LYAPUNOV-TYPE INEQUALITY FOR THIRD-ORDER
HALF-LINEAR DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we give a proof of a Lyapunov-type inequality for third-order half-linear differential equations. Then some applications, e.g. the distance between consecutive zeros of a solution, are studied with the help of the inequality.

1. Introduction

In this paper, we generalize the Lyapunov inequality for linear third-order differential equation

\[ y''' + p(t) y = 0 \]  

for a class of half-linear differential equations of the third-order. It is well known [6] that if \( p \in C[a, b] \) and \( x(t) \) is nonzero solution of (1) s.t. \( x(a) = x(b) = 0 \) (\( a < b \)) and there exist \( d \in [a, b] \) s.t. \( y''(d) = 0 \), then the following inequality holds:

\[ \int_a^b |p(t)| \, ds > \frac{4}{(b-a)^2}. \]  

Such result has found many practical uses in problems as oscillation theory or eigenvalue problems (spectral properties of differential equations). There are several generalizations in the literature. For higher-order linear differential equations see e.g. [1, 8] and for certain nonlinear higher-order differential equations see [5]. Development of theory of differential equations together with practical problems bring also delayed type of equations. If one is interested along this line see [7], where authors handle with the third-order delay differential equations. We study here a special type of nonlinear differential equations, of which the solution space possesses homogeneity property but lacks for additivity property. Second-order half-linear differential equations have been widely studied in recent years and there is a nice
overview in the monograph [2]. Less literature exists, which deals with such equations of higher-order (especially odd-order differential equations), but one can see for example [4].

2. Main results

We are concerned with the third-order half-linear differential equation

\[
\frac{1}{r_2(t)} \Phi_{\alpha_2} \left[ \left( \frac{1}{r_1(t)} \Phi_{\alpha_1} [y'] \right)' \right]' + q(t) \Phi_{\beta} [y] = 0, \tag{E}
\]

where \( \alpha_1 > 0, \alpha_2 > 0, \quad q \in C([a, b], \mathbb{R}), \) and \( \Phi_{\alpha} [x] := |x|^\alpha - 1 x, \quad \alpha > 0, \) known as signed-power function. Moreover we assume that \( (r_k)^{-1} \in C^{3-k}([a, b], (0, \infty)), \) \( k = 1, 2, \) and to preserve mentioned homogeneity property we also demand that \( \beta = \alpha_1 \alpha_2. \) Equation (E) can be written by means of quasi-derivatives with respect to the coefficients \( r_i \) and functions \( \Phi_{\alpha_i}, \ i = 1, 2. \) We will denote them as follows:

\[
\begin{align*}
D^{(0)} y(t) &= y(t), \\
D^{(i)} y(t) &= \frac{1}{r_i(t)} \Phi_{\alpha_i} \left[ \frac{d}{dt} D^{(i-1)} y(t) \right] \quad i = 1, 2 \\
D^{(3)} y(t) &= \frac{d}{dt} D^{(2)} y(t).
\end{align*}
\]

A solution of (E) is said to be oscillatory (nonoscillatory) if it has (has not) a sequence of zeros converging to infinity. Equation (E) is oscillatory if all its solutions are oscillatory and nonoscillatory otherwise. If a solution of (E) has two consecutive zeros \( a < b, \) then there can two cases occur. Either there exist a \( d \in [a, b] \) s.t. \( \frac{d}{dt} D^{(1)} y(d) = 0 \) or \( \frac{d}{dt} D^{(1)} y(t) \neq 0 \) for \( t \in [a, b]. \) The first case illustrates the following assertion.

**Theorem 2.1.** If \( y(t) \) is a nonzero solution of (E) satisfying \( y(a) = y(b) = 0 \) and there exist a \( d \in [a, b] \) s.t. \( \frac{d}{dt} D^{(1)} y(d) = 0. \) Then

\[
2 \left( \int_a^b |q(t)| dt \right)^{\frac{1}{\beta}} > \min_{c \in [a, b]} h(c),
\]

where \( h(c) = \frac{\int_a^c \frac{1}{a_1} \left( \int_a^c r_2(t) \frac{1}{a_2} dt \right)^{\frac{1}{a_1}} + \frac{1}{b_2} \left( \int_a^b r_2(t) \frac{1}{a_2} dt \right)^{\frac{1}{a_1}}}{\int_a^c \frac{1}{a_1} \left( \int_a^c r_1(t) \frac{1}{a_2} dt \right)^{\frac{1}{a_1}}} \).

