ON NEW INEQUALITIES OF SIMPSON’S TYPE FOR QUASI-CONVEX FUNCTIONS WITH APPLICATIONS

ERHAN SET, M. EMIN ÖZDEMIR AND MEHMET ZEKI SARıKAYA

Abstract. In this paper, we introduce some inequalities of Simpson’s type based on quasi-convexity. Some applications for special means of real numbers are also given.

1. Introduction

The following inequality is well known in the literature as Simpson’s inequality.

Theorem 1. Let \( f : [a, b] \to \mathbb{R} \) be a four times continuously differentiable mapping on \((a, b)\) and \( \|f^{(4)}\|_\infty = \sup_{x \in (a, b)} |f^{(4)}(x)| < \infty \). Then, the following inequality holds:

\[
\left| \frac{1}{3} \left( \frac{f(a) + f(b)}{2} + 2f \left( \frac{a + b}{2} \right) \right) - \frac{1}{b - a} \int_a^b f(x)\,dx \right| \leq \frac{1}{2880} \|f^{(4)}\|_\infty (b - a)^4.
\]

For recent refinements, counterparts, generalizations and new Simpson’s type inequalities, see ([1],[2],[4]).

In [2], Dragomir, Agarwal and Cerone proved the following some recent developments on Simpson’s inequality for which the remainder is expressed in terms of lower derivatives than the fourth.

Theorem 2. Suppose \( f : [a, b] \to \mathbb{R} \) is a differentiable mapping whose derivative is continuous on \((a, b)\) and \( f' \in L[a, b] \). Then the following inequality

\[
\left| \frac{1}{3} \left( \frac{f(a) + f(b)}{2} + 2f \left( \frac{a + b}{2} \right) \right) - \frac{1}{b - a} \int_a^b f(x)\,dx \right| \leq \frac{b - a}{3} \|f'\|_1 \tag{1.1}
\]

holds, where \( \|f'\|_1 = \int_a^b |f'(x)|\,dx \).

The bound of (1.1) for L-Lipschitzian mapping was given in [2] by \( \frac{5}{36} L(b - a) \).

Also, the following inequality was obtained in [2].

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Theorem 3. Suppose \( f : [a, b] \rightarrow \mathbb{R} \) is an absolutely continuous mapping on \([a, b]\) whose derivative belongs to \( L^p[a, b] \). Then the following inequality holds,

\[
\left| \frac{1}{3} \left[ \frac{f(a) + f(b)}{2} + 2f \left( \frac{a + b}{2} \right) \right] - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{1}{6} \left[ \frac{2q+1}{3(q+1)} \right]^{\frac{1}{q}} (b-a)^\frac{q}{3q+1} \|f'\|_p
\]

where \( \frac{1}{p} + \frac{1}{q} = 1 \).

We recall that the notion of quasi-convex functions generalized the notion of convex functions. More precisely, a function \( f : [a, b] \rightarrow \mathbb{R} \) is said to be quasi-convex on \([a, b]\) if

\[
f \left( tx + (1-t)y \right) \leq \max \{f(x), f(y)\}, \quad \forall x, y \in [a, b].
\]

Any convex function is a quasi-convex function but the reverse are not true. Because there exist quasi-convex functions which are not convex, (see for example [3])

The main aim of this paper is to establish new Simpson’s type inequalities for the class of functions whose derivatives in absolute value at certain powers are quasi-convex functions.

2. Simpson’s Type Inequalities for Quasi-Convex

In order to prove our main theorems, we need the following lemma, see [1].

Lemma 1. Let \( f : I \subset \mathbb{R} \rightarrow \mathbb{R} \) be an absolutely continuous mapping on \( I^\circ \) where \( a, b \in I \) with \( a < b \). Then the following equality holds:

\[
\left| \frac{1}{6} \left[ f(a) + 4f \left( \frac{a + b}{2} \right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq (b-a) \int_0^1 p(t) f'(tb + (1-t)a) \, dt
\]

where

\[
p(t) = \begin{cases} 
  t - \frac{1}{6}, & t \in \left[0, \frac{1}{2}\right), \\
  t - \frac{5}{6}, & t \in \left[\frac{1}{2}, 1\right].
\end{cases}
\]

A simple proof of this equality can be also done by integrating by parts in the right hand side. The details are left to the interested reader.

