NORMAL FAMILIES OF MEROMORPHIC FUNCTIONS
WITH MULTIPLE POLES

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Abstract. Let $\mathcal{F}$ be a family of meromorphic functions defined in a domain $\mathcal{D}$, and $a$, $b$ be two constants such that $a \neq 0$, $\infty$ and $b \neq \infty$. If for each $f \in \mathcal{F}$, all poles of $f(z)$ are of multiplicity at least 3 in $\mathcal{D}$, and $f'(z) + af^2(z) - b$ has at most 1 zero in $\mathcal{D}$, ignoring multiplicity, then $\mathcal{F}$ is normal in $\mathcal{D}$.

1. Introduction and main results

Let $\mathcal{D}$ be a domain in $\mathbb{C}$, and $\mathcal{F}$ be a family of meromorphic functions defined in the domain $\mathcal{D}$. $\mathcal{F}$ is said to be normal in $\mathcal{D}$, in the sense of Montel, if for every sequence $\{f_n\} \subseteq \mathcal{F}$ contains a subsequence $\{f_{n_j}\}$ such that $f_{n_j}$ converges spherically uniformly on compact subsets of $\mathcal{D}$ (See [1, Definition 3.1.1]).

$\mathcal{F}$ is said to be normal at a point $z_0 \in \mathcal{D}$ if there exists a neighborhood of $z_0$ in which $\mathcal{F}$ is normal. It is well known that $\mathcal{F}$ is normal in a domain $\mathcal{D}$ if and only if it is normal at each of its points (see [1, Theorem 3.3.2]).

Let $f(z)$ and $g(z)$ be two meromorphic functions in a domain $\mathcal{D} \subseteq \mathbb{C}$, and let $a$ be a finite complex number. If $f(z) - a$ and $g(z) - a$ assume the same zeros, then we say that $f$ and $g$ share the value $a$ in $\mathcal{D}$ IM (ignoring multiplicity)(see [2, pp.115-116]).

In 1959, W. K. Hayman [3] proved the following well-known result.

**Theorem A.** Let $f$ be a non-constant meromorphic function in the complex plane $\mathbb{C}$, $n$ be a positive integer and $a$, $b$ be two constants such that $n \geq 5$, $a \neq 0, \infty$ and $b \neq \infty$. If $f' - af^n \neq b$, then $f$ is a constant.

Corresponding to Theorem A there are the following theorems which confirmed a Hayman’s well-known conjecture about normal families in [4, Problem 5.14].

Received March 9, 2013, accepted April 9, 2014.
2010 Mathematics Subject Classification. 30D35, 30D45.
Key words and phrases. Meromorphic functions, normal family, multiplicity.
Theorem B. Let $F$ be a family of meromorphic functions in a complex domain $D$, $n$ be a positive integer and $a$, $b$ be two constants such that $n \geq 3$, $a \neq 0, \infty$ and $b \neq \infty$. If $f' - af^n \neq b$, then $F$ is normal in $D$.

Theorem C. Let $F$ be a family of holomorphic functions in a complex domain $D$, $n$ be a positive integer and $a$, $b$ be two constants such that $n \geq 2$, $a \neq 0, \infty$ and $b \neq \infty$. If $f' - af^n \neq b$, then $F$ is normal in $D$.

Theorem B is due to S. Li [5, $n \geq 5$], X. Li [6, $n \geq 5$], J. Langley [7, $n \geq 5$], X. Pang [8, $n = 4$], H. Chen and M. Fang [9, $n = 3$] and L. Zalcman [10, $n = 3$] independently. Theorem C is due to D. Drasin [11, $n \geq 3$] and Y. Ye [12, $n = 2$].

Generally speaking, for $n = 2$, the result of Theorem B is not valid. For examples we refer the reader to [13]. However, in [13], M. Fang and W. Yuan had

Theorem D. Let $F$ be a family of meromorphic functions in a complex domain $D$, and $a \neq 0, \infty$, $b \neq \infty$. If, for every $f \in F$, $f' - af^2 \neq b$ and the poles of $f$ are of multiplicity at least 3, then $F$ is normal in $D$.

It is natural to ask whether the condition in Theorem D that $f' - af^2 \neq b$ can be relaxed. In this paper we investigate this problem and prove the following result.