**Proof.** We first define functions \( y_k, \ k = 0, 1, 2, \) as follows:

\[
\begin{align*}
y_0 &= D^{(0)} y, \\
y_i &= D^{(i)} y(t) = \frac{1}{r_i(t)} \Phi_{\alpha_i} \left[ \frac{d}{dt} y_{i-1} \right], \quad i = 1, 2
\end{align*}
\]
Equation (E) is then equivalent to the following differential system:

\[
y'_i = r_{i+1}(t) \Phi_{a_{i+1}}^{-1} [y_{i+1}], \quad i = 0, 1
\]

\[
y'_2 = -q(t) \Phi_{\beta} [y_0].
\]

(7)

Condition \(y_0(a) = y_0(b) = 0\) gives us existence of \(c \in (a, b)\) s.t. \(y'_0(c) = 0\) and \(|y_0(c)| = \max_{t \in [a, b]} |y_0(t)|\).

It follows from the latter that \(y_1(c) = 0\). By integrating the first equation of the system (7) from \(a\) to \(c\) we obtain

\[
y_0(c) = \int_a^c r_{1}^{\frac{1}{a_1}}(t) \Phi_{a_1}^{-1} |y_1(t)| \, dt,
\]

which implies

\[
|y_0(c)| \leq \int_a^c r_{1}^{\frac{1}{a_1}}(t) |y_1(t)| \, dt.
\]

(8)

Now let \(t\) be in \([a, c]\). From the fact that \(y_1(c) = 0\) and \(y_1(t) = \int_c^t y'_1(s) \, ds\) we get

\[
|y_1(t)| \leq \int_a^c r_{2}^{\frac{1}{a_2}}(t) |y_2(t)| \, dt.
\]

(9)

Further, from the second condition of the assumptions and relations (6) or (7), we know that \(y_2(d) = 0\), which implies \(y_2(t) = -\int_d^t q(s) \Phi_{\beta} [y_0(s)] \, ds\) for \(t \in [a, c]\). Moreover we have

\[
|y_2(t)| < |y_0(c)|^\beta \int_a^b |q(t)| \, dt.
\]

(10)

Combining inequalities (8)-(10), we get

\[
|y_0(c)| < |y_0(c)| \int_a^c r_{1}^{\frac{1}{a_1}}(t) \, dt \left( \int_a^c r_{2}^{\frac{1}{a_2}}(t) \, dt \right)^{\frac{1}{a_1}} \left( \int_a^b |q(t)| \, dt \right)^{\frac{1}{\beta}}.
\]

which implies

\[
1 < \int_a^c r_{1}^{\frac{1}{a_1}}(t) \, dt \left( \int_a^c r_{2}^{\frac{1}{a_2}}(t) \, dt \right)^{\frac{1}{a_1}} \left( \int_a^b |q(t)| \, dt \right)^{\frac{1}{\beta}}.
\]

(11)

Analogously, we can get

\[
1 < \int_c^b r_{1}^{\frac{1}{a_1}}(t) \, dt \left( \int_c^b r_{2}^{\frac{1}{a_2}}(t) \, dt \right)^{\frac{1}{a_1}} \left( \int_a^b |q(t)| \, dt \right)^{\frac{1}{\beta}}.
\]

(12)

But (11) and (12) together imply (5). Moreover, it is obvious that \(h\) takes its minimum in \((a, b)\), since it is continuous there and \(\lim_{c \to a^+} h(c) = \lim_{c \to b^-} h(c) = \infty\).

In the case that \(\frac{d}{dt} D^{(1)} y(t) \neq 0\) for \(t \in [a, b]\), we consider three consecutive zeros of \(y(t)\). We give only sketch of the proof as it is almost copy of the previous one.
**Theorem 2.2.** If \( y(t) \) is a nonzero solution of (E) satisfying \( y(a) = y(b) = y(0) = 0 \), \( \frac{d}{dt}D(1) y(t) \neq 0 \) for \( t \in [a,b] \) and \( y(t) \neq 0 \) for \( t \in (a,b) \cup (b,d) \). Then inequality (5) holds.

