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ASYMPTOTIC ANALYSIS OF A MONOSTABLE EQUATION IN PERIODIC MEDIA

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Abstract. We consider a multidimensional monostable reaction-diffusion equation whose nonlinearity involves periodic heterogeneity. This serves as a model of invasion for a population facing spatial heterogeneities. As a rescaling parameter tends to zero, we prove the convergence to a limit interface, whose motion is governed by the minimal speed (in each direction) of the underlying pulsating fronts. This dependance of the speed on the (moving) normal direction is in contrast with the homogeneous case and makes the analysis quite involved. Key ingredients are the recent improvement [4] of the well-known spreading properties [32], [9], and the solution of a Hamilton-Jacobi equation.

1. Introduction

We consider the Cauchy problem

$$(P^\varepsilon) \qquad \begin{cases} \partial_t u^\varepsilon = \varepsilon \Delta u^\varepsilon + \dfrac{1}{\varepsilon} f\left(\dfrac{x}{\varepsilon}, u^\varepsilon\right) & \text{in } (0,\infty)\times\mathbb{R}^N \\ u^\varepsilon(0,x) = g(x) & \text{in } \mathbb{R}^N, \end{cases}$$

where u will typically denote a population density, and the nonlinearity $f(x,u)$ is periodic in $x \in \mathbb{R}^N$ and of the monostable type. The parameter $\varepsilon > 0$ measures the thickness of the diffuse interfacial layer, which will account for the invasion front of the population. Our goal is to study the asymptotic behavior — or the *singular limit*, or the *sharp interface limit* — of (P^ε) as $\varepsilon \to 0$.

The reaction-diffusion equation in problem (P^ε) arises from the hyperbolic space-time rescaling $u^\varepsilon(t,x) := u\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right)$ of the heterogeneous equation

$$\partial_t u = \Delta u + f(x,u). \tag{1}$$

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Let us emphasize that the understanding of the long time behavior of (1) is not equivalent to that of the sharp interface limit of (P^ε). Roughly speaking, the former one deals with the stabilization of the interface into a predetermined shape after a long time, whereas the latter one keeps the memory of the shape of the initial data. In other words, the singular limit procedure describes some transient states, during which geometry is quite relevant.

Let us now state the assumptions on the nonlinearity $f(x,u)$. Let $L_1,\dots,L_N$ be given positive constants. A function $h:\mathbb{R}^N\to\mathbb{R}$ is said to be *periodic* if

$$h(x_1,\dots,x_k+L_k,\dots,x_N)=h(x_1,\dots,x_N),$$

for all $1\le k\le N$, all $(x_1,\dots,x_N)\in\mathbb{R}^N$. In such case, $(0,L_1)\times\cdots\times(0,L_N)$ is called the cell of periodicity. Throughout this work, we assume that

$$\text{for all } u\in\mathbb{R}_+,\ f(\cdot,u):\mathbb{R}^N\to\mathbb{R} \text{ is periodic.} \tag{2}$$

Our second main assumption on the nonlinearity f is the following.

Assumption 1.1 (Monostable nonlinearity)**.** *The function $f:\mathbb{R}^N\times\mathbb{R}_+\to\mathbb{R}$ is of class $C^{1,\alpha}$ in (x,u) and C^2 in u, and nonnegative on $\mathbb{R}^N\times[0,1]$. Concerning the steady states of the periodic equation* (1)*, we assume that*

- (i) *the constants 0 and 1 are steady states (that is, $f(\cdot,0)\equiv f(\cdot,1)\equiv 0$ in $\mathbb{R}^N$);*
- (ii) $\forall u\in(0,1),\ \exists x\in\mathbb{R}^N,\ \ f(x,u)>0.$
- (iii) *there exists some $\rho>0$ such that $f(x,u)$ is nonincreasing with respect to u in the set $\mathbb{R}^N\times(1-\rho,1]$.*

Notice that, if $0\le p(x)\le 1$ is a periodic stationary state, then $p\equiv 0$ or $p\equiv 1$. Indeed, since $f(x,p)\ge 0$, the strong maximum principle enforces p to be identically equal to its minimum, thus constant and, by (ii), the constant has to be 0 or 1. Hence, under the above hypotheses, equation (1) is often referred to as the monostable equation. Typical examples are of the form $f(x,u)=p(x)\tilde f(u)$, where $p(x)$ is positive and periodic, and $\tilde f$ is a homogeneous nonlinearity possibly of the following types: $\tilde f_1(u)=u(1-u)$ (Fisher-KPP), $\tilde f_2(u)=u^r(1-u)$ with $r>1$ (weak Allee effect), $\tilde f_3(u)=e^{-1/u}(1-u)$ (Arrhenius nonlinearity), or $\tilde f_4(u)=u(e^{1-u}-1)$ (Nicholson's blowflies equation).

The monostable problem (1) arises in various fields of physics and the life sciences, and especially in population dynamics models where propagation phenomena are involved. Indeed, a particular feature of this equation is the formation of traveling fronts, that is particular solutions describing the transition at a constant speed from one stationary solution to another one. Such solutions have proved in numerous situations their utility in describing the dynamics of a population modelled by a reaction-diffusion equation.

Equation (1) is a heterogeneous version of the reaction-diffusion equation

$$\partial_t u = \Delta u + f(u), \tag{3}$$

with f of the monostable type. Among monostable nonlinearities, one can distinguish those satisfying the Fisher-KPP assumption, namely $u \mapsto \frac{f(u)}{u}$ is maximal at 0, the most famous example $f(u) = \tilde{f}_1(u) = u(1-u)$ being introduced by Fisher [14] and Kolmogorov, Petrovsky and Piskunov [24] to model the spreading of advantageous genetic features in a population. The KPP assumption means that the growth is only slowed down by the intra-specific competition, so that the growth per capita is maximal at small densities. Due for instance to the lack of genetic diversity at low density, this assumption may be unrealistic. To take into account such a *weak Allee effect*, one may use the growth function $f(u) = \tilde{f}_2(u) = u^r(1-u)$, $r > 1$. The nonlinearity $f(u) = \tilde{f}_4(u) = u(e^{1-u} - 1)$ is commonly used [19] to explain oscillations of a population of Australian sheep blowflies, *Lucilia Cuprina*, described by Nicholson [29]. Let us notice that our work stands in the class of monostable nonlinearities, and therefore covers all these examples coming from population dynamics models, and the Arrhenius case $f(u) = \tilde{f}_3(u) = e^{-1/u}(1-u)$ which comes from combustion models.

Nevertheless, the environment is rarely homogeneous and may depend in a non trivial way on the position in space (patches, periodic media, or more general heterogeneity...), so that one should take into account heterogeneities. We refer to the seminal book of Shigesada and Kawasaki [30], and the enlightening introduction in [10] where the reader can find very precise and various references. For example such heterogeneities are very relevant in some epidemiology models, where different treatments (antibiotics or insecticides) are tested, aiming at finding an optimal combination.

In the periodic framework of equation (3), traveling fronts from the homogeneous equation (1) are replaced by the so-called *pulsating* traveling fronts. As far as the rescaled equation in (P^ε) is concerned, fronts become sharper as $\varepsilon \to 0$, and we therefore have to deal with the so-called interfaces. Also, as explained above, the singular limit analysis of (1) describes a transient state where the geometry of the initial habitat of the population is an insightful information.

In this paper, we aim at looking at the way those interfaces are generated and propagate, hence providing some accurate connection between the behavior of solutions $u^\varepsilon(t,x)$ in the fast reaction and low diffusion regime and some free boundary problem. One of the originality of this work is that we allow the equation to be spatially heterogeneous, which as recalled above is essential in realistic biological models. More precisely, we restrict ourselves to the spatially periodic case, which provides insightful information on the role and influence of the heterogeneity on the propagation, as well as a slightly more common mathematical framework.

We will describe in subsection 2.1 what is known as far as front-like solutions of (1) are concerned. In particular, we will see that the outcome of the heterogeneity is some new dynamics, that do not appear in the homogeneous case, where the speed of the propagation depends on its direction. This feature is the origin of new technical difficulties when retrieving the interface motion.

As far as initial data $g(x)$ appearing in (P^ε) are concerned, we make the following hypotheses.