Theorem 1. Let $F$ be a family of meromorphic functions defined in a domain $D$, and $a$, $b$ be two constants such that $a \neq 0, \infty$ and $b \neq \infty$. If for each $f \in F$, all poles of $f(z)$ are of multiplicity at least 3 in $D$, and $f'(z) + af^2(z) - b$ has at most 1 zero in $D$, ignoring multiplicity, then $F$ is normal in $D$.

By the idea of shared values, Q. Zhang [14, Theorem 2] proved.

Theorem E. Let $F$ be a family of holomorphic functions defined in a domain $D$ and $a$, $b$ be two constants such that $a \neq 0, \infty$ and $b \neq \infty$. If for every pair of functions $f$, $g \in F$, $f' + af^2$ and $g' + ag^2$ share the value $b$ in $D$, then $F$ is normal in $D$.

It is natural to ask whether Theorems D can be improved by the idea of shared values. In this paper, we study the problem and obtain the following theorem.

Theorem 2. Let $F$ be a family of meromorphic functions defined in a domain $D$. Let $a \neq 0$, $b$ be two finite complex number. If for each $f \in F$, all poles of $f(z)$ are of multiplicity at least 3 in $D$, and if $f' + af^2$ and $g' + ag^2$ share the value $b$ in $D$ for every pair of functions $f$, $g \in F$, then $F$ is normal in $D$.

Example 1. Let $D = \{z : |z| < 1\}$. Let $F = \{f_m\}$ where $f_m := \frac{1}{mz^2}$, then $f'_m + f^2 = \frac{1 - kmz^{k-1}}{m^2z^{2k}}$, so $f'_m + f^2$ has exactly one zero for $k = 2$ and $f'_m + f^2$ has two distinct zeros for $k = 3$. However, it is easily obtained that $F$ is not normal at the point $z = 0$. 
This shows that the condition that all poles of \( f(z) \) are of multiplicity at least 3 and \( f'(z) + af^2(z) \) has at most 1 zero in Theorems 1 is sharp.

2. Some lemmas

To prove our results, we need some preliminary results.

**Lemma 1** ([15], [16] Lemma 1 (Zalcman’s Lemma)). Let \( \mathcal{F} \) be a family of functions meromorphic on a domain \( \mathcal{D} \), all of whose poles have multiplicity at least \( j \); Then if \( \mathcal{F} \) is not normal at \( z_0 \in \mathcal{D} \), there exist, for each \( j < \alpha < 1 \),

(a) points \( z_n, z_n \to z_0 \);

(b) functions \( f_n \in \mathcal{F} \); and

(c) positive numbers \( \rho_n \to 0^+ \)

such that \( \rho_n f_n(z_n + \rho_n \xi) = g_n(\xi) \to g(\xi) \) locally uniformly with respect to the spherical metric, where \( g(\xi) \) is a nonconstant meromorphic function on \( \mathbb{C} \). Moreover, the order of \( g(\xi) \) is less than 2 and the poles of \( g(\xi) \) are of multiplicity at least \( j \).

Here, as usual, \( g^\#(\xi) = \frac{|g'(\xi)|}{1 + |g(\xi)|} \) is the spherical derivative.

**Lemma 2** ([17], Theorem 2). Let \( f(z) \) be a transcendental meromorphic function in \( \mathbb{C} \). If all zeros of \( f(z) \) have multiplicity at least 3, for any positive integer \( k \), then \( f^{(k)}(z) \) assumes every non-zero finite value infinitely often.

**Lemma 3** ([17]). Let \( f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 + q(z)/p(z) \), where \( a_0, a_1, \ldots, a_n \) are constants with \( a_n \neq 0 \), and \( q \) and \( p \) are two co-prime polynomials, neither of which vanishes identically, with \( \deg q < \deg p \); and let \( k \) be a positive integer and \( b \) a nonzero complex number. If \( f^{(k)} \neq b \), and the zeros of \( f \) all have multiplicity at least \( k + 1 \), then

\[
 f(z) = \frac{b(z - d)^{k+1}}{k!(z - c)},
\]

where \( c \) and \( d \) are distinct complex numbers.

**Lemma 4** ([19], Lemma 4). Let \( f \) be a nonconstant rational function and \( k, m \) be positive integers. If \( f \) has no zeros in \( \mathbb{C} \) while all poles of \( f \) have multiplicity at least \( m \), then \( f^{(k)} - 1 \) has at least \( k + m \) distinct zeros in \( \mathbb{C} \).