**Proof.** Conditions \( y_0(a) = y_0(d) = y_0(b) = 0 \) give us existence of \( c_1 \in (a,d)\), \( c_2 \in (d,b) \), s.t. \( y_1(c_1) = y_1(c_2) = 0 \) and further application of Rolle’s theorem gives us existence of \( e \in (c_1,c_2) \), s.t. \( y_2(e) = 0 \). Denoting by \( c \in (a,d) \cup (d,b) \) a point where \( \max_{t \in [a,b]} |y_0(t)| = |y_0(c)| \) and using previous procedure it can be proved that inequality (5) holds. Notice, that there can not be problem with continuity of \( h \) on \((a,b)\). \( \square \)

**Remark 2.1.** Put \( r_1(t) = r_2(t) = 1 \). Since \( h \) attains minimum at \( c = \frac{a+b}{2} \), then (5) reduces to
\[
\int_a^b |q(t)| \, dt > \left( \frac{2}{b-a} \right)^{a(\alpha_1+1)}.
\]
Notice in case \( \alpha_1 = \alpha_2 = 1 \) this inequality reduces to (2), which appears in the classical result.

**Remark 2.2.** Put \( r_1(t) = r_2(t) = r(t) \) and \( \alpha_1 = \alpha_2 = \alpha \) then (5) reduces to
\[
\int_a^b |q(t)| \, dt > \left( \frac{2}{\int_a^b r(t) \, dt} \right)^{\alpha(\alpha+1)} = \int_a^b |r(t)| \, dt.
\]

3. Applications

Further we introduce some applications of the previous results for reduced equation (E).

**Theorem 3.1.** Let \( y(t) \) be an oscillatory solution of the reduced \((r_1(t) = r_2(t) = 1)\) equation (E) with increasing sequence of zeros \( \{t_k\}_{k=1}^\infty \) and \( q \in L^\mu([0,\infty), \mathbb{R}) \), \( \mu \in [1,\infty) \). Then distances between consecutive zeros \( \{t_{k+1} - t_k\} \) or \( \{t_{k+2} - t_k\} \) goes to infinity.

**Proof.** In a proof by contradiction we suppose that, in the case that \( \frac{d}{dt}D(1) y(t) = 0 \) for some \( t \in [t_k, t_{k+1}] \) for every large \( k \), is not true that \( \{t_{k+1} - t_k\} \to \infty \). Hence, there exist a subsequence \( \{t_{k_n}\}_{n=1}^\infty \) s.t. \( t_{k_{n+1}} - t_{k_n} < K \) for every \( n \), \( (K > 0) \). Let \( \frac{d}{dt}D(1) y(c_{k_n}) = 0 \) for \( c_{k_n} \in [t_{k_n}, t_{k_{n+1}}] \) and \( \max_{t \in [t_{k_n}, t_{k_{n+1}}]} |y(t)| = |y(d_{k_n})| \) where \( d_{k_n} \in (t_{k_n}, t_{k_{n+1}}) \). Without loss of generality we can assume that \( c_{k_n} < d_{k_n} \). Then it follows that
\[
\int_{t_{k_n}}^{t_{k_{n+1}}} |q(t)| \, dt > \left( \frac{2}{d_{k_n} - t_{k_n}} \right)^{\beta + \alpha_2}.
\]
From integrability of $q$ we have

$$\int_{t_{kn}}^{\infty} |q(t)|^\mu \, dt < \left( \frac{2\beta + \alpha_2}{K^{\beta + \alpha_2 + \frac{1}{\mu}}} \right)^\mu,$$

for sufficiently large $n$ and $\frac{1}{\mu} + \frac{1}{\beta} = 1$. Therefore using Hölder inequality we obtain

