A REFINEMENT OF THE GRÜSS INEQUALITY AND APPLICATIONS

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Abstract. A sharp refinement of the Grüss inequality in the general setting of measurable spaces and abstract Lebesgue integrals is proven. Some consequential particular inequalities are mentioned.

1. Introduction

Let \((\Omega, \mathcal{A}, \mu)\) be a measurable space consisting of a set \(\Omega\), a \(\sigma\)–algebra \(\mathcal{A}\) of parts of \(\Omega\) and a countably additive and positive measure \(\mu\) on \(\mathcal{A}\) with values in \(\mathbb{R} \cup \{\infty\}\).

For a \(\mu\)-measurable function \(w : \Omega \to \mathbb{R}\), with \(w(x) \geq 0\) for \(\mu\)–a.e. \(x \in \Omega\), consider the Lebesgue space \(L_w(\Omega, \mu) := \{f : \Omega \to \mathbb{R}, f\) is \(\mu\)-measurable and \(\int_{\Omega} w(x)|f(x)|d\mu(x) < \infty\}\). Assume \(\int_{\Omega} w(x)d\mu(x) > 0\).

If \(f, g : \Omega \to \mathbb{R}\) are \(\mu\)-measurable functions and \(f, g, fg \in L_w(\Omega, \mu)\), then we may consider the Čebyšev functional

\[
T_w(f, g) := \frac{1}{\int_{\Omega} w(x)d\mu(x)} \int_{\Omega} w(x)f(x)g(x)d\mu(x) - \frac{1}{\int_{\Omega} w(x)d\mu(x)} \int_{\Omega} w(x)f(x)d\mu(x) \times \frac{1}{\int_{\Omega} w(x)d\mu(x)} \int_{\Omega} w(x)g(x)d\mu(x). \tag{1.1}
\]

The following result is known in the literature as the Grüss inequality

\[
|T_w(f, g)| \leq \frac{1}{4}(\Gamma - \gamma)(\Delta - \delta), \tag{1.2}
\]

provided

\[
-\infty < \gamma \leq f(x) \leq \Gamma < \infty, \quad -\infty < \delta \leq g(x) \leq \Delta < \infty \tag{1.3}
\]

for \(\mu\)–a.e. \(x \in \Omega\).

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The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller constant.

Note that if $\Omega = \{1, \ldots, n\}$ and $\mu$ is the discrete measure on $\Omega$, then we obtain the discrete Grüss inequality

$$\left| \frac{1}{W_n} \sum_{i=1}^{n} w_i x_i y_i - \frac{1}{W_n} \sum_{i=1}^{n} w_i x_i \cdot \frac{1}{W_n} \sum_{i=1}^{n} w_i y_i \right| \leq \frac{1}{4} (\Gamma - \gamma)(\Delta - \delta),$$

(1.4)

provided $\gamma \leq x_i \leq \Gamma, \delta \leq y_i \leq \Delta$ for each $i \in \{1, \ldots, n\}$ and $w_i \geq 0$ with $W_n := \sum_{i=1}^{n} w_i > 0$.

The following result was proved in Cheng and Sun [4].

**Theorem 1.** Let $f, g : [a, b] \to \mathbb{R}$ be two integrable functions such that $\delta \leq g(x) \leq \Delta$ for some constants $\delta, \Delta$ for all $x \in [a, b]$, then

$$\left| \frac{1}{b-a} \int_{a}^{b} f(x)g(x)dx - \frac{1}{(b-a)^2} \int_{a}^{b} f(x)dx \int_{a}^{b} g(x)dx \right| \leq \frac{\Delta - \delta}{2} \frac{1}{b-a} \int_{a}^{b} \left| f(x) - \frac{1}{b-a} \int_{a}^{b} f(y)dy \right| dx. \quad (1.5)$$

They used the result (1.5) to obtain perturbed trapezoidal rules.

In the current paper we obtain bounds for $|T_w(f, g)|$ under the general setting expressed in (1.1). A bound which is shown to be sharp is obtained in Section 2. The sharpness of (1.5) was not demonstrated in [4]. Sharp results were obtained for a perturbed interior point rule (Ostrowski-Grüss) inequalities in Cheng [3]. Some particular instances of the results in Section 2 are investigated in Sections 4 and 5, recapturing earlier work. Results are presented in Section 3, for Lebesgue measurable functions and for a discrete weighted Čebyšev functional involving $n$–tuples.