Assumption 1.2 (Structure of initial data)**.**

(i) *Let Ω_0 be a nonempty, open and bounded set of $\mathbb{R}^N$. Let $\widetilde{g}:\overline{\Omega_0}\to[0,1)$ be a map of the class C^2 on $\overline{\Omega_0}$, positive on Ω_0 and such that $\widetilde{g}(x)=0$ for all $x\in\partial\Omega_0$. Define the map $g:\mathbb{R}^N\to\mathbb{R}$ by*
$$g(x)=\begin{cases}\widetilde{g}(x) & \text{if } x\in\overline{\Omega_0}\\ 0 & \text{if } x\notin\overline{\Omega_0}.\end{cases}$$

(ii) *We assume that Ω_0 is convex and has a smooth boundary $\Gamma_0:=\partial\Omega_0$.*

Notice that the assumption $g(x)<1$ becomes unnecessary if one assumes further that there is no steady state for (1) above 1. Also, rather than compactly supported initial data, one may allow $g(x)$ to have tails that are "consistent" with those of the pulsating fronts (see [2] for the homogeneous case with "tails"). For the sake of simplicity, we do not consider here such cases. The convexity assumption (ii) will allow to describe explicitly the limit interface (obtained via a Hamilton-Jacobi approach) in Proposition 2.5 and then to use a family of planar supersolutions in Section 7.

Before stating our results, let us now comment on related works. First, there is a large literature on the singular limit of (generalizations of)

$$\partial_t u^\varepsilon=\varepsilon\Delta u^\varepsilon+\frac{1}{\varepsilon}f\left(x,u^\varepsilon\right). \tag{4}$$

Observe that (4) arises after a hyperbolic rescaling of

$$\partial_t u=\Delta u+f(\varepsilon x,u),$$

whereas Problem (P^ε) under consideration follows from (1). First results are due to Freidlin [15, 16] using probabilistic methods. Later, Evans and Souganidis [13] used PDE technics, Hamilton-Jacobi framework to be more precise, to study (4). In this context, we also refer to [6], [7] and, for an overview, to [31]. Let us also mention the related work [27] which is linked with homogenization processes [25]. As far as (generalizations of) the considered problem (P^ε) is concerned, we refer to [26, Section 9] where Hamilton-Jacobi and homogenization technics are combined. Nevertheless, notice that all these results hold under the KPP assumption, that is $f(x,u) \leq f_u(x,0)u$, whereas we stand in the larger class of monostable nonlinearities.

In the homogeneous case $f(x,u) = f(u)$, the sharp interface limit of (4) has been recently revisited using specific reaction-diffusion tools, such as the comparison principle and traveling wave solutions, which allows to capture accurate convergence rates [1, 2]. Hence, the introduction of a delay effect has been handled in [3], via such methods.

Our analysis of the introduction of heterogeneity in (P^ε) stands mainly in this latter framework. It relies on accurate "local" subsolutions combined with improved spreading speeds properties [4], and on a family of planar supersolutions whose envelop solves the limit Hamilton-Jacobi equation.

2. Some known results

Before stating our main results in Section 3, we need to say a few words on monostable pulsating fronts and spreading speeds (in subsection 2.1), and on the limit free boundary problem (P^0_{HJ}) (in subsection 2.2), which is expected to describe the motion of the transition layers of the solutions $u^\varepsilon(t,x)$ of (P^ε), as $\varepsilon \to 0$.

2.1. Monostable pulsating fronts and spreading properties

The definition of the so-called pulsating traveling wave was introduced by Xin [33] in the framework of flame propagation. It is the natural extension, in the periodic framework, of classical traveling waves. Due to the interest of taking into account the role of the heterogeneity of the medium on the propagation of solutions, a lot of attention was later drawn on this subject. As far as monostable pulsating fronts are concerned, we refer to the seminal works of Weinberger [32], Berestycki and Hamel [9]. Let us also mention [11], [20], [21], [28] for related results.

For the sake of completeness, let us first recall the definition of a pulsating traveling wave for the monostable equation (1), as stated in [9].

Definition 2.1 (Pulsating traveling wave)**.** A pulsating traveling wave solution, with speed $c > 0$ in the direction $n \in \mathbb{S}^{N-1}$, is an entire solution $u(t,x)$ — $t \in \mathbb{R}$, $x \in \mathbb{R}^N$— of (1) satisfying

$$\forall k \in \prod_{i=1}^{N} L_i\mathbb{Z}, \qquad u(t,x) = u\left(t + \frac{k\cdot n}{c}, x+k\right),$$

for any $t \in \mathbb{R}$ and $x \in \mathbb{R}^N$, along with the asymptotics

$$u(-\infty,\cdot) = 0 < u(\cdot,\cdot) < u(+\infty,\cdot) = 1,$$

where the convergences in $\pm\infty$ are understood to hold locally uniformly in the space variable.

One can easily check that, for any $c>0$ and $n\in\mathbb{S}^{N-1}$, $u(t,x)$ is a pulsating traveling wave with speed c in the direction n if and only if it can be written in the form

$$u(t,x)=U(x\cdot n-ct,x),$$

where $U(z,x)$ — $z\in\mathbb{R}$, $x\in\mathbb{R}^N$— satisfies

$$\text{for all } z\in\mathbb{R},\ U(z,\cdot):\mathbb{R}^N\to\mathbb{R} \text{ is periodic,}$$

$$U(-\infty,\cdot)=1<U(\cdot,\cdot)<U(+\infty,\cdot)=0 \quad \text{uniformly w.r.t. the space variable,}$$

along with the following equation

$$(\partial_{zz}+\Delta_x)U+2\nabla_x\partial_z U\cdot n+c\partial_z U+f(x,U)=0 \quad \text{on } \mathbb{R}\times\mathbb{R}^N. \tag{5}$$

We can now recall the result of [9], [32], on existence of pulsating traveling waves for the spatially periodic monostable equations: in any direction there is a minimal speed $c^*(n)>0$ which allows existence. Precisely, the following holds.

Theorem 2.2 (Monostable pulsating fronts, [9], [32])**.** *Assume that f is of the spatially periodic monostable type, i.e. f satisfies* (2) *and Assumption* 1.1.

Then for any $n\in\mathbb{S}^{N-1}$, there exists $c^*(n)>0$ such that traveling waves with speed c in the n-direction exist if and only if $c\geq c^*(n)$. Furthermore, any pulsating traveling wave is increasing in time.

In the Fisher-KPP case the continuity of the velocity map $n\mapsto c^*(n)$, even if not explicitly stated, seems to follow from the characterization of $c^*(n)$ (see [32], [9]). In the more general monostable case, such a property was recently proved.

Theorem 2.3 (Continuity of minimal speeds, [4])**.** *The mapping $n\in\mathbb{S}^{N-1}\mapsto c^*(n)$ is continuous.*

The introduction of these pulsating traveling waves was motivated by their expected role in describing the large time behavior of solutions of (1) for a large class of initial data. In this context, let us recall the seminal result of [32]: for any planar-like initial data in some direction n, the associated solution of (1) spreads in the n direction with speed $c^*(n)$. Actually, for our singular limit analysis, it turns out that we need the stronger property that this spreading is uniform with respect to the direction n. This was the purpose of our previous work [4].

Theorem 2.4 (Uniform spreading, [4]). *Assume that f is of the spatially periodic monostable type, i.e. f satisfies (2) and Assumption 1.1. Let a family of nonnegative initial data $(u_{0,n})_{n\in\mathbb{S}^{N-1}}$ be such that*

$$\exists C>0, \qquad \forall n\in\mathbb{S}^{N-1}, \qquad x\cdot n\geq C \Longrightarrow u_{0,n}(x)=0,$$

$$\exists \mu>0, \qquad \exists K>0, \qquad \inf_{n\in\mathbb{S}^{N-1},\, x\cdot n\leq -K} u_{0,n}(x)\geq \mu,$$

$$\inf_{n\in\mathbb{S}^{N-1}}\inf_{x\in\mathbb{R}^N} 1-u_{0,n}(x)>0.$$

We denote by $(u_n)_{n\in\mathbb{S}^{N-1}}$ the associated family of solutions of (1).

Let $\alpha>0$ and $\eta>0$ be given. Then, there exists $\tau>0$ such that for all $t\geq\tau$,

$$\sup_{n\in\mathbb{S}^{N-1}}\ \sup_{x\cdot n\leq(c^*(n)-\alpha)t} |1-u_n(t,x)|\leq\eta, \tag{6}$$

$$\sup_{n\in\mathbb{S}^{N-1}}\ \sup_{x\cdot n\geq(c^*(n)+\alpha)t} u_n(t,x)\leq\eta. \tag{7}$$

Let us notice that, under suitable assumptions such as those in [9], [4], the above results are also available for more general spatially periodic and monostable equations which include heterogeneous diffusion and advection terms. We restrict ourselves to Problem (P^ε) to simplify the presentation, but our argument easily extends to such a framework.