$$1 < \left( \frac{d_{kn} - t_{kn}}{2} \right)^{\beta + \alpha_2} \int_{t_{kn}}^{t_{kn+1}} |q(t)| \, dt \leq \frac{(t_{kn+1} - t_{kn})^{\beta + \alpha_2 + \frac{1}{\mu}}}{2^{\beta + \alpha_2}} \left( \int_{t_{kn}}^{t_{kn+1}} |q(t)|^\mu \, dt \right)^{\frac{1}{\mu}} \leq \frac{(t_{kn+1} - t_{kn})^{\beta + \alpha_2 + \frac{1}{\mu}}}{2^{\beta + \alpha_2}} \left( \int_{t_{kn}}^{\infty} |q(t)|^\mu \, dt \right)^{\frac{1}{\mu}} < \frac{K^{\beta + \alpha_2 + \frac{1}{\mu}}}{2^{\beta + \alpha_2}} \left( \frac{2\beta + \alpha_2}{K^{\beta + \alpha_2 + \frac{1}{\mu}}} \right) = 1,$$

a contradiction.

Now suppose that $\frac{d}{dt} D^{(1)} y(t) \neq 0$ for $t \in [t_k, t_{k+1}]$ (for some large $k$). In this case we consider three consecutive zeros $t_k < t_{k+1} < t_{k+2}$. Suppose that there exist subsequence $\{t_{kn}\}_{n=1}^{\infty}$ s.t. $(t_{kn+1} - t_{kn}) < M$ for every $n$, $(M > 0)$ and $\frac{d}{dt} D^{(1)} y(t) \neq 0$ for $t \in [t_{kn}, t_{kn+1}]$. Since $y(t_{kn+2}) = 0$, there exists a $c_k \in (t_{kn+1}, t_{kn+2})$ such that $\frac{d}{dt} D^{(1)} y(c_k) = 0$. Now set $\max_{t \in [t_{kn}, t_{kn+2}]} |y(t)| = |y(d_{kn})|$ where $d_{kn} \in (t_{kn}, t_{kn+1}) \cup (t_{kn+1}, t_{kn+2})$.

If $d_{kn} \in (t_{kn}, t_{kn+1})$, then we can proceed as in the previous part of the proof. If $d_{kn} \in (t_{kn+1}, t_{kn+2})$, then it follows that

$$\int_{t_{kn}}^{t_{kn+2}} |q(t)| \, dt > \left( \frac{2}{d_{kn} - t_{kn}} \right)^{\beta + \alpha_2}.$$

Therefore using Hölder inequality we obtain a contradiction as in the first part of the proof.

The following theorems give us an estimation (upper bound) of the number of zeros of an oscillatory solution of reduced equation (E) on bounded interval $[0, T]$.

**Theorem 3.2.** If $y(t)$ is an oscillatory solution of reduced ($r_1(t) = r_2(t) = 1$) equation (E) with zeros $0 < t_1 < t_2 < \cdots < t_N \leq T$ and $\frac{d}{dt} D^{(1)} y(e_k) = 0$ for some $e_k \in [t_k, t_{k+1}]$, $k = 1, 2, \ldots, N - 1$. Moreover let $\beta + \alpha_2 \geq 1$, then

$$T^{\beta + \alpha_2} \int_0^T |q(t)| \, dt > 2^{\beta + \alpha_2} (N - 1)^{\beta + \alpha_2 + 1}.$$

**Proof.** We know that

$$\int_{t_k}^{t_{k+1}} |q(t)| \, dt > \left( \frac{2}{t_{k+1} - t_k} \right)^{\beta + \alpha_2}, \quad k = 1, \ldots, N - 1.$$

Thus,

$$\int_0^T |q(t)| \, dt \geq \int_{t_1}^{t_N} |q(t)| \, dt \geq \sum_{k=1}^{N-1} \left( \frac{2}{t_{k+1} - t_k} \right)^{\beta + \alpha_2}.$$
Now, using known inequality for the power mean with exponent $\beta + \alpha_2$ and arithmetic mean
and inequality $\frac{1}{n} \sum_{i=1}^{n} A_i \geq \left( \prod_{i=1}^{n} A_i \right)^{1/n} \geq \left( \frac{1}{n} \sum_{i=1}^{n} \frac{1}{A_i} \right)^{-1}$, $A_i > 0$, $1 \leq i \leq n$, we obtain

$$\sum_{k=1}^{N-1} \left( \frac{1}{t_{k+1} - t_k} \right)^{\beta + \alpha_2} \geq (N-1) \left( \frac{1}{N-1} \sum_{k=1}^{N-1} \frac{1}{t_{k+1} - t_k} \right)^{\beta + \alpha_2} \geq (N-1)^{\beta + \alpha_2 + 1} \left( \sum_{k=1}^{N-1} (t_{k+1} - t_k) \right)^{-\beta - \alpha_2}$$

$$= (N-1)^{\beta + \alpha_2 + 1} (t_N - t_1)^{-\beta - \alpha_2} > \frac{(N-1)^{\beta + \alpha_2 + 1}}{T^{\beta + \alpha_2}}.$$ 