**2. An integral inequality**

With the assumptions as presented in the Introduction and if $f \in L_w(\Omega, \mu)$ then we may define

$$D_w(f) := D_{w,1}(f)$$

$$:= \frac{1}{\int_{\Omega} w(x)d\mu(x)} \int_{\Omega} w(x) \left| f(x) - \frac{1}{\int_{\Omega} w(y)d\mu(y)} \int_{\Omega} w(y)f(y)d\mu(y) \right| d\mu(x). \quad (2.1)$$

The following fundamental result holds.

**Theorem 2.** Let $w, f, g : \Omega \to \mathbb{R}$ be $\mu$-measurable functions with $w \geq 0 \mu$-a.e. on $\Omega$ and $\int_{\Omega} w(y)d\mu(y) > 0$. If $f, g, fg \in L_w(\Omega, \mu)$ and there exists the constants $\delta, \Delta$ such that

$$-\infty < \delta \leq g(x) \leq \Delta < \infty \text{ for } \mu \text{- a.e. } x \in \Omega,$$

(2.2)
then we have the inequality

$$|T_w(f, g)| \leq \frac{1}{2}(\Delta - \delta)D_w(f).$$

(2.3)

The constant \(\frac{1}{2}\) is sharp in the sense that it cannot be replaced by a smaller quantity.

**Proof.** Obviously, we have

$$T_w(f, g) = \frac{1}{\int_{\Omega} w(x)d\mu(x)} \int_{\Omega} w(x) \times \left(f(x) - \frac{1}{\int_{\Omega} w(y)d\mu(y)} \int_{\Omega} w(y)f(y)d\mu(y)\right)g(x)d\mu(x).$$

(2.4)

Consider the measurable subsets \(\Omega_+\) and \(\Omega_-\), of \(\Omega\), defined by

\[\Omega_+ := \{x \in \Omega \mid f(x) - \frac{1}{\int_{\Omega} w(y)d\mu(y)} \int_{\Omega} w(y)f(y)d\mu(y) \geq 0\}\]

and

\[\Omega_- := \{x \in \Omega \mid f(x) - \frac{1}{\int_{\Omega} w(y)d\mu(y)} \int_{\Omega} w(y)f(y)d\mu(y) < 0\}\]

Obviously, \(\Omega = \Omega_+ \cup \Omega_-\), \(\Omega_+ \cap \Omega_- = \emptyset\) and if we define

\[I_+(f, g, w) := \frac{1}{\int_{\Omega} w(y)d\mu(y)} \int_{\Omega_+} w(x) \times \left(f(x) - \frac{1}{\int_{\Omega} w(y)d\mu(y)} \int_{\Omega} w(y)f(y)d\mu(y)\right)g(x)d\mu(x)\]

and

\[I_-(f, g, w) := \frac{1}{\int_{\Omega} w(y)d\mu(y)} \int_{\Omega_-} w(x) \times \left(f(x) - \frac{1}{\int_{\Omega} w(y)d\mu(y)} \int_{\Omega} w(y)f(y)d\mu(y)\right)g(x)d\mu(x)\]

then we have

$$T_w(f, g) = I_+(f, g, w) + I_-(f, g, w).$$

(2.5)

Since \(-\infty < \delta \leq g(x) \leq \Delta < \infty\) for \(\mu\)-a.e. \(x \in \Omega\) and \(w(x) \geq 0\) for \(\mu\)-a.e. \(x \in \Omega\), we may write:

\[I_+(f, g, w) \leq \frac{\Delta}{\int_{\Omega} w(y)d\mu(y)} \int_{\Omega_+} w(x) \times \left(f(x) - \frac{1}{\int_{\Omega} w(y)d\mu(y)} \int_{\Omega} f(y)w(y)d\mu(y)\right)d\mu(x)\]

(2.6)
On the other hand,

\[
I_{-}(f, g, w) \leq \frac{\delta}{\int_{\Omega} w(y) d\mu(y)} \int_{\Omega} w(x) \times \left( f(x) - \frac{1}{\int_{\Omega} w(y) d\mu(y)} \int_{\Omega} f(y) w(y) d\mu(y) \right) d\mu(x). \tag{2.7}
\]