2.2. On limit free boundary problems

We recall that we aim at investigating the $\varepsilon\to 0$ limit of $u^\varepsilon(t,x)$ the solution of (P^ε). Then the limit solution $\tilde{u}^\varepsilon(t,x)$ will be a step function, taking the value 1 on one side of a moving interface which we will denote by Γ_t, and 0 on the other side. This sharp interface, if smooth, obeys the law of motion

$$(P^0)\qquad \begin{cases} V_n=c^*(n) & \text{on } \Gamma_t\\ \Gamma_t|_{t=0}=\Gamma_0, \end{cases}$$

where V_n denotes the normal velocity of Γ_t in the exterior direction n, the unit outer normal of Γ_t at each point $x\in\Gamma_t$. Here $c^*(n)$ denotes the minimal speed of the underlying monostable pulsating wave traveling in the n-direction.

As we only know the mapping $n\mapsto c^*(n)$ to be continuous, the smoothness of the interface, and hence the well posedness of (P^0), is not guaranteed even for small positive times.

A classical way to overcome the lack of smoothness is to define the limit interface via the level sets of the viscosity solution of a Hamilton-Jacobi problem

$$(P^0_{HJ})\qquad \begin{cases} \partial_t w+|\nabla w|c^*\left(\frac{\nabla w}{|\nabla w|}\right)=0 & \text{in } (0,\infty)\times\mathbb{R}^N\\ w(0,x)=w_0(x) & \text{in } \mathbb{R}^N. \end{cases}$$

Here $w_0 : \mathbb{R}^N \to \mathbb{R}$ is any uniformly continuous function such that

$$\Omega_0 = \{x : \; w_0(x) < 0\}, \quad \Gamma_0 = \{x : \; w_0(x) = 0\}. \tag{8}$$

Thanks to the continuity of $c^*(n)$ with respect to $n \in \mathbb{S}^{N-1}$, namely Theorem 2.3, the Hamilton-Jacobi problem admits a unique viscosity solution $w \in C((0,\infty) \times \mathbb{R}^N)$, and

$$\Omega_t := \{x : \; w(t,x) < 0\}, \quad \Gamma_t := \{x : \; w(t,x) = 0\}$$

do not depend on the choice of w_0 as above (see Theorems 4.3.5 and 4.3.6 in [17]). As long as (P^0) admits a smooth solution, both motions coincide, which is why we still denote it by Γ_t. However, the Hamilton-Jacobi approach does not require smoothness as (P^0) does, and therefore enables to define the zero level set Γ_t as the limit interface for all $t \geq 0$.

The literature on this level set approach via viscosity solutions of Hamilton-Jacobi equations is rather large. The reader may consult [12] or the book of Giga [17] and the references therein.

Thanks to the convexity of the initial set Ω_0, a so-called Hopf formula [23] is actually available and provides an explicit depiction of the motion, as stated in the following result.

Proposition 2.5 (The limit interface explicitly)**.** *Let Assumption 1.2 (ii) hold. Let the limit interface Γ_t be defined via the Hamilton-Jacobi problem (P^0_{HJ}) as above.*

Then, for all time $t \geq 0$, the set Γ_t is the zero level set of the convex function

$$v(t,x) := \max_{y \in \Gamma_0} (x-y) \cdot n_y - c^*(n_y)t,$$

where n_y denotes the outward unit normal vector of Γ_0 at point y. In particular, for all time $t \geq 0$, the set Γ_t remains sharp, in the sense that it does not develop an interior, and the bounded domain Ω_t delimited by Γ_t remains convex.

Roughly speaking, this proposition means that the motion can be described by first looking at Γ_0 as the envelop of some half-spaces, and by then letting each of those half-spaces move at the speed $c^*(n)$ corresponding to its normal direction. We refer to [5] where the Hopf formula was revisited in the context of viscosity solutions, and obtained using the more general theory of differential games. We propose a direct proof of Proposition 2.5 in subsection 4.1.

3. Main result

We are now in the position to state our main result of convergence of (P^ε) to the interface motion defined via the level sets of solutions of (P^0_{HJ}). Together with Proposition 2.5, the theorem below provides a precise depiction of the shape of solutions or, in other words, of the expansion of the habitat of the population.

Theorem 3.1 (Convergence to a propagating interface)**.** *Let the nonlinearity f be of the spatially periodic monostable type, i.e. f satisfies* (2) *and Assumption 1.1, and let the initial data g in Problem (P^ε) satisfy Assumption 1.2. For any $\varepsilon > 0$, let $u^\varepsilon : [0,\infty) \times \mathbb{R}^N \to \mathbb{R}$ be the solution of (P^ε). Let Γ_t and Ω_t be defined via the Hamilton-Jacobi problem (P^0_{HJ}) as in subsection* 2.2.

Then, the following convergence results hold.

(i) *For any $0 < \tau \leq T < +\infty$ and small $\beta > 0$, we have*

$$\sup_{\tau \leq t \leq T} \sup_{\{x : d(t,x) \leq -\beta\}} \left|1 - u^\varepsilon(t,x)\right| \to 0 \quad \text{as } \varepsilon \to 0;$$

(ii) *For any $0 < T < +\infty$ and small $\beta > 0$, we have*

$$\sup_{0 \leq t \leq T} \sup_{\{x : d(t,x) \geq \beta\}} |u^\varepsilon(t,x)| \to 0 \quad \text{as } \varepsilon \to 0.$$

Here $d(t,\cdot)$ denotes the signed distance to the set Γ_t, which is chosen to be negative in Ω_t and positive in $\mathbb{R}^N \setminus (\Gamma_t \cup \Omega_t)$.

The rest of the paper is devoted to the proof of Theorem 3.1 and is organized as follows.

We start, in Section 4, by some results on the motion of the limit interface which are crucial to our analysis of the parabolic problem (P^ε), but are also of independent interest for the Hamilton-Jacobi problem (P^0_{HJ}). On the one hand, we prove Proposition 2.5, hence providing an explicit description of the limit interface. On the other hand, we approximate the motion defined via (P^0_{HJ}) by a smooth motion, which preserves all its essential geometric properties.

To prove the control from below (i) of Theorem 3.1, we distinguish two regimes. First, we prove in Section 5 the emergence of transition layers for $u^\varepsilon(t,x)$ in very small times. The propagation of the layers (from below) that occurs in later times is then studied in Section 6. The heterogeneity rises some technical difficulties since pulsating fronts depend non trivially on the direction of propagation. Roughly speaking, we construct "local" subsolutions and combine the uniform spreading properties of Theorem 2.4 with an iteration procedure. The construction of such subsolutions requires smoothness of the interface, which insures that the motion is locally governed by the planar dynamics of the rescaled equation (1). Hence, we actually apply the above procedure to the smooth approximated motion defined in Section 4.

Last, in Section 7, to prove the control from above (ii) of Theorem 3.1, we construct a family of planar supersolutions — whose envelop coincides with the explicit characterization of Proposition 2.5 — and use again the uniform spreading properties of Theorem 2.4.

4. Some results on the motion of the limit interface

In this section, we are only concerned with the limit interface motion (P^0_{HJ}). We first prove the explicit description of Proposition 2.5, and then proceed to an approximation of the motion (P^0_{HJ}) by a smooth motion. As mentioned before, smoothness will play an essential role in the convergence of solutions of (P^ε), and more specifically in Section 6.

4.1. Characterization of the motion

We begin by recalling that

$$v(t,x) := \max_{y\in\Gamma_0} \{(x-y)\cdot n_y - c^*(n_y)t\},$$

where n_y is the outward unit normal of Γ_0 at point y. The zero level sets of $v(t,x)$ are obtained by "intersecting all the half-planes arising from $y \in \Gamma_0$ and propagating with speed $c^*(n_y)$ in direction n_y". We will prove that, at least for its level sets lying above some small $-\delta < 0$, the function v is a viscosity solution of the Hamilton-Jacobi problem (P^0_{HJ}). As the motion of interface is defined by the zero level set of the viscosity solution, this will be enough to infer that its zero level set defines the appropriate interface Γ_t, that is Proposition 2.5.