The next theorem gives a new result of the Simpson inequality for quasi-convex functions.
Theorem 4. Let \( f : I \subseteq \mathbb{R} \to \mathbb{R} \) be a differentiable mapping on \( I^o \), such that \( f' \in L[a,b] \), where \( a, b \in I \) with \( a < b \). If \( |f'| \) is quasi-convex on \( [a,b] \), then the following inequality holds:

\[
\left| \frac{1}{6} \left[ f(a) + 4f \left( \frac{a+b}{2} \right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{5(b-a)}{36} \max \{|f'(a)|, |f'(b)|\}. \tag{2.2}
\]

Proof. From Lemma 1, and since \( |f'| \) is quasi-convex, we have

\[
\left| \frac{1}{6} \left[ f(a) + 4f \left( \frac{a+b}{2} \right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) \, dx \right| = (b-a) \int_0^1 p(t) f' \left( tb + (1-t) a \right) \, dt \leq (b-a) \int_0^{1/2} \left| t - \frac{1}{6} \right| |f'(tb + (1-t) a)| \, dt + (b-a) \int_{1/2}^1 \left| t - \frac{5}{6} \right| |f'(tb + (1-t) a)| \, dt = (b-a) \int_0^{1/6} \left( \frac{1}{6} - t \right) \max \{|f'(a)|, |f'(b)|\} \, dt + (b-a) \int_{1/6}^{1/2} \left( t - \frac{1}{6} \right) \max \{|f'(a)|, |f'(b)|\} \, dt + (b-a) \int_{1/2}^{5/6} \left( \frac{5}{6} - t \right) \max \{|f'(a)|, |f'(b)|\} \, dt + (b-a) \int_{5/6}^1 \left( t - \frac{5}{6} \right) \max \{|f'(a)|, |f'(b)|\} \, dt = \frac{5(b-a)}{36} \max \{|f'(a)|, |f'(b)|\}
\]

which completes the proof. \( \square \)

Corollary 1. In Theorem 4, if \( f(a) = f \left( \frac{a+b}{2} \right) = f(b) \), then we have

\[
\left| \frac{1}{b-a} \int_a^b f(x) \, dx - f \left( \frac{a+b}{2} \right) \right| \leq \frac{5(b-a)}{36} \max \{|f'(a)|, |f'(b)|\}.
\]

A similar results is embodied in the following theorem.

Theorem 5. Let \( f : I \subseteq \mathbb{R} \to \mathbb{R} \) be a differentiable mapping on \( I^o \), such that \( f' \in L[a,b] \), where \( a, b \in I \) with \( a < b \). If \( |f'|^q \) is quasi-convex on \( [a,b] \) and \( q > 1 \), then the following inequality
holds:

\[
\left| \frac{1}{6} \left[ f(a) + 4f \left( \frac{a+b}{2} \right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) \, dx \right|
\leq \frac{1}{6} (b-a) \left( \frac{1+2^{p+1}}{3(p+1)} \right)^{\frac{1}{p}} \left\{ \max \{ |f'(a)|^q, |f'(b)|^q \} \right\}^{\frac{1}{q}}
\]  

(2.3)

where \( \frac{1}{p} + \frac{1}{q} = 1 \).