**Lemma 5.** Let \( f \) be a non-constant meromorphic function, and \( a \neq 0 \) be a finite complex number. Let all poles of \( f(z) \) have multiplicity at least 3, then \( f' + af^2 \) has at least two distinct zeros.
Proof. Set $f = \frac{1}{a \varphi}$, then all zeros of $\varphi$ have multiplicity at least 3 and $f' + af^2 = -\frac{\varphi'-1}{a \varphi^2}$.

Case 1. If $\frac{\varphi'-1}{a \varphi^2}$ has only one zero $z_0$, then $z_0$ is a multiple pole of $\varphi$, or else a zero of $\varphi' - 1$. If $z_0$ is a multiple pole of $\varphi$, since $\frac{\varphi'-1}{a \varphi^2}$ has only one zero, then $\varphi' - 1 \neq 0$. By Lemma 2 and Lemma 3, this is a contradiction.

So $\varphi$ has no multiple pole and $\varphi' - 1$ has just one unique zero $z_0$. By the Lemma 2, $\varphi$ is not any transcendental function.

Case 1.1. $\varphi$ is a non-constant polynomial.

Since $\varphi' - 1$ has only unique one zero $z_0$, set

$$\varphi' - 1 = A(z - z_0)^l$$

where $A$ is non-zero constant, $l$ is a positive integer. Then

$$\varphi'' = Al(z - z_0)^{l-1}$$

Note that $\varphi$ is a non-constant polynomial and all zeros of $\varphi$ have multiplicity at least 3, so $l \geq 3 - 1 = 2$. But $\varphi''$ has only one zero $z_0$, so $\varphi'$ has only the same zero $z_0$ too. Hence $\varphi'(z_0) = 0$, which contradicts $\varphi'(z_0) = 1 \neq 0$. Therefore $\varphi$ is a rational function which is not a polynomial.

Case 1.2. If $\varphi$ is rational but not a polynomial and has at least one zero.

Under the conditions of Lemma 5 on the rational function $\varphi$,

$$\varphi(z) = \frac{A(z + \alpha_1)^{m_1}(z + \alpha_2)^{m_2} \cdots (z + \alpha_s)^{m_s}}{(z + \beta_1)(z + \beta_2) \cdots (z + \beta_t)}$$

(2.1)

where $A$ is a non-zero constant, $m_i \geq 3$ ($i = 1, 2, \ldots, s$), $\alpha_i (i = 1, 2, \ldots, s)$, and $\beta_j (j = 1, 2, \ldots, t)$ are distinct complex numbers.

For simplicity, we denote

$$m_1 + m_2 + \cdots + m_s = M \geq 3s.$$ 

(2.2)

Let

$$g(z) = z - \varphi(z)$$

We use $\text{deg}(g)$ to denote the degree of a polynomial. Next we use some results from complex dynamics, cf. [18, Chapter 3, 19 Lemma 5]. Since $\varphi' - 1$ has only unique one zero, so that $g$ has only unique one critical point. Since $m_i \geq 3$, we also see that every zero point $\alpha_i = -z_i$ of $-\varphi$ is a fixed point of $g$ of multiplicity $m_i$. Moreover, since near $z_i$

$$g(z) = z + c_i(z - z_i)^{m_i}[1 + o(1)]$$
there are \( m_1 - 1 \) parabolic basins associated with the fixed point \( z_i \).

Case 1.2.1. If \( \infty \) is not a fixed point of \( g \), each of these parabolic basins, with at most one exception, contains a critical point of \( g \) which is not a pole of \( g \), so the function \( g(z) = z - \varphi(z) \) has \( 1 \geq M - s - 1 \) parabolic basins associated with the zero points of \( \varphi \), we deduce that \( s = 1 \) and \( m_1 = 3 \). We note that \( \infty \) is not a fixed point of \( g \), so

\[
\varphi(z) = \frac{(z + \alpha_1)^3}{(z + \beta_1)(z + \beta_2)} \tag{2.3}
\]

In order to simplify the calculation, set \( Z = z + \beta_1 \). From (2.3), we have

\[
\varphi(Z) = \frac{(Z + \alpha)^3}{Z(Z + \beta)}
\]

where \( \alpha = \alpha_1 - \beta_1 \neq 0 \) and \( \beta = \beta_2 - \beta_1 \neq 0 \).