Since

\[
0 = \int_{\Omega} w(x) \left( f(x) - \frac{1}{\int_{\Omega} w(y) d\mu(y)} \int_{\Omega} f(y) w(y) d\mu(y) \right) d\mu(x)
\]

\[
= \int_{\Omega_{+}} w(x) \left( f(x) - \frac{1}{\int_{\Omega} w(y) d\mu(y)} \int_{\Omega} f(y) w(y) d\mu(y) \right) d\mu(x)
\]

\[
+ \int_{\Omega_{-}} w(x) \left( f(x) - \frac{1}{\int_{\Omega} w(y) d\mu(y)} \int_{\Omega} f(y) w(y) d\mu(y) \right) d\mu(x)
\]

we get

\[
\int_{\Omega_{-}} w(x) \left( f(x) - \frac{1}{\int_{\Omega} w(y) d\mu(y)} \int_{\Omega} f(y) w(y) d\mu(y) \right) d\mu(x)
\]

\[
= - \int_{\Omega_{+}} w(x) \left( f(x) - \frac{1}{\int_{\Omega} w(y) d\mu(y)} \int_{\Omega} f(y) w(y) d\mu(y) \right) d\mu(x)
\]

and thus, from (2.6), we deduce

\[
I_{-}(f, g, w) \leq \frac{-\delta}{\int_{\Omega} w(y) d\mu(y)} \int_{\Omega} w(x) \times \left( f(x) - \frac{1}{\int_{\Omega} w(y) d\mu(y)} \int_{\Omega} w(y) f(y) d\mu(y) \right) d\mu(x). \tag{2.8}
\]

Consequently, by adding (2.7) with (2.8), we deduce

\[
T_{w}(f, g) \leq \frac{\Delta - \delta}{\int_{\Omega} w(y) d\mu(y)} \int_{\Omega} w(x) \times \left( f(x) - \frac{1}{\int_{\Omega} w(y) d\mu(y)} \int_{\Omega} w(y) f(y) d\mu(y) \right) d\mu(x). \tag{2.9}
\]

On the other hand,

\[
\int_{\Omega} w(x) \left| f(x) - \frac{1}{\int_{\Omega} w(y) d\mu(y)} \int_{\Omega} f(y) w(y) d\mu(y) \right| d\mu(x)
\]

\[
= \int_{\Omega_{+}} w(x) \left| f(x) - \frac{1}{\int_{\Omega} w(y) d\mu(y)} \int_{\Omega} f(y) w(y) d\mu(y) \right| d\mu(x)
\]

\[
+ \int_{\Omega_{-}} w(x) \left| f(x) - \frac{1}{\int_{\Omega} w(y) d\mu(y)} \int_{\Omega} f(y) w(y) d\mu(y) \right| d\mu(x)
\]
\[
\begin{align*}
&= \int_{\Omega^+} w(x) \left( f(x) - \frac{1}{\int_{\Omega} w(y) d\mu(y)} \int_{\Omega} f(y) w(y) d\mu(y) \right) d\mu(x) \\
&\quad - \int_{\Omega^-} w(x) \left( f(x) - \frac{1}{\int_{\Omega} w(y) d\mu(y)} \int_{\Omega} f(y) w(y) d\mu(y) \right) d\mu(x) \\
&= 2 \int_{\Omega^+} w(x) \left( f(x) - \frac{1}{\int_{\Omega} w(y) d\mu(y)} \int_{\Omega} f(y) w(y) d\mu(y) \right) d\mu(x),
\end{align*}
\]
and thus, by (2.9) we deduce
\[
T_w(f, g) \leq \frac{1}{2} (\Delta - \delta) \frac{1}{\int_{\Omega} w(y) d\mu(y)} \int_{\Omega} w(x) \bigg| f(x) - \frac{1}{\int_{\Omega} w(y) d\mu(y)} \int_{\Omega} f(y) w(y) d\mu(y) \bigg| d\mu(x). \tag{2.10}
\]
Now, if we write the inequality (2.10) for \(-f\) instead of \(f\) and taking into account that \(T_w(-f, g) = -T_w(f, g)\), we deduce
\[
-T(f, g) \leq \frac{1}{2} (\Delta - \delta) \frac{1}{\int_{\Omega} w(y) d\mu(y)} \int_{\Omega} w(x) \bigg| f(x) - \frac{1}{\int_{\Omega} w(y) d\mu(y)} \int_{\Omega} f(y) w(y) d\mu(y) \bigg| d\mu(x), \tag{2.11}
\]
giving the desired inequality.

To prove the sharpness of the constant \(\frac{1}{2}\), assume that (2.12) holds for \(\Omega = [a, b]\) and \(w \equiv 1\), with a constant \(C > 0\). That is,
\[
|T(f, g)| \leq C(\Delta - \delta) \frac{1}{b - a} \int_{a}^{b} \bigg| f(x) - \frac{1}{b - a} \int_{a}^{b} f(y) dy \bigg| dx, \tag{2.12}
\]
where
\[
T(f, g) = \frac{1}{b - a} \int_{a}^{b} f(x) g(x) dx - \frac{1}{b - a} \int_{a}^{b} f(x) dx \cdot \frac{1}{b - a} \int_{a}^{b} g(x) dx
\]
and the integral \(\int_{a}^{b}\) is the usual Lebesgue integral on \([a, b]\).