**Proof.** From Lemma 1, using the well known Hölder integral inequality, we have

\[
\left| \frac{1}{6} \left[ f(a) + 4f \left( \frac{a+b}{2} \right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) \, dx \right|
= (b-a) \left| \int_0^1 p(t) f'(tb + (1-t) a) \, dt \right|
\leq (b-a) \int_0^{1/2} \left| t - \frac{1}{6} \right| \left| f'(tb + (1-t) a) \right| \, dt
+ (b-a) \int_{1/2}^1 \left| t - \frac{5}{6} \right| \left| f'(tb + (1-t) a) \right| \, dt
\leq (b-a) \left( \int_0^{1/2} \left| t - \frac{1}{6} \right|^p \, dt \right)^{\frac{1}{p}} \left( \int_0^{1/2} \left| f'(tb + (1-t) a) \right|^q \, dt \right)^{\frac{1}{q}}
+ (b-a) \left( \int_{1/2}^1 \left| t - \frac{5}{6} \right|^p \, dt \right)^{\frac{1}{p}} \left( \int_{1/2}^1 \left| f'(tb + (1-t) a) \right|^q \, dt \right)^{\frac{1}{q}}
= (b-a) \left( \int_0^{1/6} \left( \frac{1}{6} - t \right)^p \, dt + \int_{1/6}^{1/2} \left( t - \frac{1}{6} \right)^p \, dt \right)^{\frac{1}{p}}
\times \left( \int_0^{1/2} \left| f'(tb + (1-t) a) \right|^q \, dt \right)^{\frac{1}{q}}
+ (b-a) \left( \int_{1/2}^{5/6} \left( \frac{5}{6} - t \right)^p \, dt + \int_{5/6}^1 \left( t - \frac{5}{6} \right)^p \, dt \right)^{\frac{1}{p}}
\times \left( \int_{1/2}^1 \left| f'(tb + (1-t) a) \right|^q \, dt \right)^{\frac{1}{q}}
\]

Since \( |f'|^q \) is quasi-convex, we have

\[
\left| f'(tb + (1-t) a) \right|^q \leq \max \{ |f'(b)|^q, |f'(a)|^q \}
\]

hence

\[
\left| \frac{1}{6} \left[ f(a) + 4f \left( \frac{a+b}{2} \right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) \, dx \right|
\leq 2(b-a) \left( \frac{1+2^{p+1}}{6^{p+1}(p+1)} \right)^{\frac{1}{p}} \left( \frac{\max \{ |f'(a)|^q, |f'(b)|^q \} \right)^{\frac{1}{q}}
\]
\[ \leq 2^\frac{1}{p} (b-a) \left( \frac{1+2^{p+1}}{6^{p+1}(p+1)} \right)^\frac{1}{p} \left( \max \{|f'(a)|^q, |f'(b)|^q \} \right)^\frac{1}{q} \]

where we use the fact that

\[ \int_0^{1/6} \left( \frac{6}{1-t} \right)^p dt + \int_{1/6}^{1/2} \left( 1 - \frac{t}{1} \right)^p dt = \int_{1/6}^{5/6} \left( \frac{6}{1-t} \right)^p dt + \int_{5/6}^1 \left( 1 - \frac{5}{6} \right)^p dt = \frac{1+2^{p+1}}{6^{p+1}(p+1)} \]

which completes the proof. \( \square \)

**Corollary 2.** In Theorem 5, if \( f(a) = f\left(\frac{a+b}{2}\right) = f(b) \), then we have

\[ \left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| \leq \frac{1}{6} (b-a) \left( \frac{1+2^{p+1}}{3(p+1)} \right)^\frac{1}{p} \left( \max \{|f'(a)|^q, |f'(b)|^q \} \right)^\frac{1}{q}. \]

**Corollary 3.** In Theorem 5, if \( f(a) = f\left(\frac{a+b}{2}\right) = f(b) \) and \( p = 2 \), then we have

\[ \left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| \leq \frac{b-a}{6} \max \{|f'(a)|^2, |f'(b)|^2 \}. \]

A more general inequality is given using Lemma 1, as follows.