Remark 4.1. Write $v(t,x) = \max_{y\in\Gamma_0} \psi(t,x,y)$ where

$$\psi : (t,x,y) \in (0,\infty)\times\mathbb{R}^N\times\Gamma_0 \mapsto (x-y)\cdot n_y - c^*(n_y)t,$$

is continuous with respect to $y \in \Gamma_0$, smooth and convex (since linear) with respect to $t > 0$ and $x \in \mathbb{R}^N$. For a given $(t,x) \in (0,\infty)\times\mathbb{R}^N$, let us denote by $Y(t,x)$ the set of $y \in \Gamma_0$ that maximize $\psi(t,x,\cdot)$, that is

$$Y(t,x) = \{y \in \Gamma_0 : \ v(t,x) = \psi(t,x,y)\}.$$

If, for a given (t_0,x_0), the set $Y(t_0,x_0)$ reduces to a singleton y_0 then it follows from classical results of convex analysis (see [22, Corollary 4.4.5]) that v is differentiable at (t_0,x_0), and

$$\partial_t v(t_0,x_0) = \partial_t\psi(t_0,x_0,y_0) = -c^*(n_{y_0}), \quad \nabla_x v(t_0,x_0) = \nabla_x\psi(t_0,x_0,y_0) = n_{y_0},$$

so that v satisfies the Hamilton-Jacobi equation $\partial_t v + |\nabla v| c^*\left(\frac{\nabla v}{|\nabla v|}\right) = 0$ in the classical sense at (t_0,x_0). However, we have to deal with the case where $Y(t_0,x_0)$ is not a singleton. As we will see, this can be performed in the set $\{v \geq -\delta\}$ for some small enough $\delta > 0$, and requires to cut-off the set $\{v < -\delta\}$.

We prove the following proposition, of which Proposition 2.5 is an immediate corollary.

Proposition 4.2 (A solution of the Hamilton-Jacobi problem)**.** *For any small enough $\delta > 0$, the function*

$$v_\delta(t,x) := \max(-\delta; v(t,x)),$$

is a (viscosity) solution of the equation of the limit problem (P^0_{HJ}), that is

$$\partial_t v_\delta + |\nabla v_\delta| c^* \left(\frac{\nabla v_\delta}{|\nabla v_\delta|} \right) = 0, \quad \text{in } (0,\infty) \times \mathbb{R}^N, \tag{9}$$

and, by convexity, $v_\delta(0,x)$ is an admissible initial datum for (P^0_{HJ}) in the sense of (8).

Proof. Recall that at time $t = 0$, Γ_0 is a smooth hypersurface, and that the bounded set Ω_0 delimited by Γ_0 is convex. Hence, for $\delta > 0$ small enough, one can define a smooth hypersurface $\Gamma_0^{-\delta}$ as

$$\Gamma_0^{-\delta} := \{x \in \mathbb{R}^N \ : \ d(0,x) = -\delta\} = \{y - \delta n_y \ : y \in \Gamma_0\},$$

where $d(0,\cdot)$ denotes the signed distance to Γ_0, which is negative in the bounded set Ω_0, and positive in $\mathbb{R}^N \setminus \overline{\Omega_0}$. Notice also that, when $x \in \Omega_0$, we can write $v(0,x) = -\min_{y \in \Gamma_0} dist(x, H_y)$, where H_y is the hyperplane going through y and with normal vector n_y. As a result, the convexity assumption yields

$$\Gamma_0^{-\delta} = \{x \in \mathbb{R}^N \ : \ v(0,x) = -\delta\}.$$

In particular, since the function v is convex with respect to x, the bounded set $\Omega_0^{-\delta}$ delimited by $\Gamma_0^{-\delta}$ is still convex. Moreover, as we have chosen δ small enough so that $\Gamma_0^{-\delta}$ is smooth, it is straightforward that the outward unit normal vector of $\Gamma_0^{-\delta}$ at $y - \delta n_y$ is also n_y. Therefore, by some slight abuse of notation, when $y \in \Gamma_0^{-\delta}$, n_y will denote the outward unit normal vector of $\Gamma_0^{-\delta}$ at point y. Then

$$\begin{aligned} v_\delta(t,x) &= \max\{-\delta, \max_{y \in \Gamma_0} \{(x-y)\cdot n_y - c^*(n_y)t\}\} \\ &= \max\{-\delta, \max_{y \in \Gamma_0} \{(x-(y-\delta n_y))\cdot n_y - c^*(n_y)t - \delta\}\} \\ &= \max\{0, \max_{y \in \Gamma_0^{-\delta}} \{(x-y)\cdot n_y - c^*(n_y)t\}\} - \delta. \end{aligned}$$

Therefore, $v_\delta(t,x)$ is a solution of (P^0_{HJ}) if and only if

$$\bar{v}_\delta(t,x) := \max\{0, \max_{y \in \Gamma_0^{-\delta}} \{(x-y)\cdot n_y - c^*(n_y)t\}\}$$

is. For convenience, denote

$$\psi^\delta : (t,x,y) \in (0,\infty) \times \mathbb{R}^N \times \Gamma_0^{-\delta} \mapsto (x-y)\cdot n_y - c^*(n_y)t,$$

which is continuous with respect to $y \in \Gamma_0^{-\delta}$, smooth and linear with respect to $t > 0$ and $x \in \mathbb{R}^N$, and introduce also

$$w_\delta(t,x) := \max_{y \in \Gamma_0^{-\delta}} \psi^\delta(t,x,y),$$

so that $\bar{v}_\delta(t,x) = \max(0, w_\delta(t,x))$.

Let us now prove that $\bar{v}_\delta$ is a solution of (9). First, the null function and each function $(t,x) \mapsto \psi^\delta(t,x,y)$ solve (9) so that $\bar{v}_\delta(t,x)$ — as a supremum of solutions — is a viscosity subsolution of (9).

To prove that $\bar{v}_\delta(t,x)$ is also a supersolution, let φ be a smooth test function such that $\bar{v}_\delta - \varphi$ has a zero local minimum at some point $(t_0,x_0) \in (0,\infty) \times \mathbb{R}^N$. We need to prove that

$$\partial_t \varphi(t_0,x_0) + |\nabla\varphi(t_0,x_0)| c^* \left(\frac{\nabla\varphi(t_0,x_0)}{|\nabla\varphi(t_0,x_0)|} \right) \geq 0. \tag{10}$$

If $w_\delta(t_0,x_0) < 0$, then $\bar{v}_\delta \equiv 0$ in a neighborhood of (t_0,x_0) and (10) is clear. Let us now assume $0 \leq w_\delta(t_0,x_0) = \bar{v}_\delta(t_0,x_0)$. Since $\bar{v}_\delta - \varphi$ has a zero local minimum at (t_0,x_0), the time-space gradient of φ at (t_0,x_0) must belong to the time-space subdifferential of $\bar{v}_\delta$ at (t_0,x_0), which is given by

$$\partial \bar{v}_\delta(t_0,x_0) = \begin{cases} \partial w_\delta(t_0,x_0) & \text{if } w_\delta(t_0,x_0) > 0 \\ \text{Co } \{(0_{\mathbb{R}}, 0_{\mathbb{R}^N}) \cup \partial w_\delta(t_0,x_0)\} & \text{if } w_\delta(t_0,x_0) = 0, \end{cases}$$

where Co A denotes the convex hull of the set A. It also follows from [22, Theorem 4.4.2] that

$$\partial w_\delta(t_0,x_0) = \text{Co } \{(\partial_t \psi^\delta(t_0,x_0,y), \nabla_x \psi^\delta(t_0,x_0,y)) = (-c^*(n_y), n_y) \in \mathbb{R} \times \mathbb{R}^N : y \in Y(t_0,x_0)\},$$

where $Y(t_0,x_0)$ is the set of $y \in \Gamma_0^{-\delta}$ that maximize $\psi^\delta(t_0,x_0,\cdot)$. Hence, in any case, one can write

$$\partial_t \varphi(t_0,x_0) = \sum_{i=1}^p -\lambda_i c^*(n_i), \quad \nabla\varphi(t_0,x_0) = \sum_{i=1}^p \lambda_i n_i,$$

for some $y_1,\ldots,y_p$ in $Y(t_0,x_0)$, and n_i the outward unit normal of $\Gamma_0^{-\delta}$ at point y_i, and some nonnegative $\lambda_1,\ldots,\lambda_p$ such that $\sum_{i=1}^p \lambda_i \leq 1$. Therefore our goal (10) is recast as

$$c^* \left(\frac{\sum_{i=1}^p \lambda_i n_i}{|\sum_{i=1}^p \lambda_i n_i|} \right) \geq \frac{\sum_{i=1}^p \lambda_i c^*(n_i)}{|\sum_{i=1}^p \lambda_i n_i|}. \tag{11}$$