Choose in (2.12) \(g = f\) and \(f : [a, b] \to \mathbb{R}\) defined by
\[
f(x) = \begin{cases} 
-1 & \text{if } x \in \left[a, \frac{a+b}{2}\right], \\
1 & \text{if } x \in \left(\frac{a+b}{2}, b\right],
\end{cases}
\]
then, obviously,
\[
T(f, f) = \frac{1}{b - a} \int_{a}^{b} f^2(x) dx - \left( \frac{1}{b - a} \int_{a}^{b} f(x) dx \right)^2 = 1,
\]
\[ D(f) = \frac{1}{b-a} \int_a^b \left| f(x) - \frac{1}{b-a} \int_a^b f(y) \, dy \right| \, dx = 1, \]
\[ \delta = -1, \quad \Delta = 1, \]
and by (2.12) we get \( 2C \geq 1 \) giving \( C \geq \frac{1}{2} \).

For \( f \in L_{p,w}(\Omega, \mathcal{A}, \mu) := \{ f : \Omega \to \mathbb{R}, \int_{\Omega} w(x) |f(x)|^p \, d\mu(x) < \infty \}, p \geq 1 \) we may also define
\[
D_{w,p}(f) := \left[ \frac{1}{\int_{\Omega} w(x) \, d\mu(x)} \int_{\Omega} w(x) \right]
\times \left[ \left| f(x) - \frac{1}{\int_{\Omega} w(y) \, d\mu(y)} \int_{\Omega} w(y) f(y) \, d\mu(y) \right|^p \, d\mu(x) \right]^\frac{1}{p}
= \left\| f - \frac{1}{\int_{\Omega} w \, d\mu} \int_{\Omega} w f \, d\mu \right\|_{\Omega,p}.
\tag{2.13}
\]
where \( \| \cdot \|_{\Omega,p} \) is the usual \( p \)-norm on \( L_{p,w}(\Omega, \mathcal{A}, \mu) \), namely,
\[
\| h \|_{\Omega,p} := \left( \int_{\Omega} w |h|^p \, d\mu \right)^\frac{1}{p}, \quad p \geq 1.
\]

Using Hölder’s inequality we get
\[
D_{w,1}(f) \leq D_{w,p}(f) \quad \text{for } p \geq 1, f \in L_{p,w}(\Omega, \mathcal{A}, \mu);
\tag{2.14}
\]
and, in particular for \( p = 2 \)
\[
D_{w,1}(f) \leq D_{w,2}(f) = \left[ \frac{\int_{\Omega} w f^2 \, d\mu}{\int_{\Omega} w \, d\mu} - \left( \frac{\int_{\Omega} w f \, d\mu}{\int_{\Omega} w \, d\mu} \right)^2 \right]^\frac{1}{2},
\tag{2.15}
\]
if \( f \in L_{2,w}(\Omega, \mathcal{A}, \mu) \).

For \( f \in L_{\infty}(\Omega, \mathcal{A}, \mu) := \{ f : \Omega \to \mathbb{R}, \| f \|_{\Omega,\infty} := \text{ess sup}_{x \in \Omega} |f(x)| < \infty \} \) we also have
\[
D_{w,p}(f) \leq D_{w,\infty}(f) := \left\| f - \frac{1}{\int_{\Omega} w \, d\mu} \int_{\Omega} w f \, d\mu \right\|_{\Omega,\infty}.
\tag{2.16}
\]
The following corollary may be useful in practice.