We have

\[
\varphi'(Z) = \frac{(Z + \alpha)^2(Z^2 + 2(\beta - \alpha)Z - \alpha\beta)}{Z^2(Z + \beta)^2} \tag{2.4}
\]

Since \( \varphi'(z) - 1 \) has exactly one zero \( z_0 \), from (2.3) we obtain

\[
\varphi'(Z) = 1 + \frac{B(Z - Z_0)^l}{Z^2(Z + \beta)^2} = \frac{Z^2(Z + \beta)^2 + B(Z - Z_0)^l}{Z^2(Z + \beta)^2} \tag{2.5}
\]

where \( B \) is a non-zero constant and \( Z_0 = z_0 - \beta_1 \) and \( l \) is a positive integer. From (2.4) and (2.5), we have \( l \leq 3 \).

From (2.4) we have

\[
\varphi''(Z) = 2(Z + \alpha)[(\beta^2 + 3\alpha^2 - 3\alpha\beta)Z^2 + (3\alpha^2 \beta - \alpha\beta^2)Z + \alpha^2 \beta^2]}{Z^3(Z + \beta)^3} \tag{2.6}
\]

From (2.5), we have

\[
\varphi''(Z) = B \frac{(Z - Z_0)^{l-1}[(l - 4)Z^2 + (l\beta + 4Z_0 - 2\beta)Z + 2\beta Z_0]}{Z^3(Z + \beta)^3} \tag{2.7}
\]

If \( l = 1 \), from (2.4) and (2.5), we have

\[
(Z + \alpha)^2(Z^2 + 2(\beta - \alpha)Z - \alpha\beta) = Z^2(Z + \beta)^2 + B(Z - Z_0)
\]

By comparing coefficients of both sides, we have

\[
\begin{align*}
3\alpha(\beta - \alpha) &= \beta^2 \\
-2\alpha^3 &= B \\
-\alpha^3 \beta &= BZ_0
\end{align*}
\]
i.e.
\[
\begin{cases}
3\alpha\beta - 3\alpha^2 - \beta^2 = 0 \\
B = -2\alpha^3 \\
Z_0 = \beta/2
\end{cases}
\]

By (2.6), we have \(\varphi''(-\alpha) = 0\). From (2.7), we get
\[
(1 - 4)(-\alpha)^2 + (\beta + 4\frac{\beta}{2} - 2\beta)(-\alpha) + 2\beta\frac{\beta}{2} = 0
\]
i.e.
\[\alpha\beta + 3\alpha^2 - \beta^2 = 0\]
Combining this with \(3\alpha\beta - 3\alpha^2 - \beta^2 = 0\) yields
\[\alpha = \beta = 0\]
which is a contradiction.

If \(l = 2\), from (2.4) and (2.5), we have
\[
(Z + \alpha)^2(Z^2 + 2(\beta - \alpha)Z - \alpha\beta) = Z^2(Z + \beta)^2 + B(Z - Z_0)^2
\]
By comparing coefficients of both sides, we have
\[
\begin{cases}
3\alpha(\beta - \alpha) = \beta^2 + B \\
-2\alpha^3 = -2BZ_0 \\
-\alpha^2\beta = BZ_0^2
\end{cases}
\]
i.e.
\[
\begin{cases}
3\alpha\beta^2 - 3\alpha^2\beta - \beta^3 + \alpha^3 = 0 \\
B = -\frac{\alpha^3}{\beta} \\
Z_0 = -\beta
\end{cases}
\]
By (2.6), we have \(\varphi''(-\alpha) = 0\). From (2.7), we get
\[
\varphi''(-\alpha) = -\frac{\alpha^3}{\beta} \frac{(\beta - \alpha)((2 - 4)(-\alpha)^2 + (2\beta - 4\beta - 2\beta)(-\alpha) - 2\beta^2)}{(-\alpha)^3(\beta - \alpha)^3} = -\frac{2}{\beta} \neq 0
\]
which is a contradiction. If \(l = 3\), then \(\text{deg}((Z + \alpha)((\beta^2 + 3\alpha^2 - 3\alpha\beta)Z^2 + (3\alpha^2\beta - \alpha\beta^2)Z + \alpha^2\beta^2)) < \text{deg}((Z - Z_0)^{l-1}((1 - 4)Z^2 + (1\beta + 4Z_0 - 2\beta)Z + 2\beta Z_0))\), which is a contradiction.

Case 1.2.2. If \(\infty\) is a fixed point of \(g\), each parabolic basin contains a critical point of \(g\) which is not a pole of \(g\). Thus each parabolic basins contains a zero point of \(g'\), and hence \(2 \leq M - s \leq 1\), which is a contradiction.
Case 1.3. If \( \phi \) is rational but not a polynomial and has no zero. By Lemma 4, we have \( \phi' - 1 \) has at least two distinct zeros, which contradicts the fact that \( \phi' - 1 \) has just one unique zero \( z_0 \).