Let us define

$$n_0 := \frac{\sum_{i=1}^p \lambda_i n_i}{|\sum_{i=1}^p \lambda_i n_i|} \in \mathbb{S}^{N-1},$$

and pick a $y_0 \in \Gamma_0^{-\delta}$ such that $n_{y_0} = n_0$. Note that such a y_0 necessarily exists from the smoothness of the bounded hypersurface $\Gamma_0^{-\delta}$. One must then have

$$\psi^\delta(t_0,x_0,y_0) = (x_0 - y_0)\cdot n_0 - c^*(n_0)\, t_0 \leq w_\delta(t_0,x_0),$$

so that

$$\begin{aligned} c^*(n_0)t_0 &\geq (x_0-y_0)\cdot n_0 - w_\delta(t_0,x_0) \\ &= \frac{\sum_{i=1}^p \lambda_i (x_0-y_0)\cdot n_i}{|\sum_{i=1}^p \lambda_i n_i|} - w_\delta(t_0,x_0) \\ &\geq \frac{\sum_{i=1}^p \lambda_i (x_0-y_i)\cdot n_i}{|\sum_{i=1}^p \lambda_i n_i|} - w_\delta(t_0,x_0). \end{aligned}$$

Here we used the convexity of $\Omega_0^{-\delta}$, so that $(y_i-y_0)\cdot n_i \geq 0$ for all $1\leq i\leq p$. Next, as each y_i belongs to $Y(t_0,x_0)$, we have $w_\delta(t_0,x_0)=(x_0-y_i)\cdot n_i - c^*(n_i)t_0$, so that

$$c^*(n_0)t_0 \geq \frac{\sum_{i=1}^p \lambda_i c^*(n_i)}{|\sum_{i=1}^p \lambda_i n_i|} t_0 + w_\delta(t_0,x_0)\left(\frac{\sum_{i=1}^p \lambda_i}{|\sum_{i=1}^p \lambda_i n_i|} - 1\right) \geq \frac{\sum_{i=1}^p \lambda_i c^*(n_i)}{|\sum_{i=1}^p \lambda_i n_i|} t_0,$$

since $w_\delta(t_0,x_0)\geq 0$ (notice that this is where it fails if no cut-off is performed). This proves (11) and concludes the proof of Proposition 4.2. □

4.2. Regularization of the motion

We now construct, by the vanishing viscosity method, a smooth hypersurface Γ_t^α which approximates the interface Γ_t as $\alpha\to 0$. Moreover, the motion of this smooth hypersurface is always "slower" than that of the original interface Γ_t: this will allow us, in Section 6, to construct subsolutions of (P^ε) which fully cover the bounded set delimited by Γ_t^α.

Proposition 4.3 (Approximated smooth motion)**.** *Fix $\alpha_0>0$ small enough and, for any $0<\alpha\leq\alpha_0$, let $F^\alpha:\mathbb{R}^N\to\mathbb{R}$ be a smooth function such that*

$$0\leq F^\alpha(p)\leq |p|(c^*(p/|p|)-\alpha), \quad \text{for all } p\in\mathbb{R}^N,$$

and, as $\alpha\to 0$,

$$F^\alpha(p)\to |p|c^*(p/|p|), \quad \text{locally uniformly in } \mathbb{R}^N.$$

Let $v_0^\alpha(x)$ be a smooth and strictly convex function such that

$$\|\nabla v_0^\alpha\|_\infty \leq 1, \quad v_\delta(0,\cdot)+\alpha \leq v_0^\alpha \leq v_\delta(0,\cdot)+2\alpha,$$

where v_δ is the explicit viscosity solution of (P_{HJ}^0) with initial datum $v_\delta(0,x)$, as defined in Proposition 4.2.

Then, the solution v^α of the parabolic equation

$$\begin{cases} \partial_t v^\alpha + F^\alpha(\nabla v^\alpha) - \alpha\Delta v^\alpha = 0 & \text{in } (0,\infty)\times\mathbb{R}^N \\ v^\alpha(0,x)=v_0^\alpha(x) & \text{in } \mathbb{R}^N, \end{cases}$$

is smooth, convex w.r.t. space, and converges locally uniformly to v_δ as $\alpha \to 0$. In particular, for any $T>0$ and up to reducing α, the zero level set $\Gamma_t^\alpha := \{x \in \mathbb{R}^N : v^\alpha(t,x) = 0\}$ is a smooth hypersurface for any $0 \le t \le T$, and is such that

$$\sup_{0 \le t \le T} d_{\mathcal{H}}(\Gamma_t^\alpha, \Gamma_t) \to 0 \quad \text{as } \alpha \to 0, \tag{12}$$

where $d_{\mathcal{H}}(A,B) := \max\{\sup_{a\in A} dist(a,B), \sup_{b \in B} dist(b,A)\}$ denotes the Hausdorff distance between two compact sets A and B. Last, v^α satisfies

$$\partial_t v^\alpha + |\nabla v^\alpha| \left(c^*\left(\frac{\nabla v^\alpha}{|\nabla v^\alpha|} \right) - \alpha \right) \ge 0 \quad \text{in } (0,\infty) \times \mathbb{R}^N. \tag{13}$$

Proof. One can differentiate (in any direction) the parabolic equation satisfied by v^α and, using $\|\nabla v_0^\alpha\|_\infty \le 1$ for any $0 < \alpha \le \alpha_0$, deduce from the comparison principle that

$$\|\nabla v^\alpha(t,\cdot)\|_\infty \le 1, \quad \text{for all } 0<\alpha \le \alpha_0 \text{ and } t>0.$$

In other words, the family $(v^\alpha(t,\cdot))_{0<\alpha\le\alpha_0, t\ge 0}$ is uniformly Lipschitz-continuous. As confirmed by [18], the proof of Theorem 4.6.3 in [17] still applies thanks to the above estimate, even though the solutions we consider are unbounded. Therefore, one can conclude that the family of functions v^α converges locally uniformly to the unique viscosity solution of (P_{HJ}^0) with initial datum $v_\delta(0,x)$, namely v_δ.

We now proceed by noting that, for each $0<\alpha \le \alpha_0$, the smoothness of v^α follows from standard parabolic estimates. One can then differentiate the parabolic equation twice in any given direction $e \in \mathbb{S}^{N-1}$ and deduce from the comparison principle (recall that v_0^α is convex) that $v^\alpha(t,\cdot)$ is convex for any positive time. In particular, we have $\Delta v^\alpha(t,x) \ge 0$ for all $t \ge 0$ and $x \in \mathbb{R}^N$, which proves (13).

Let us now turn to the convergence of the zero level set Γ_t^α of v^α to Γ_t. The proof again follows the steps of [17] (see the proof of Theorem 4.6.4 in the particular case of geometric motions). We fix any $\beta>0$ and $T>0$ and show that, for small enough α, $\sup_{0\le t\le T} d_{\mathcal{H}}(\Gamma_t^\alpha, \Gamma_t) \le \beta$. By (13), we get that $v^\alpha(t,x) > v_\delta(t,x)$ for all $t \ge 0$ and $x \in \mathbb{R}^N$: in particular, $\Gamma_t^\alpha \subset \Omega_t$ for all $t \ge 0$. Let now $R>0$ be large enough so that for all $0 \le t \le T$ the inclusion $\Omega_t \subset B_R$ holds, where B_R denotes the ball of radius R and centered at the origin. By the locally uniform convergence, it is clear that for any small enough α and $x \in \Omega_t$ such that $d(t,x) \le -\beta$ (recall that $d(t,\cdot)$ denotes the signed distance to Γ_t), then $v^\alpha(t,x)<0$. The convergence (12) easily follows.

Let us again fix $T>0$ and now prove that, for small enough α, the zero level set Γ_t^α is a smooth hypersurface on the time interval $[0,T]$. Note that, for any $0 \le t_0 \le T$ and $x_0 \in \Gamma_{t_0}^\alpha$, one has that $|\nabla v^\alpha(t_0,x_0)| \ne 0$ provided α is small enough. Otherwise, it would follow from the convexity of $v^\alpha(t_0,\cdot)$ that $v^\alpha(t_0,\cdot) \ge 0$ in $\mathbb{R}^N$, a contradiction with the fact that it approaches $v_\delta(t_0,\cdot)$ locally uniformly. Then, as $|\nabla v^\alpha(t_0,x_0)| \ne 0$, one can apply the implicit function theorem and obtain the smoothness of $\Gamma_{t_0}^\alpha$. □

5. Rapid emergence of the layers from below

In this section we prove that, as $\varepsilon \to 0$, the solution $u^\varepsilon(t,x)$ of (P^ε) is very close to 1 in Ω_0 after a very short time. The proof relies on the spreading properties of solutions of (1) with large enough compact support at initial time [32]. Precisely, the following holds.