**Corollary 1.** With the assumptions of Theorem 2, we have

\[
|T_w(f, g)| \leq \frac{1}{2} (\Delta - \delta) D_w(f)
\leq \frac{1}{2} (\Delta - \delta) D_{w,p}(f) \quad \text{if } f \in L_p(\Omega, \mathcal{A}, \mu), 1 < p < \infty;
\]
GRÜSS INEQUALITY

\[
\leq \frac{1}{2}(\Delta - \delta) \left\| f - \frac{1}{\int_{\Omega} w d\mu} \int_{\Omega} w f d\mu \right\|_{\infty} \quad \text{if } f \in L_\infty(\Omega, \mathcal{A}, \mu). \tag{2.17}
\]

**Remark 1.** The inequalities in (2.17) are in order of increasing coarseness. If we assume that \(-\infty < \gamma \leq f(x) \leq \Gamma < \infty\) for \(\mu\)-a.e. \(x \in \Omega\), then by the Grüss inequality for \(g = f\) we have for \(p = 2\)

\[
\frac{\int_{\Omega} w f^2 d\mu}{\int_{\Omega} w d\mu} - \left( \frac{\int_{\Omega} w f d\mu}{\int_{\Omega} w d\mu} \right)^2 \leq \frac{1}{2}(\Gamma - \gamma). \tag{2.18}
\]

By (2.17), we deduce the following sequence of inequalities

\[
|T_w(f, g)| \leq \frac{1}{2}(\Delta - \delta) \frac{1}{\int_{\Omega} w d\mu} \int_{\Omega} w \left| f - \frac{1}{\int_{\Omega} w d\mu} \int_{\Omega} w f d\mu \right| d\mu
\]

\[
\leq \frac{1}{2}(\Delta - \delta) \left[ \frac{\int_{\Omega} w^2 d\mu}{\int_{\Omega} w d\mu} - \left( \frac{\int_{\Omega} w f d\mu}{\int_{\Omega} w d\mu} \right)^2 \right]^{\frac{1}{2}}
\]

\[
\leq \frac{1}{4}(\Delta - \delta)(\Gamma - \gamma) \tag{2.19}
\]

for \(f, g : \Omega \to \mathbb{R}\), \(\mu\) - measurable functions and so that \(-\infty < \gamma \leq f(x) < \Gamma < \infty\), \(-\infty < \delta \leq g(x) \leq \Delta < \infty\) for \(\mu\)-a.e. \(x \in \Omega\). Thus, the inequality (2.19) is a refinement of Grüss’ inequality (1.2).

It is well known that if \(f \in L_{2,w}(\Omega, \mathcal{A}, \mu)\), then the following Schwarz’s type inequality holds:

\[
\frac{1}{\int_{\Omega} w d\mu} \int_{\Omega} w f^2 d\mu \geq \left( \frac{1}{\int_{\Omega} w d\mu} \int_{\Omega} w f d\mu \right)^2. \tag{2.20}
\]

Using the above results, we may point out the following counterpart result.

**Proposition 1.** Assume that the \(\mu\)-measurable function \(f : \Omega \to \mathbb{R}\) satisfies the assumption:

\[-\infty < \gamma \leq f(x) \leq \Gamma < \infty \quad \text{for } \text{a.e. } x \in \Omega. \tag{2.21}\]

Then one has the inequality

\[
0 \leq \frac{1}{\int_{\Omega} w d\mu} \int_{\Omega} w f^2 d\mu - \left( \frac{1}{\int_{\Omega} w d\mu} \int_{\Omega} w f d\mu \right)^2
\]

\[
\leq \frac{1}{2}(\Gamma - \gamma) \frac{1}{\int_{\Omega} w d\mu} \int_{\Omega} w \left| f - \frac{1}{\int_{\Omega} w d\mu} \int_{\Omega} w f d\mu \right| d\mu
\]

\[
\leq \frac{1}{4}(\Gamma - \gamma)^2. \tag{2.22}\]

The constant \(\frac{1}{2}\) is sharp.
The proof follows by the inequality (2.3) for \( g = f \).

The following proposition also holds.

**Proposition 2.** Assume that the measurable functions \( f, g : \Omega \rightarrow \mathbb{R} \) satisfy (the condition in Grüss’ inequality). Then

\[
|T_w(f, g)| \leq \frac{1}{2} \left[ (\Gamma - \gamma)(\Delta - \delta) \right]^{\frac{1}{2}} \left[ D_w(f) D_w(g) \right]^{\frac{1}{2}} \\
\leq \frac{1}{4} (\Delta - \delta)(\Gamma - \gamma).
\]

(2.23)

The constant \( \frac{1}{2} \) in the first inequality and \( \frac{1}{4} \) in the second inequality are sharp.

**Proof.** By (2.19) we have

\[
|T_w(f, g)| \leq \frac{1}{2} (\Delta - \delta) D_w(f)
\]

and

\[
|T_w(f, g)| \leq \frac{1}{2} (\Gamma - \gamma) D_w(g)
\]

from which, by multiplication, gives the first part of (2.23).

The second part and the sharpness of the constants are obvious.

### 3. Some particular inequalities

The following particular inequalities are of interest.

