12a_4z^4 + 20a_5z^5 + \cdots + (2n - 1)2na_{2n}z^{2n} + \cdots)
\end{align*}
\]
\[
= [(1 - \alpha)(z + a_3z^3 + a_5z^5 + \cdots + a_{2n-1}z^{2n-1} + a_{2n+1}z^{2n+1} + \cdots)]
\]
\[+a(z + 3a_3z^3 + 5a_5z^5 + \cdots + (2n - 1)a_2n_{-1}z^{2n_{-1}} + (2n + 1)a_2n_{+1}z^{2n_{+1}} + \cdots)]
\cdot(1 + p_1z + p_2z^2 + p_3z^3 + p_4z^4 + \cdots + p_{2n-1}z^{2n-1} + p_{2n}z^{2n} + \cdots)\]

Equating the coefficients of like powers of \(z\), we have

\[2(1 + a)a_2 = p_1, \quad 2(1 + 2a)a_3 = p_2\] (3.3)

\[4(1 + 3a)a_4 = p_3 + (1 + 2a)a_3p_1\]
\[4(1 + 4a)a_5 = p_4 + (1 + 2a)a_3p_2\] (3.4)

\[2n(1 + (2n - 1)a)a_{2n} = p_{2n-1} + (1 + 2a)a_3p_{2n-3} + \cdots + (1 + (2n - 2)a)a_{2n-1}p_1\] (3.5)

\[(2n + 1)(1 + 2n)a_{2n+1} = p_{2n} + (1 + 2a)a_3p_{2n-2} + \cdots + (1 + (2n - 2)a)a_{2n-1}p_2\] (3.6)

Using lemma 2.1 and (3.3), we get

\[|a_2| \leq \frac{A-B}{2(1+a)}, \quad |a_3| \leq \frac{A-B}{2(1+2a)},\] (3.7)

Again by applying (3.6) and followed by Lemma 2.1, we get from (3.4)

\[|a_4| \leq \frac{(A-B)(A-B+2)}{(2)(4)(1+3a)}, \quad |a_5| \leq \frac{(A-B)(A-B+2)}{(2)(4)(1+4a)}\]

It follows that (3.1) and (3.2) hold for \(n = 1, 2\). We prove (3.1) using induction.

Equation (3.5) in conjunction with lemma 2.1 yield

\[|a_{2n}| \leq \frac{A-B}{2n(1 + (2n - 1)a)} \left[ 1 + \sum_{k=1}^{n-1} (1 + 2ka)|a_{2k+1}| \right].\] (3.8)

We assume that (3.1) holds for \(k = 3, 4, \ldots, (n - 1)\). Then from (3.8), we obtain

\[|a_{2n}| \leq \frac{A-B}{2n(1 + (2n - 1)a)} \left[ 1 + \sum_{k=1}^{n-1} \frac{A-B}{2^k k!} \prod_{j=1}^{k-1} (A-B+2j) \right].\] (3.9)

In order to complete the proof, it is sufficient to show that

\[\frac{A-B}{2m(1 + (2m - 1)a)} \left[ 1 + \sum_{k=1}^{m-1} \frac{A-B}{2^k k!} \prod_{j=1}^{k-1} (A-B+2j) \right] = \frac{A-B}{2^m m!(1 + (2m - 1)a)} \prod_{j=1}^{k-1} (A-B+2j), \quad (m = 3, 4, 5, \ldots, n).\] (3.10)

(3.10) is valid for \(m = 3\).

Let us suppose that (3.10) is true for all \(m, 3 < m \leq (n - 1)\). Then from (3.9)

\[\frac{(A-B)}{2n(1 + (2n - 1)a)} \left[ 1 + \sum_{k=1}^{n-1} \frac{A-B}{2^k k!} \prod_{j=1}^{k-1} (A-B+2j) \right] \]
Theorem 3.2. Let \( f \in M_c(\alpha, A, B) \). Then for \( n \geq 1, 0 \leq \alpha \leq 1 \),

\[
|a_{2n}| \leq \frac{(A - B)}{(2n - 1)! (1 + (2n - 1)\alpha)} \prod_{j=1}^{n-2} (A - B + j), \\
|a_{2n+1}| \leq \frac{(A - B)}{(2n)! (1 + 2n\alpha)} \prod_{j=1}^{n-1} (A - B + j).
\]

Proof. From (1.3) and (1.4), we have

\[
(z + 2a_2z^2 + 3a_3z^3 + 4a_4z^4 + 5a_5z^5 + \cdots + 2na_{2n}z^{2n} + \cdots) \\
+ \alpha (2a_2z^2 + 6a_3z^3 + 12a_4z^4 + 20a_5z^5 + \cdots + (2n-1)2na_{2n}z^{2n} + \cdots) \\
= [(1 - \alpha)(z + 2a_2z^2 + 3a_3z^3 + 4a_4z^4 + 5a_5z^5 + \cdots + a_{2n}z^{2n} + \cdots) \\
+ \alpha (z + 2a_2z^2 + 3a_3z^3 + 4a_4z^4 + 5a_5z^5 + \cdots + 2na_{2n}z^{2n} + \cdots) ] \\
\cdot (1 + p_1z + p_2z^2 + p_3z^3 + p_4z^4 + \cdots + p_{2n-1}z^{2n-1} + \cdots)
\]