This completes the proof. □

**Theorem 3.3.** If $y(t)$ is an oscillatory solution of reduced ($r_1(t) = r_2(t) = 1$) equation (E) with zeros $0 < t_1 < t_2 < \cdots < t_{2N+1} \leq T$ and $\frac{d}{dt} D^{(1)} y(t) \neq 0$ for $t \in [t_{2k-1}, t_{2k}]$, $k = 1, 2, \ldots, N$. Moreover let $\beta + \alpha_2 \geq 1$, then

$$T^{\beta + \alpha_2} \int_0^T |q(t)| \, dt > 2^{\beta + \alpha_2} N^{\beta + \alpha_2 + 1}.$$ 

The proof can be omitted as one can proceed similarly as in Theorem 3.2. We left the case $\beta + \alpha_2 \in (0, 1)$ as an open problem.

**Example 3.1.** For simplicity we consider exponents $\alpha_1$, $\alpha_2$ to be the quotients of two odd numbers. Moreover, let the coefficients of the quasi-derivatives be identically constant. We study the following generalized Euler’s differential equation on $\mathbb{R}^+$

$$\left( \left[ \left( \left[ y' \right]^{\alpha_1} \right]^{\alpha_2} \right)' \right)^{\gamma} + \frac{\gamma}{(t + 1)^{\beta + \alpha_2 + 1}} y^{\beta} = 0. \quad (13)$$

We can proceed using the analogy with the linear Euler differential equation. If we denote as $D = (\alpha_1 + \alpha_2)^2 + 4 \beta (\beta + \alpha_2)$, then the roots of an algebraic (indical) equation corresponding to a solution $t^\lambda$ are

$$\lambda_{\pm} = \frac{\alpha_1 + 2\beta (1 + \alpha_1) + \alpha_2 \pm \sqrt{D}}{2 \alpha_1 (1 + \beta + \alpha_2)}.$$ 

Although it has not been proven yet and it is only a conjecture, see [3], we believe that it can be shown the following. Constants $\gamma_{\pm} = \lambda_{\pm}^{\beta} (\lambda_{\pm} - 1)^{\alpha_2} a_1^{\alpha_1} (\beta (\lambda_{\pm} - 1) - \alpha_2)$ decide whether (13) is oscillatory or not (notice, that in the linear case $\gamma_{\pm} = 2\sqrt{3}/9$). Be more precise, a conjecture states that if $\gamma \in [\gamma_-, \gamma_+]$ then equation (13) is nonoscillatory, otherwise it is oscillatory. Thus we can state at least estimate for $\gamma$. So, from the previous reflections we have

$$\frac{|\gamma| (1 - (T + 1)^{-\beta - \alpha_2}) T^{\beta + \alpha_2}}{\beta + \alpha_2} > 2^{\beta + \alpha_2} (N-1)^{\beta + \alpha_2 + 1}.$$
Example 3.2. Finally we give an application of the obtained result for the following eigenvalue problem

\[ D^{(3)} y \pm \lambda q(t) \Phi_\beta[y] = 0, \]
\[ y(a) = y(c) = y(b) = 0, \quad a < c < b. \]  

(14)

Let the assumptions of Theorem 2.1 be fulfilled, then it follows that

\[ |\lambda| > \frac{H^\beta}{2^\beta \int_a^b |q(t)| \, dt}, \]

where \( H = \min_{c \in [a,b]} h(c) \) and \( h \) is function defined in Theorem 2.1. Especially for the reduced problem \( (r_1(t) = r_2(t) = 1) \) we obtain

\[ |\lambda| > \frac{2^{\beta+a_2}}{(b-a)^{\beta+a_2} \int_a^b |q(t)| \, dt}. \]

References


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