1. Let \( w, f, g : [a, b] \rightarrow \mathbb{R} \) be Lebesgue measurable functions with \( w \geq 0 \) a.e. on \([a, b]\) and \( \int_a^b w(y) dy > 0 \). If \( f, g, fg \in L_w[a, b] \), where

\[
L_w[a, b] := \left\{ f : [a, b] \rightarrow \mathbb{R} \mid \int_a^b w(x) |f(x)| \, dx < \infty \right\}
\]

and

\[-\infty < \delta \leq g(x) \leq \Delta < \infty \quad \text{for a.e. } x \in [a, b], \tag{3.1}\]

then we have the inequalities

\[
\left| \int_a^b \frac{1}{w(x) dx} \int_a^b w(x) f(x) g(x) \, dx \\
- \frac{1}{\int_a^b w(x) dx} \int_a^b w(x) f(x) \, dx \cdot \frac{1}{\int_a^b w(x) dx} \int_a^b w(x) g(x) \, dx \right| \\
\leq \frac{1}{2} (\Delta - \delta) \frac{1}{\int_a^b w(x) dx} \int_a^b w(x) \left| f(x) - \frac{1}{\int_a^b w(y) dy} \int_a^b w(y) f(y) \, dy \right| \, dx
\]
The following counterpart of Schwarz’s inequality holds

\[ 0 \leq \frac{1}{f_a^b} \int_a^b w(x) \int_a^b w(y) f(x) dy dx - \left( \frac{1}{f_a^b} \int_a^b w(x) dy \int_a^b w(y) f(x) dx \right)^2 \]

\[ \leq \frac{1}{2} (\Delta - \gamma) \int_a^b w(x) f(x) dx \left( \frac{1}{f_a^b} \int_a^b w(x) dy \int_a^b w(y) f(y) dy \right) dx \]

\[ \left( \leq \frac{1}{4} (C - \gamma)^2 \right), \quad (3.3) \]

provided \(-\infty < \gamma \leq f(x) \leq C < \infty\) for a.e. \(x \in [a, b]\). The constant \(\frac{1}{4}\) is sharp.

If \(w(x) = 1\), \(x \in [a, b]\), then we recapture the result in (1.5) as depicted here by (3.2).

2. Let \(\bar{a} = (a_1, \ldots, a_n)\), \(\bar{b} = (b_1, \ldots, b_n)\), \(\bar{p} = (p_1, \ldots, p_n)\) be \(n\)-tuples of real numbers with \(p_i \geq 0\) \((i \in \{1, \ldots, n\})\) and \(\sum_{i=1}^n p_i = 1\). If

\[ b \leq b_i \leq B, \quad i \in \{1, \ldots, n\}, \quad (3.4) \]

then one has the inequality

\[ \left| \sum_{i=1}^n p_i a_i b_i - \sum_{i=1}^n p_i a_i \cdot \sum_{i=1}^n p_i b_i \right| \leq \frac{1}{2} (B - b) \left| \sum_{i=1}^n p_i a_i - \sum_{j=1}^n p_j a_j \right|^{\frac{1}{p}} \]

\[ \leq \frac{1}{2} (B - b) \max_{i=1,n} \left| a_i - \sum_{j=1}^n p_j a_j \right|. \quad (3.5) \]

The constant \(\frac{1}{2}\) is sharp in the first inequality.

If \(p_i = 1\), \(i \in \{1, \ldots, n\}\), the following unweighted inequality may be stated

\[ 0 \leq \frac{1}{n} \sum_{i=1}^n a_i b_i - \frac{1}{n} \sum_{i=1}^n a_i \cdot \frac{1}{n} \sum_{i=1}^n b_i \]
\[
\leq \frac{1}{2} (B - b) \left| \frac{1}{n} \sum_{i=1}^{n} a_i - \frac{1}{n} \sum_{j=1}^{n} a_j \right| ^p \leq \frac{1}{2} (B - b) \left( \frac{1}{n} \sum_{i=1}^{n} a_i - \frac{1}{n} \sum_{j=1}^{n} a_j \right) \]

\[
\leq \frac{1}{2} (B - b) \max_{i=1,n} \left| a_i - \frac{1}{n} \sum_{j=1}^{n} a_j \right| .
\]

The following counterpart of Schwarz’s inequality also holds

\[
0 \leq \sum_{i=1}^{n} p_i a_i^2 - \left( \sum_{i=1}^{n} p_i a_i \right)^2 \leq \frac{1}{2} (A - a) \sum_{i=1}^{n} p_i \left| a_i - \sum_{j=1}^{n} p_j a_j \right| \leq \frac{1}{4} (A - a)^2,
\]

provided \( a \leq a_i \leq A \) for each \( i \in \{1, \ldots, n\} \) and \( \sum_{i=1}^{n} p_i = 1 \). The constant \( \frac{1}{2} \) is sharp.