Proposition 5.1 (Emergence of the layers from below)**.** *Let the nonlinearity f be of the spatially periodic monostable type, i.e. f satisfies* (2) *and Assumption 1.1. Let the initial datum g in Problem* (P^ε) *satisfy Assumption 1.2.*

Then, for any small $\eta > 0$ and small $\alpha > 0$, there is a time $t_\alpha > 0$ such that the following holds: there is $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$,

$$x \in \Omega_0,\ dist(x, \partial\Omega_0) > \alpha \implies 1 - \eta \le u^\varepsilon(t_\alpha \varepsilon, x) \le 1. \tag{14}$$

Proof. Since 1 solves the reaction-diffusion equation in (P^ε) and since $u^\varepsilon(0,\cdot) = g(\cdot) \le 1$, the comparison principle implies $u^\varepsilon(t,x) \le 1$, which proves the upper bound in (14). We next prove the lower bound.

We begin by recalling the following result on the spreading of solutions with initial compact support [32, Theorem 2.3]: for any $\sigma \in (0,1)$, there is $R_\sigma > 0$ large enough so that the solution v of (1) with initial datum $v_0 = \sigma \chi_{B_{R_\sigma}}$ converges locally uniformly to 1 as $t \to +\infty$. Here, χ denotes the characteristic function and B_R the ball of radius R and centered at the origin. Note that Weinberger's result [32] also provides a positive spreading speed in any direction; however, it is not required to prove Proposition 5.1.

Let us now fix some $\eta > 0$ and $\alpha > 0$. From Assumption 1.2 on the initial data g, there is $\sigma_1 \in (0,1)$ such that, for all $\varepsilon > 0$,

$$x \in \Omega_0,\ dist(x, \partial\Omega_0) > \alpha \implies u^\varepsilon(0,x) = g(x) \ge \sigma_1. \tag{15}$$

We can now let $t_\alpha > 0$ be such that the solution v of (1) with initial datum $v_0 = \sigma_1 \chi_{B_{R_{\sigma_1}}}$ satisfies

$$v(t_\alpha, x) \ge 1 - \eta, \quad \forall x \in B_{3R_{\sigma_1}}. \tag{16}$$

We assume without loss of generality that $R_{\sigma_1} > 2\sqrt{N} \max_i L_i$.

Let us now fix $x^* \in \Omega_0$ such that $dist(x^*, \partial\Omega_0) > \alpha$. We are going to prove

$$u^\varepsilon(t_\alpha \varepsilon, x^*) \ge 1 - \eta, \tag{17}$$

for $\varepsilon \in (0, \varepsilon_0)$, where $\varepsilon_0 > 0$ has to be independent on the point x^* chosen as above. We let $x_0 \in \partial\Omega_0$ such that $dist(x^*, \partial\Omega_0) = |x^* - x_0|$. Since $R_{\sigma_1} > 2\sqrt{N} \max_i L_i$, there exists $k^*_\varepsilon = (k^*_{1,\varepsilon}, \ldots, k^*_{N,\varepsilon}) \in \mathbb{Z}^N$ such that

$$x^* - 2R_{\sigma_1} \varepsilon \frac{x_0 - x^*}{|x_0 - x^*|} \in \varepsilon k^*_\varepsilon L + B_{\frac{R_{\sigma_1} \varepsilon}{2}}, \tag{18}$$

where we denote $k_\varepsilon^* L := (k_{1,\varepsilon}^* L_1, \dots, k_{N,\varepsilon}^* L_N)$. Also, provided α and $\varepsilon_0 > 0$ are small enough depending only on $0 < \max_{y\in\Gamma_0} \gamma(y) < +\infty$ with $\gamma(y)$ the mean curvature (positive by convexity) of Γ_0 at point y, we have, for all $\varepsilon \in (0, \varepsilon_0)$,

$$\varepsilon k_\varepsilon^* L \in \Omega_0 \text{ and } dist(\varepsilon k_\varepsilon^* L, \partial\Omega_0) > \alpha + R_{\sigma_1}\varepsilon. \tag{19}$$

Observe that if $x \notin \varepsilon k_\varepsilon^* L + B_{\varepsilon R_{\sigma_1}}$ then $v_0\left(\frac{x-\varepsilon k_\varepsilon^* L}{\varepsilon}\right) = \sigma_1 \chi_{B_{R_{\sigma_1}}}\left(\frac{x-\varepsilon k_\varepsilon^* L}{\varepsilon}\right) = 0$, and that if $x \in \varepsilon k_\varepsilon^* L + B_{\varepsilon R_{\sigma_1}}$ then (19) implies that $x \in \Omega_0$ and $dist(x, \partial\Omega_0) > \alpha$. Hence, it follows from (15) that

$$g(x) \geq v\left(0, \frac{x-\varepsilon k_\varepsilon^* L}{\varepsilon}\right) \text{ for all } x \in \mathbb{R}^N.$$

Since $v(\frac{t}{\varepsilon}, \frac{x-\varepsilon k_\varepsilon^* L}{\varepsilon})$ solves the parabolic equation in (P^ε), the comparison principle implies in particular that

$$u^\varepsilon(t_\alpha \varepsilon, x^*) \geq v\left(t_\alpha, \frac{x^* - \varepsilon k_\varepsilon^* L}{\varepsilon}\right).$$

In view of (16) and (18), the above estimate implies (17). The proposition is proved. □

The above argument also shows that, roughly speaking, the solution of (P^ε) may only expand, which is rather natural from the dynamics of the monostable equation. Precisely the following holds.

Lemma 5.2 (Expansion)**.** *Let $\eta > 0$ be given. Let $(\tilde\Omega_t)_{0\leq t\leq T}$ be a family of bounded and convex domains with smooth boundaries $\tilde\Gamma_t := \partial\tilde\Omega_t$. Then, for any $\sigma \in (0,1)$ there is a time $t_\sigma > 0$ such that the following holds: there is $\varepsilon_0 > 0$ — depending only on $0 < \max_{0\leq t\leq T} \max_{y\in\tilde\Gamma_t} \gamma_t(y) < +\infty$ with $\gamma_t(y)$ the mean curvature of $\tilde\Gamma_t$ at point y— such that, for any $0 \leq t_0 < T$, any $\varepsilon \in (0, \varepsilon_0)$,*

$$u^\varepsilon(t_0, x) \geq \sigma, \quad \forall x \in \tilde\Omega_{t_0} \Longrightarrow u^\varepsilon(t,x) \geq 1-\eta, \quad \forall x \in \tilde\Omega_{t_0}, \forall t \geq t_0 + t_\sigma \varepsilon.$$

6. Propagation of the layers from below

We now begin the analysis of the motion of interface. In this section, we prove the lower estimate on the motion of level sets of the solutions $u^\varepsilon(t,x)$, namely statement (i) of Theorem 3.1.

To that purpose, we fix some times $0 < \tau < T$, and a small $\beta > 0$. We then let $\alpha > 0$ be small enough so that the hypersurfaces $(\Gamma_t^\alpha)_{0\leq t\leq T+1}$, as defined in subsection 4.2, are smooth and such that

$$\sup_{0\leq t\leq T+1} d_{\mathcal{H}}(\Gamma_t^\alpha, \Gamma_t) \leq \frac{\beta}{2}. \tag{20}$$

We also denote, in this section, by Ω_t^α the region enclosed by Γ_t^α.

6.1. Lower estimates in small canisters

We start by looking, for any fixed time t_0, at the “local motion" of the interface. By “local motion", we mean that we will investigate the motion of the solution on small neighborhoods of any point of $\Gamma^\alpha_{t_0}$. Precisely, the following holds.

Lemma 6.1 (Lower estimates in small canisters)**.** *Let $\eta > 0$ be given. Fix some time $t_0 \in (0,T)$, and assume that*

$$x \in \Omega^\alpha_{t_0} \Longrightarrow u^\varepsilon(t_0,x) \geq 1-\eta. \tag{21}$$

Then there are two positive constants A_1 and A_2, independent on t_0 and $\varepsilon > 0$ (provided it is small enough), such that

$$u^\varepsilon(t_0 + A_1\sqrt{\varepsilon}, x) \geq 1-\eta,$$

for all $x \in D := \cup_{x_0 \in \Gamma^\alpha_{t_0}} \mathscr{C}(x_0)$, where $\mathscr{C}(x_0)$ is the finite cylinder, or canister, made of the points x such that

$$|(x-x_0)\cdot n| \leq A_1\left(c^*(n) - \frac{\alpha}{2}\right)\sqrt{\varepsilon} \text{ and } |(x-x_0)\cdot n^\perp| \leq A_2\frac{\sqrt{\varepsilon}}{2}. \tag{22}$$

Here n denotes the unit outer normal of $\Gamma^\alpha_{t_0}$ at point x_0, and $(x-x_0)\cdot n^\perp$ denotes the orthogonal projection of $x - x_0$ on the hyperplane $(\mathbb{R}n)^\perp$.