**Theorem 6.** Let \( f : I \subset \mathbb{R} \to \mathbb{R} \) be a differentiable mapping on \( I^p \), such that \( f' \in L[a,b] \), where \( a,b \in I \) with \( a < b \). If \( |f'|^q \) is quasi-convex on \( |a,b| \) and \( q \geq 1 \), then the following inequality holds:

\[ \left| \frac{1}{6} \left[ f(a) + 4 f \left( \frac{a+b}{2} \right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{5(b-a)}{36} \left( \max \{|f'(a)|^q, |f'(b)|^q \} \right)^\frac{1}{q}. \]  

**Proof.** Suppose that \( q \geq 1 \). From Lemma 1 and using the well known power mean inequality, we have

\[ \left| \frac{1}{6} \left[ f(a) + 4 f \left( \frac{a+b}{2} \right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \]

\[ = (b-a) \left| \int_0^1 p(t) f'(tb+(1-t)a) dt \right| \]

\[ \leq (b-a) \int_0^{1/2} \left| t - \frac{1}{6} \right| f'(tb+(1-t)a) dt \]

\[ + (b-a) \int_{1/2}^1 \left| t - \frac{5}{6} \right| f'(tb+(1-t)a) dt \]

\[ \leq (b-a) \left( \int_0^{1/2} \left| t - \frac{1}{6} \right| dt \right)^{-\frac{1}{q}} \left( \int_0^{1/2} \left| t - \frac{1}{6} \right| \left| f'(tb+(1-t)a) \right|^q dt \right)^\frac{1}{q} \]
\[ + (b - a) \left( \int_{1/2}^{1} \left| t - \frac{5}{6} \right| dt \right)^{\frac{1}{q}} \left( \int_{1/2}^{1} \left| f'(tb + (1 - t)a) \right|^{q} dt \right)^{\frac{1}{q}}. \]

Since \( |f'|^q \) is quasi-convex, we have

\[
\left| \frac{1}{6} \left[ f(a) + 4f \left( \frac{a + b}{2} \right) + f(b) \right] - \frac{1}{b - a} \int_{a}^{b} f(x) dx \right| \\
\leq 2 (b - a) \left( \frac{5}{72} \right)^{\frac{1}{q}} \left( \frac{5}{72} \max \{|f'(a)|^q, |f'(b)|^q\} \right)^{\frac{1}{q}} \\
= \frac{5(b - a)}{36} \left( \max \{|f'(a)|^q, |f'(b)|^q\} \right)^{\frac{1}{q}}.
\]

Also, we note that

\[
\int_{0}^{1/2} \left| t - \frac{1}{6} \right| dt = \int_{1/2}^{1} \left| t - \frac{5}{6} \right| dt = \frac{5}{72}.
\]

Therefore, the proof is completed. \( \square \)

**Remark 1.** Theorem 6 is equal to Theorem 4 for \( q = 1 \).

**Remark 2.** In Theorem 5, since

\[
\lim_{p \to \infty} \left( \frac{1 + 2^{p+1}}{3(p + 1)} \right)^{\frac{1}{p}} = 2 \quad \text{and} \quad \lim_{p \to 1^+} \left( \frac{1 + 2^{p+1}}{3(p + 1)} \right)^{\frac{1}{p}} = \frac{5}{6}
\]

we have

\[
\frac{5}{6} < \left( \frac{1 + 2^{p+1}}{3(p + 1)} \right)^{\frac{1}{p}} < 2 \quad p \in (1, \infty),
\]

so for \( q > 1 \), Theorem 6 is an improvement of Theorem 5.

**Corollary 4.** In Theorem 6, if \( f(a) = f \left( \frac{a + b}{2} \right) = f(b) \), then we have

\[
\left| \frac{1}{b - a} \int_{a}^{b} f(x) dx - f \left( \frac{a + b}{2} \right) \right| \leq \frac{5(b - a)}{36} \left( \max \{|f'(a)|^q, |f'(b)|^q\} \right)^{\frac{1}{q}}.
\]

### 3. Applications to Special Means

We now consider the applications of above Theorems to the following special means:

(a) The arithmetic mean: \( A = A(a, b) := \frac{a + b}{2} \), \( a, b \geq 0 \),

(b) The harmonic mean:

\[ H = H(a, b) := \frac{2ab}{a + b}, \quad a, b > 0, \]