Equating the coefficients of like powers of \( z \), we have

\[
(1 + \alpha) a_2 = p_1, \quad 2(1 + 2\alpha) a_3 = p_2 + (1 + \alpha) a_2 p_1, \\
3(1 + 3\alpha) a_4 = p_3 + (1 + \alpha) a_2 a_2 p_2 + (1 + 2\alpha) a_3 p_1, \quad (1.14) \\
4(1 + 4\alpha) a_5 = p_4 + (1 + \alpha) a_2 a_2 p_3 + (1 + 2\alpha) a_3 a_2 p_2 + (1 + 3\alpha) a_4 p_1, \quad (1.15) \\
(2n - 1)(1 + (2n - 1)\alpha) a_{2n} = p_{2n-1} + (1 + \alpha) a_2 p_{2n-2} + \cdots + (1 + (2n - 2)\alpha) a_{2n-1} p_1, \quad (1.16) \\
2n(1 + 2n\alpha) a_{2n+1} = p_{2n} + (1 + \alpha) a_2 p_{2n-1} + \cdots + (1 + (2n - 2)\alpha) a_{2n} p_1. \quad (1.17)
\]
By using Lemma 2.1 and (3.13), we get

\[ |a_2| \leq \frac{(A - B)}{1 + \alpha}, \quad |a_3| \leq \frac{(A - B)(A - B + 1)}{2(1 + 2\alpha)}. \]  

(3.18)

Again by applying (3.18) and followed by Lemma 2.1, we get from (3.14) and (3.15), we have

\[ |a_4| \leq \frac{(A - B)(A - B + 1)(A - B + 2)}{(2)(3)(1 + 3\alpha)}, \]

\[ |a_5| \leq \frac{(A - B)^2 + 6(A - B)^3 + 11(A - B)^2 + 6(A - B)}{(2)(3)(4)(1 + 4\alpha)}. \]

It follows that (3.11) hold for \( n = 1, 2 \). We now prove (3.11) using induction.

Equation (3.16) in conjunction with Lemma 2.1 yield

\[ |a_{2n}| \leq \frac{(A - B)}{(2n - 1)(1 + (2n - 1)\alpha)} \left[ 1 + \sum_{k=1}^{n-1} |a_{2k}| + \sum_{k=1}^{n-1} |a_{2k+1}| \right]. \]  

(3.19)

We assume that (3.11) holds for \( k = 3, 4, \ldots, (n - 1) \). Then from (3.19), we obtain

\[ |a_{2n}| \leq \frac{(A - B)}{(2n - 1)(1 + (2n - 1)\alpha)} \left[ 1 + \sum_{k=1}^{n-1} \frac{(A - B)}{2(k-1)!} \prod_{j=1}^{2k-2} (A - B + j) 
+ \sum_{k=1}^{n-1} \frac{(A - B)}{(2k)!} \prod_{j=1}^{2k-1} (A - B + j) \right]. \]  

(3.20)

In order to complete the proof, it is sufficient to show that

\[ \frac{(A - B)}{(2m - 1)(1 + (2m - 1)\alpha)} \left[ 1 + \sum_{k=1}^{m-1} \frac{(A - B)}{2(k-1)!} \prod_{j=1}^{2k-2} (A - B + j) + \sum_{k=1}^{m-1} \frac{(A - B)}{(2k)!} \prod_{j=1}^{2k-1} (A - B + j) \right] \]

\[ = \frac{(A - B)}{(2m - 1)!((1 + (2m - 1)\alpha)} \prod_{j=1}^{2m-2} (A - B + j), \quad (m = 3, 4, 5, \ldots, n). \]  

(3.21)

(3.21) is valid for \( m = 3 \).

Let us suppose that (3.21) is true for all \( m, 3 < m \leq (n - 1) \). Then from (3.20)

\[ \frac{(A - B)}{(2n - 1)(1 + (2n - 1)\alpha)} \left[ 1 + \sum_{k=1}^{n-1} \frac{(A - B)}{2(k-1)!} \prod_{j=1}^{2k-2} (A - B + j) + \sum_{k=1}^{n-1} \frac{(A - B)}{(2k)!} \prod_{j=1}^{2k-1} (A - B + j) \right] \]

\[ = \frac{(2n - 3)}{(2n - 1)} \left( \frac{(A - B)}{(2(n-1) - 1)(1 + (2n - 1)\alpha)} \left[ 1 + \sum_{k=1}^{n-2} \frac{(A - B)}{2(k-1)!} \prod_{j=1}^{2k-2} (A - B + j) + \sum_{k=1}^{n-2} \frac{(A - B)}{(2k)!} \prod_{j=1}^{2k-1} (A - B + j) \right] + \sum_{k=1}^{n-2} \frac{(A - B)}{(2k)!} \prod_{j=1}^{2k-1} (A - B + j) \right) \]

\[ + \frac{(A - B)}{(2n - 1)(1 + (2n - 1)\alpha)} \frac{(A - B)}{(2(n-1) - 1)!} \prod_{j=1}^{2n-4} (A - B + j) \]
Thus (3.21) holds for \( m = n \) and hence (3.11) follows. Similarly we can prove (3.12).

On specializing the values of \( \alpha \) in Theorem 3.1 and Theorem 3.2, we get the following.

**Remark 3.1.** In Theorem 3.1, if we set \( \alpha = 0 \), we get starlike functions with respect to symmetric points and if we set \( \alpha = 1 \), we get convex functions with respect to symmetric points.

**Remark 3.2.** In Theorem 3.2, if we set \( \alpha = 0 \), we get starlike functions with respect to conjugate points and if we set \( \alpha = 1 \), we get convex functions with respect to conjugate points. For other values of \( \alpha \) the transition is smooth.

**References**


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