Proof. First, let $\gamma > 0$ be large enough so that, for all $t \in [0,T]$, all $y \in \Gamma^\alpha_t$ with n_y the associated unit outer normal, we have the inclusion

$$B_{\frac{1}{\gamma}}\left(y - \frac{1}{\gamma}n_y\right) \subset \Omega^\alpha_t, \tag{23}$$

where $B_r(z)$ denotes the open ball of center z, radius r. By convexity, it suffices to take γ as the maximal curvature (in absolute value) of Γ^α_t in the time interval $[0,T]$.

Let $\eta > 0$ and $0 < t_0 < T$ be given. Let $x_0 \in \Gamma^\alpha_{t_0}$ be given and n the associated unit outer normal. For the lemma to be proved notice that constants A_1 and A_2, that we need to determine, have to be independent on t_0, small $\varepsilon > 0$ but also on x_0 and n. By assumption (21) and inclusion (23), we have

$$\forall x \in B_{\frac{1}{\gamma}}\left(x_0 - \frac{1}{\gamma}n\right), \quad u^\varepsilon(t_0,x) \geq 1-\eta.$$

We fix a constant $C > 2\sqrt{N}\max_i L_i$ and, proceeding similarly as in Section 5, we can find some point $\varepsilon k_\varepsilon L := \varepsilon(k_{1,\varepsilon}L_1, \ldots, k_{N,\varepsilon}L_N)$, where $k_{i,\varepsilon} \in \mathbb{Z}$ for all $1 \leq i \leq N$, and such that

$$x_0 - \frac{n}{\gamma} \in \varepsilon k_\varepsilon L + B_{C\varepsilon}. \tag{24}$$

Then

$$\forall x \in B_{\frac{1}{\gamma}-C\varepsilon}(\varepsilon k_\varepsilon L), \quad u^\varepsilon(t_0,x) \geq 1-\eta. \tag{25}$$

This leads us to study the solution $u(t,x)$ of (1) with initial datum

$$u_0(x) := (1-\eta) \times \chi_{B_{\frac{1}{\gamma\varepsilon}-C}}(x), \tag{26}$$

where B_r denotes the open ball centered at the origin and of radius r. Note that this initial datum has compact support, so that Theorem 2.4 does not apply. In fact, the solution $u(t,x)$ does not spread with speed $c^*(n)$ in the n-direction as $t \to +\infty$, but rather with some minimum of the $\frac{c^*(n')}{n\cdot n'}$ over all $n' \in \mathbb{S}^{N-1}$. However, as the radius of the initial support is very large, we can exhibit some transient dynamics where the solution does spread, in any direction n, with speed $c^*(n)$ the minimal speed of pulsating traveling waves. Let us make this sketch precise.

We first note that, provided that ε is small depending only on C and γ, the finite cylinder

$$D_0 := \left\{ x \in \mathbb{R}^N \ : \ |x\cdot n| \leq \frac{1}{\gamma\varepsilon} - 2C \text{ and } |x \cdot n^\perp| \leq \sqrt{\frac{C}{\gamma\varepsilon}} \right\}$$

is a subset of $B_{\frac{1}{\gamma\varepsilon}-C}$ thanks to Pythagoras' theorem. In order to apply Theorem 2.4, which is concerned with planar-shaped initial data, it is more convenient to consider a box-shaped initial support. With this in mind, we introduce $(n_1,\dots,n_{N-1})$ an orthonormalized basis of $(\mathbb{R}n)^\perp$, and define the finite box

$$D_1 := \left\{ x \in \mathbb{R}^N \ : \ |x\cdot n| \leq \frac{1}{\gamma\varepsilon} - 2C \text{ and } \ \forall 1 \leq i \leq N-1, \ |x\cdot n_i| \leq \sqrt{\frac{C}{(N-1)\gamma\varepsilon}} \right\},$$

which is a subset of D_0.

We can now begin our investigation of the spreading of u, the solution of (1) with initial datum (26). By the parabolic comparison principle, we have

$$u \geq \underline{u},$$

where $\underline{u}$ is the solution of (1) with initial datum

$$\underline{u}_0(x) := (1-\eta) \times \chi_{D_1}(x).$$

We let $\tilde{u}(t,x;n)$ denote the solution of (1) with initial datum

$$\tilde{u}_0(x;n) := (1-\eta) \times \chi_{\{x\cdot n \leq \frac{1}{\gamma\varepsilon}-2C\}}(x),$$

which is planar-shaped so that $\tilde{u}(t,x;n)$ spreads in the direction n with speed $c^*(n)$. Precisely, recalling $C > 2\sqrt{N}\max_i L_i$, we can find some point $\tilde{k}_\varepsilon L := (\tilde{k}_{1,\varepsilon}L_1,\dots,\tilde{k}_{N,\varepsilon}L_N)$, where $\tilde{k}_{i,\varepsilon} \in \mathbb{Z}$ for all $1 \leq i \leq N$, and such that $\frac{n}{\gamma\varepsilon} \in B_C(\tilde{k}_\varepsilon L)$. Then observe that

$$\tilde{v}_0(x;n) := \tilde{u}_0(x + \tilde{k}_\varepsilon L; n) \geq (1-\eta) \times \chi_{\{x\cdot n \leq -3C\}}(x).$$

We can now apply Theorem 2.4 with the family of functions in the right-hand side member above (which do not depend on ε) as the family of initial data. Then, applying the comparison principle, we get that there exists $\tau > 0$ (which does not depend on ε) such that

$$\inf_{t\geq\tau} \inf_{x\cdot n\leq (c^*(n)-\frac{1}{4}\alpha)t} \tilde{v}(t,x;n) \geq 1-\frac{\eta}{2},$$

where $\tilde{v}(t,x;n)$ denotes the solution of (1) with initial datum $\tilde{v}_0(x;n)$. Then, since $\tilde{v}(t,x;n) = \tilde{u}(t,x+\tilde{k}_\varepsilon L;n)$ thanks to the spatial periodicity, the above estimate implies

$$\inf_{t\geq\tau} \inf_{x\cdot n\leq \frac{1}{\gamma\varepsilon}-3C+(c^*(n)-\frac{1}{4}\alpha)t} \tilde{u}(t,x;n) \geq 1-\frac{\eta}{2}. \tag{27}$$

We emphasize that $\tau > 0$ can also be chosen independently of $n \in \mathbb{S}^{N-1}$: this is the exact purpose of our improvement of Weinberger's spreading result [32], namely Theorem 2.4.

We now estimate the difference $w := \tilde{u} - \underline{u} \geq 0$, which satisfies $\partial_t w - \Delta w - g(t,x)w = 0$, where

$$g(t,x) := \begin{cases} \dfrac{f(x,\tilde{u}) - f(x,\underline{u})}{\tilde{u}-\underline{u}} & \text{if } w(t,x) \neq 0, \\ \partial_u f(x,\tilde{u}) & \text{if } w(t,x) = 0. \end{cases}$$

From Assumption 1.1, $g(t,x)$ is uniformly bounded by some K which only depends on f. Then w satisfies

$$\partial_t w - \Delta w - Kw \leq 0. \tag{28}$$

As this parabolic equation is linear, we infer that $w(t,x) \leq \sum_{i=0}^{2N-2} w_i(t,x)$, where w_0 is the solution of (28) with initial datum

$$w_0(0,x) = \begin{cases} 1-\eta & \text{if } x\cdot n \leq -\frac{1}{\gamma\varepsilon}+2C, \\ 0 & \text{otherwise,} \end{cases}$$

and the w_{2i-1} and w_{2i}'s, $1 \leq i \leq N-1$, are the solutions of (28) with initial data

$$w_{2i-1}(0,x) = \begin{cases} 1-\eta & \text{if } x\cdot n \leq \frac{1}{\gamma\varepsilon}-2C \text{ and } x\cdot n_i \geq \sqrt{\frac{C}{(N-1)\gamma\varepsilon}}, \\ 0 & \text{otherwise,} \end{cases}$$

$$w_{2i}(0,x) = \begin{cases} 1-\eta & \text{if } x\cdot n \leq \frac{1}{\gamma\varepsilon}-2C \text{ and } x\cdot n_i \leq -\sqrt{\frac{C}{(N-1)\gamma\varepsilon}}, \\ 0 & \text{otherwise.} \end{cases}$$