(c) The logarithmic mean:

\[ L = L(a, b) := \begin{cases} 
   a & \text{if } a = b \\
   \frac{b - a}{\ln b - \ln a} & \text{if } a \neq b
\end{cases}, \quad a, b > 0, \]
(d) The \( p \)-logarithmic mean:

\[
L_p = L_p(a, b) := \begin{cases} 
\left[ \frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right]^\frac{1}{p} & \text{if } a \neq b \\
\frac{a}{p} & \text{if } a = b 
\end{cases} \quad , \quad p \in \mathbb{R} \setminus \{-1, 0\}; \ a, b > 0.
\]

It is well known that \( L_p \) is monotonic nondecreasing over \( p \in \mathbb{R} \) with \( L_{-1} := L \) and \( L_0 := I \). In particular, we have the following inequalities

\[
H \leq L \leq A.
\]

Now, using the results of Section 2, some new inequalities is derived for the above means.

**Proposition 7.** Let \( a, b \in \mathbb{R}, 0 < a < b \) and \( n \in \mathbb{N}, n \geq 2 \). Then, we have

\[
\left| \frac{1}{3} A(a^n, b^n) + \frac{2}{3} A^n(a, b) - L_n^n(a, b) \right| \leq n \frac{(b-a)}{36} \max \{ a^{n-1}, b^{n-1} \}.
\]

**Proof.** The assertion follows from Theorem 4 applied to the quasi-convex mapping \( f(x) = x^n \), \( x \in [a, b] \) and \( n \in \mathbb{N} \). \( \square \)

**Proposition 8.** Let \( a, b \in \mathbb{R}, 0 < a < b \). Then, for all \( p > 1 \), we have

\[
\left| \frac{1}{3} H^{-1}(a, b) + \frac{2}{3} A^{-1}(a, b) - L^{-1}(a, b) \right| \leq \frac{1}{6} (b-a) \left( \frac{1+2p+1}{3(p+1)} \right)^{\frac{1}{p}} \left( \max \{ a^{-2q}, b^{-2q} \} \right)^{\frac{1}{q}}.
\]

**Proof.** The assertion follows from Theorem 5 applied to the quasi-convex mapping \( f(x) = 1/x, x \in [a, b] \). \( \square \)

**Proposition 9.** Let \( a, b \in \mathbb{R}, 0 < a < b \) and \( n \in \mathbb{N}, n \geq 2 \). Then, we have

\[
\left| \frac{1}{3} A(a^n, b^n) + \frac{2}{3} A^n(a, b) - L_n^n(a, b) \right| \leq n \frac{(b-a)}{6} \left( \frac{1+2p+1}{3(p+1)} \right)^{\frac{1}{p}} \left( \max \{ a^{q(n-1)}, b^{q(n-1)} \} \right)^{\frac{1}{q}}.
\]

**Proof.** The assertion follows from Theorem 5 applied to the quasi-convex mapping \( f(x) = x^n, x \in [a, b] \) and \( n \in \mathbb{N} \). \( \square \)

**Proposition 10.** Let \( a, b \in \mathbb{R}, 0 < a < b \) and \( n \in \mathbb{N}, n \geq 2 \). Then, for all \( q > 1 \), we have

\[
\left| \frac{1}{3} A(a^n, b^n) + \frac{2}{3} A^n(a, b) - L_n^n(a, b) \right| \leq n \frac{5(b-a)}{36} \left( \max \{ a^{q(n-1)}, b^{q(n-1)} \} \right)^{\frac{1}{q}}.
\]

**Proof.** The assertion follows from Theorem 6 applied to the quasi-convex mapping \( f(x) = x^n, x \in [a, b] \) and \( n \in \mathbb{N} \). \( \square \)

**Remark 3.** Proposition 10 is equal to Proposition 7 for \( q = 1 \).

**Remark 4.** Because of Remark 2, Proposition 10 is an improvement of Proposition 9 for \( q = 1 \).
References


Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey.
E-mail: erhanset@yahoo.com

Atatürk University, K.K. Education Faculty, Department of Mathematics, 25240, Campus, Erzurum, Turkey.
E-mail: emos@atauni.edu.tr

Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey.
E-mail: sarikayamz@gmail.com