Case 2. If \( \frac{\phi'-1}{\alpha \phi^2} \neq 0 \).

Case 2.1. Since all zeros of \( \phi(z) \) have the multiple at least 3 and \( \phi \) is a non-constant function, it easily obtained that \( \phi \) is not a polynomial.

Case 2.2. If \( \phi \) is rational but not a polynomial. If there exist a point \( z_0 \) such that \( \phi'(z_0) = 1 \), we note that \( \frac{\phi'-1}{\alpha \phi^2} \neq 0 \), so \( \phi(z_0) = 0 \). Since all zeros of \( \phi(z) \) have the multiple at least 3, we have \( \phi'(z_0) = 0 \), we get a contradiction.

If \( \phi' \neq 1 \), by Lemma 3, we have

\[
\phi = \frac{(z - d)^2}{(z - c)},
\]

where \( c \) and \( d \) are distinct complex numbers. This contradicts the fact that all zeros of \( \phi(z) \) have the multiple at least 3.

The proof is complete.

3. Proofs of Theorems

Proof of Theorem 1. Suppose that \( \mathcal{F} \) is not normal in \( \mathcal{D} \). Then there exists at least one point \( z_0 \) such that \( \mathcal{F} \) is not normal at the point \( z_0 \in \mathcal{D} \). Without loss of generality we assume that \( z_0 = 0 \). By Zalcman’s lemma, there exist:

(a) points \( z_n, z_n \to z_0 \);
(b) functions \( f_n \in \mathcal{F} \); and
(c) positive numbers \( \rho_n \to 0^+ \)

such that

\[
g_j(\xi) = \rho_j f_j(z_j + \rho_j \xi) \to g(\xi)
\]

spherically uniformly on compact subsets of \( \mathbb{C} \), where \( g(\xi) \) is a non-constant meromorphic function in \( \mathbb{C} \) and the poles of \( g(\xi) \) are of multiplicity at least 3.

From (3.1), we get

\[
g_j'(\xi) = \rho_j^2 f_j'(z_j + \rho_j \xi) \to g'(\xi)
\]

also locally uniformly with respect to the spherical metric.

Thus

\[
g_j'(\xi) + a g_{nj}(\xi) - \rho_j^2 b = \rho_j^2 (f_j'(z_j + \rho_j \xi) + a f_j^2(z_j + \rho_j \xi) - b) \to g'(\xi) + a g^2(\xi)
\]
also locally uniformly with respect to the spherical metric.

We claim that \( g'(\xi) + ag^2 \) has at most 1 zero ignoring multiplicity.

If \( g'(\xi) + ag^2 \equiv 0 \), then \( g(\xi) \equiv \frac{1}{az + c} \), this contradicts the fact that the poles of \( g(\xi) \) are of multiplicity at least 3. So \( g'(\xi) + ag^2 \neq 0 \).

Suppose that \( g'(\xi) + ag^2 \) has two distinct zeros \( \xi_0 \) and \( \xi_0^* \) and choose \( \delta(> 0) \) small enough such that \( D(\xi_0, \delta) \cap D(\xi_0^*, \delta) = \emptyset \) where \( D(\xi_0, \delta) = \{ \xi | |\xi - \xi_0| < \delta \} \) and \( D(\xi_0^*, \delta) = \{ \xi | |\xi - \xi_0^*| < \delta \} \).

From (3.3), by Hurwitz's theorem, there exist points \( \xi_j \in D(\xi_0, \delta), \xi_j^* \in D(\xi_0^*, \delta) \) such that for sufficiently large \( j \)

\[
\begin{align*}
f_j'(z_j + \rho_j \xi_j) + a f_j^2(z_j + \rho_j \xi_j) - b &= 0. \\
f_j'(z_j + \rho_j \xi_j^*) + a f_j^2(z_j + \rho_j \xi_j^*) - b &= 0.
\end{align*}
\]

Since \( z_j \to 0 \) and \( \rho_j \to 0^+ \), we have \( z_j + \rho_j \xi_j \in D(\xi_0, \delta) \) and \( z_j + \rho_j \xi_j^* \in D(\xi_0^*, \delta) \) for sufficiently large \( j \), so \( f_j'(z) + af_j^2 - b \) has two distinct zeros, which contradicts the fact that \( f_j'(z) + af_j^2 - b \) has at most 1 zero.