4. Applications for Ostrowski’s inequality

If \( \varphi : [a, b] \to \mathbb{R} \) is an absolutely continuous function on \( [a, b] \) such that \( \varphi' \in L_{\infty} [a, b] \), then the following inequality is known in the literature as Ostrowski’s inequality

\[
\left| \varphi(x) - \frac{1}{b - a} \int_{a}^{b} \varphi(t) dt \right| \leq \frac{1}{4} + \left( \frac{x - a + b}{b - a} \right)^2 \| \varphi' \|_{\infty}, \quad x \in [a, b],
\]

where \( \| \varphi' \|_{\infty} := \text{ess sup}_{\alpha \in [a,b]} |\varphi'(x)| \). The constant \( \frac{1}{4} \) is best possible.

A simple proof of this fact, as mentioned in [1], may be accomplished by the use of the Montgomery identity

\[
\varphi(x) = \frac{1}{b - a} \int_{a}^{b} \varphi(t) dt + \frac{1}{b - a} \int_{a}^{b} K(x, t) \varphi'(t) dt,
\]

where the kernel \( K : [a, b]^2 \to \mathbb{R} \) is defined by

\[
K(x, t) := \begin{cases} 
  t - a & \text{if } a \leq t \leq x \\
  t - b & \text{if } a \leq x < t \leq b.
\end{cases}
\]
We will now use the unweighted version of the inequality (3.2), namely, (1.5) (obtained by Cheng and Sun [4]) to procure the next result concerning a perturbed version of Ostrowski’s inequality (4.1).

The following result also obtained by Cheng [3] is recaptured in a simpler manner. A weighted version of this result was obtained by Roumeliotis [5].

**Theorem 3.** Assume that \( \varphi : [a, b] \to \mathbb{R} \) is an absolutely continuous function on \([a, b]\) such that \( \varphi' : [a, b] \to \mathbb{R} \) satisfies the condition

\[
-\infty < \gamma \leq \varphi'(x) \leq \Gamma < \infty \quad \text{for a.e. } x \in [a, b].
\] (4.4)

Then we have the inequality

\[
\left| \varphi(x) - \frac{1}{b - a} \int_a^b \varphi(t) \, dt - \left( x - \frac{a + b}{2} \right) [\varphi; a, b] \right| \leq \frac{1}{8} (b - a) (\Gamma - \gamma) \] (4.5)

for any \( x \in [a, b] \), where \([\varphi; a, b] = \frac{\varphi(b) - \varphi(a)}{b - a}\) is the divided difference. The constant \( \frac{1}{8} \) is best possible.

**Proof.** We apply inequality (3.1) for the choices \( w(t) = 1, f(t) = K(x, t) \) defined by (4.3), \( g(t) = \varphi'(t), t \in [a, b] \) to get

\[
\left| \frac{1}{b - a} \int_a^b K(x, t) \varphi'(t) \, dt - \frac{1}{b - a} \int_a^b K(x, t) \, dt \right| \left| \frac{1}{b - a} \int_a^b \varphi'(t) \, dt \right| \leq \frac{1}{2} (\Gamma - \gamma) \left| \frac{1}{b - a} \int_a^b K(x, t) - \frac{1}{b - a} \int_a^b K(x, s) \, ds \right| \, dt. \] (4.6)

We obviously have,

\[
\frac{1}{b - a} \int_a^b K(x, t) \, dt = x - \frac{a + b}{2}
\]

and

\[
\frac{1}{b - a} \int_a^b \varphi'(t) \, dt = \frac{\varphi(b) - \varphi(a)}{b - a}.
\]