However, by Lemma 5, there does not exist non-constant meromorphic functions satisfying above properties such that our claim holds. This contradiction shows that \( \mathcal{F} \) is normal in \( \mathcal{D} \) and hence Theorem 1 is proved.

**Proof of Theorem 2.** Suppose that \( \mathcal{F} \) is not normal in \( \mathcal{D} \). Then there exists at least one point \( z_0 \) such that \( \mathcal{F} \) is not normal at the point \( z_0 \in \mathcal{D} \). Without loss of generality we assume that \( z_0 = 0 \). By Zalcman's lemma, there exist:

(a) points \( z_n, z_n \to z_0 \); 
(b) functions \( f_n \in \mathcal{F} \); and
(c) positive numbers \( \rho_n \to 0^+ \)

such that

\[
g_j(\xi) = \rho_j f_j(z_j + \rho_j \xi) \to g(\xi)
\]

spherically uniformly on compact subsets of \( \mathbb{C} \), where \( g(\xi) \) is a non-constant meromorphic function in \( \mathbb{C} \) and the poles of \( g(\xi) \) are of multiplicity at least 3.

Proceeding as in the proof of Theorem 1, we also have (3.3).

If \( g'(\xi) + ag^2 \equiv 0 \), then \( g(\xi) \equiv \frac{1}{az + c} \), this contradicts the fact that the poles of \( g(\xi) \) are of multiplicity at least 3. So \( g'(\xi) + ag^2 \neq 0 \).

Since \( g \) is a non-constant meromorphic function, by Lemma 5, we deduce that \( g'(\xi) + ag^2 \) has at least two distinct zeros.

We claim that \( g'(\xi) + ag^2(\xi) \) has just a unique zero.
Suppose that there exist two distinct zeros $\xi_0$ and $\xi_0^*$ and choose $\delta(>0)$ small enough such that $D(\xi_0,\delta) \cap D(\xi_0^*,\delta) = \emptyset$ where $D(\xi_0,\delta) = \{\xi | |\xi - \xi_0| < \delta\}$ and $D(\xi_0^*,\delta) = \{\xi | |\xi - \xi_0^*| < \delta\}$.

From (3.4), by Hurwitz’s theorem, there exist points $\xi_j \in D(\xi_0,\delta)$, $\xi_j^* \in D(\xi_0^*,\delta)$ such that for sufficiently large $j$

$$f'_j(z_j + \rho_j \xi_j) + af_j^2(z_j + \rho_j \xi_j) - b = 0.$$  

$$f'_j(z_j + \rho_j \xi_j^*) + af_j^2(z_j + \rho_j \xi_j^*) - b = 0.$$  

By the assumption that $f' + af^2$ and $g' + ag^2$ share $b$ in $D$ for each pair $f$ and $g$ in $F$, we know that for any integer $m$

$$f'_m(z_j + \rho_j \xi_j) + af_m^2(z_j + \rho_j \xi_j) - b = 0.$$  

$$f'_m(z_j + \rho_j \xi_j^*) + af_m^2(z_j + \rho_j \xi_j^*) - b = 0.$$  

We fix $m$ and note that $z_j + \rho_j \xi_j \to 0$, $z_j + \rho_j \xi_j^* \to 0$ if $j \to \infty$. From this we deduce

$$f'_m(0) + af_m^2(0) - b = 0.$$  

Since the zeros of $f'_m(z) + af_m^2(z) - b$ have no accumulation point, for sufficiently large $j$, we have

$$z_j + \rho_j \xi_j = 0, \quad z_j + \rho_j \xi_j^* = 0.$$  

Hence

$$\xi_j = -\frac{z_j}{\rho j}, \quad \xi_j^* = -\frac{z_j}{\rho j}.$$  

This contradicts the fact that $\xi_j \in D(\xi_0,\delta)$, $\xi_j^* \in D(\xi_0^*,\delta)$ and $D(\xi_0,\delta) \cap D(\xi_0^*,\delta) = \emptyset$. So $g'(\xi) + ag^2(\xi)$ has just a unique zero. This contradicts the fact that $g'(\xi) + ag^2(\xi)$ has at least two distinct zeros.

This proves the theorem.

**Acknowledgement**

This work was supported by Foundation of Shaanxi Railway Institute (Grant No. 2013–12). The authors thank the reviewer(s) for reading the manuscript very carefully and making a number of valuable and kind comments which improved the presentation.


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