Also

\[
I(x) := \frac{1}{b - a} \int_a^b \left| K(x, t) - \left( x - \frac{a + b}{2} \right) \right| \, dt
\]

\[
= \frac{1}{b - a} \left[ \int_a^x \left| t - a + x - \frac{a + b}{2} \right| \, dt + \int_x^b \left| t - b - x + \frac{a + b}{2} \right| \, dt \right]
\]

\[
= \frac{1}{b - a} \left[ \int_a^x \left| t - x + \frac{b - a}{2} \right| \, dt + \int_x^b \left| t - x + \frac{b - a}{2} \right| \, dt \right].
\]
Straight forward substitution of \( u = t - x + \frac{b-a}{2} \) and \( v = t - x - \frac{b-a}{2} \) gives

\[
I(x) = \frac{1}{b-a} \left[ \int_{\frac{b-a}{2}}^{\frac{b+a}{2} - x} |u| \, du + \int_{\frac{b-a}{2}}^{\frac{b+a}{2} - x} |v| \, dv \right]
\]

\[
= \frac{1}{b-a} \int_{-\frac{b-a}{2}}^{\frac{b-a}{2}} |u| \, du = \frac{2}{b-a} \int_{0}^{\frac{b-a}{2}} u \, du = \frac{b-a}{4}.
\]

Substitution of the above into (4.6) produces (4.5). The sharpness of the constant was proved in [3].

5. Application for the Generalised trapezoid inequality

If \( \varphi : [a, b] \to \mathbb{R} \) is an absolutely continuous function on \([a, b]\) so that \( \varphi' \in L_\infty [a, b] \), then the following inequality is known as the generalised trapezoid inequality

\[
\left| (x-a)\varphi(a) + (b-x)\varphi(b) - \int_{a}^{b} \varphi(t) \, dt \right|
\]

\[
\leq \frac{1}{4} (b-a)^2 + \left( x - \frac{a + b}{2} \right)^2 \| \varphi' \|_\infty
\]

(5.1)

for any \( x \in [a, b] \). The constant \( \frac{1}{4} \) is best possible.

A simple proof of this fact is accomplished by using the identity [2]

\[
\int_{a}^{b} \varphi(t) \, dt = (x-a)\varphi(a) + (b-x)\varphi(b) + \int_{a}^{b} (x-t) \varphi'(t) \, dt.
\]

(5.2)

Utilising the inequality (5.1) we may point out the following perturbed version of (5.1).

**Theorem 4.** Assume that \( \varphi : [a, b] \to \mathbb{R} \) is an absolutely continuous function on \([a, b]\) so that \( \varphi' : [a, b] \to \mathbb{R} \) satisfies the condition (4.4). Then we have the inequality

\[
\left| \frac{1}{b-a} \int_{a}^{b} \varphi(t) \, dt - \left[ \left( x - \frac{a}{b-a} \right) \varphi(a) + \left( \frac{b-x}{b-a} \right) \varphi(b) \right] - \left( x - \frac{a + b}{2} \right) [\varphi; a, b] \right|
\]

\[
\leq \frac{1}{8} (b-a)(\Gamma - \gamma)
\]

(5.3)

for any \( x \in [a, b] \), where \([\varphi; a, b]\) is the divided difference. The constant \( \frac{1}{8} \) is sharp.

**Proof.** We apply inequality (5.2) for the choices \( f(t) = (x-t) \), \( g(t) = \varphi'(t) \), \( w(t) = 1 \), \( t \in [a, b] \), to get

\[
\left| \frac{1}{b-a} \int_{a}^{b} (x-t) \varphi'(t) \, dt - \frac{1}{b-a} \int_{a}^{b} (x-t) \, dt \cdot \frac{1}{b-a} \int_{a}^{b} \varphi'(t) \, dt \right|
\]
\[ \leq \frac{1}{2} (\Gamma - \gamma) \frac{1}{b-a} \int_a^b (x-t) - \frac{1}{b-a} \int_a^b (x-s) \, ds \, dt. \quad (5.4) \]

Since
\[ \frac{1}{b-a} \int_a^b (x-t) \, dt = \left( x - \frac{a + b}{2} \right), \]
\[ \frac{1}{b-a} \int_a^b \varphi'(t) \, dt = \frac{\varphi(b) - \varphi(a)}{b-a} = [\varphi; a, b] \]

and
\[ \frac{1}{b-a} \int_a^b \left( x-t \right) \, dt \left( x-t \right) - \frac{1}{b-a} \int_a^b (x-s) \, ds \, dt \]
\[ = \frac{1}{b-a} \int_a^b \left| x-t - x + \frac{a+b}{2} \right| \, dt \]
\[ = \frac{1}{b-a} \int_a^b \left| t - \frac{a+b}{2} \right| \, dt \]
\[ = \frac{b-a}{4}, \]

from (5.4) we deduce the desired inequality (5.3).

The sharpness of the constant may be shown on choosing \( x = \frac{a+b}{2} \) and \( \varphi(t) = \left| t - \frac{a+b}{2} \right|, t \in [a, b]\). We omit the details.

**References**


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