Note that, for any $e \in \mathbb{S}^{N-1}$ and any positive constant M, $(t,x) \mapsto Me^{-\sqrt{K}(x\cdot e-2\sqrt{K}t)}$ is a supersolution of the linear equation (28). It therefore follows that

$$w_0(t,x) \leq e^{-\sqrt{K}(x\cdot n+\frac{1}{\gamma\varepsilon}-2C-2\sqrt{K}t)},$$

and, for any integer $1 \leq i \leq N-1$,

$$w_{2i-1}(t,x) \leq e^{\sqrt{K}(x\cdot n_i - \sqrt{\frac{C}{(N-1)\gamma\varepsilon}} + 2\sqrt{K}t)},$$

$$w_{2i}(t,x) \leq e^{-\sqrt{K}(x\cdot n_i + \sqrt{\frac{C}{(N-1)\gamma\varepsilon}} - 2\sqrt{K}t)}.$$

Then, we conclude that

$$0 \leq (\tilde{u} - \underline{u})\left(\frac{A_1}{\sqrt{\varepsilon}}, x\right) = w\left(\frac{A_1}{\sqrt{\varepsilon}}, x\right) \leq \sum_{i=0}^{2N-2} w_i\left(\frac{A_1}{\sqrt{\varepsilon}}, x\right) \leq \frac{\eta}{2}, \tag{29}$$

where

$$A_1 := \frac{1}{4}\sqrt{\frac{C}{K(N-1)\gamma}},$$

for all x satisfying the two following inequalities:

$$x \cdot n \geq -\frac{1}{\gamma\varepsilon} + 2C + 2A_1\sqrt{\frac{K}{\varepsilon}} - \frac{1}{\sqrt{K}}\ln\left(\frac{\eta}{4N}\right) = -\frac{1}{\gamma\varepsilon} + O\left(\frac{1}{\sqrt{\varepsilon}}\right), \tag{30}$$

$$|x\cdot n_i| \leq \sqrt{\frac{C}{(N-1)\gamma\varepsilon}} - 2A_1\sqrt{\frac{K}{\varepsilon}} + \frac{1}{\sqrt{K}}\ln\left(\frac{\eta}{4N}\right) = \frac{1}{2}\sqrt{\frac{C}{(N-1)\gamma\varepsilon}} + O(1),$$

for $1 \leq i \leq N-1$. The second inequality is in particular satisfied, for $\varepsilon > 0$ small enough, if

$$|x\cdot n^{\perp}| \leq \frac{1}{3}\sqrt{\frac{C}{(N-1)\gamma\varepsilon}} =: \frac{A_2}{\sqrt{\varepsilon}}. \tag{31}$$

Combining the spreading property (27) of $\tilde{u}$ and inequality (29), we conclude that

$$\underline{u}(t_\varepsilon, x) \geq 1-\eta, \quad t_\varepsilon := \frac{A_1}{\sqrt{\varepsilon}}, \tag{32}$$

for any x satisfying both inequalities (30) and (31), as well as

$$x\cdot n \leq \frac{1}{\gamma\varepsilon} - 3C + (c^*(n) - \frac{1}{4}\alpha)t_\varepsilon. \tag{33}$$

We can now go back to our original problem (P^ε). Notice that both $\underline{u}(\frac{t}{\varepsilon}, \frac{x}{\varepsilon})$ and $u^\varepsilon(t_0 + t, \varepsilon k_\varepsilon L + x)$ solve the equation in (P^ε). Using $D_1 \subset B_{\frac{1}{\gamma\varepsilon} - C}$ and (25), we see that $\underline{u}(0, \frac{x}{\varepsilon}) \leq u^\varepsilon(t_0, \varepsilon k_\varepsilon L + x)$ so that

$$u^\varepsilon(t_0 + t, \varepsilon k_\varepsilon L + x) \geq \underline{u}\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right),$$

where $\varepsilon k_\varepsilon L$ satisfies (24). Thus, we get

$$u^\varepsilon(t_0 + \varepsilon t_\varepsilon, x) \geq \underline{u}\left(t_\varepsilon, \frac{x - \varepsilon k_\varepsilon L}{\varepsilon}\right) \geq 1-\eta, \tag{34}$$

provided that $\frac{x-\varepsilon k_\varepsilon L}{\varepsilon}$ satisfies (30), (31), (33) (so that (32) holds). Now, assume that x satisfies (22). Combining the first part of (22) and (24), we see that $\frac{x-\varepsilon k_\varepsilon L}{\varepsilon}$ satisfies both (30) and (33). Combining the second part of (22), $n\cdot n^\perp=0$ and (24), we see that $\frac{x-\varepsilon k_\varepsilon L}{\varepsilon}$ satisfies (31). Hence, (34) holds true and is the desired conclusion that $u^\varepsilon(t_0+A_1\sqrt{\varepsilon},x)\geq 1-\eta$.

Note that, as announced, the constants A_1 and A_2 defined above depend neither on $t_0\in(0,T)$, $x_0\in\Gamma^\alpha_{t_0}$ and the associated unit outer normal n, nor on $\varepsilon>0$. The lemma is proved. □

Remark 6.2. Let us notice that Lemma 6.1 shares some ideas with the so-called consistency assumption (H4) of Barles and Souganidis [8]. Roughly speaking, their method consists in reducing the study of the sharp interface limit to compact and smooth shapes as well as to small times, that is to consistency. In a heterogeneous and bistable context, they then proved consistency under the additional assumption that the traveling wave (which in such case is unique) depends regularly on its direction. However, such a property is far from trivial, especially in the monostable case. We therefore adopt a different approach, relying on the uniform spreading properties proved in our earlier work [4], namely Theorem 2.4.

6.2. Lower estimates for propagation of the layers

We now complete our argument by combining an iteration method and Lemma 5.2.

Proof of statement (i) of Theorem 3.1. We need to show that, for $\varepsilon>0$ small enough, we have $u^\varepsilon(t,x)\geq 1-\eta$, for all $\tau\leq t\leq T$ and all x such that $d(t,x)\leq-\beta$ (recall that $d(t,\cdot)$ denotes the signed distance function to Γ_t, negative in Ω_t).

Recalling that $\Gamma^\alpha_0\subset\Omega_0$ and $\alpha\leq d_{\mathscr{H}}(\Gamma^\alpha_0,\Gamma_0)\leq 2\alpha$ (see Proposition 4.3), it follows from Proposition 5.1 that, for $\varepsilon>0$ small enough, assumption (21) of Lemma 6.1 is satisfied at time $t_0=t_\alpha\varepsilon<\tau$. As a result

$$u^\varepsilon(t_1,x)\geq 1-\eta,\quad t_1:=t_0+A_1\sqrt{\varepsilon}, \tag{35}$$

for any $x\in D$ defined as in Lemma 6.1. Moreover, (35) also holds true if $x\in\Omega^\alpha_{t_0}$ in virtue of Lemma 5.2 (notice that the needed time to reach $1-\eta$ in Lemma 5.2 is of order ε), with Γ^α_t, Ω^α_t playing the roles of $\tilde\Gamma_t$, $\tilde\Omega_t$. Therefore, it follows from the claim

$$\Omega^\alpha_{t_1}\subset D\cup\Omega^\alpha_{t_0} \tag{36}$$

(whose proof is postponed), that

$$\forall x\in\Omega^\alpha_{t_1},\quad u^\varepsilon(t_1,x)\geq 1-\eta.$$

Proceeding by induction, we conclude that for all times

$$t_k:=t_0+kA_1\sqrt{\varepsilon},$$

up to some k such that $T < t_k < T+1$, we have

$$u^\varepsilon(t_k,x) \geq 1-\eta \quad \text{for all } \ x \in \Omega^\alpha_{t_k}.$$

In particular it follows from (20) that $u^\varepsilon(t_k,x) \geq 1-\eta$ for any x such that $d(t_k,x) \leq -\beta$.

It now only remains to consider intermediate times. Notice that, even though we stated in Lemma 5.2 that the solution u^ε may only expand, this is in fact only true when looking at interval of times of order larger than ε. Therefore, the above inequality does not guarantee that $d(t,x) \leq -\beta \Rightarrow u^\varepsilon(t,x) \geq 1-\eta$ in intervals of time $[t_k, t_k + \mathcal{O}(\varepsilon)]$. To avoid this difficulty, we can nevertheless note that for all $t \in [t_k, t_{k+1})$ with $k \geq 1$,

$$u^\varepsilon(t,x) \geq 1-\eta \quad \text{for all } \ x \in \Omega^\alpha_{t_{k-1}}. \tag{37}$$

Note that up to reducing ε, we can assume that $t_1 < \tau$. Let now any $t \in [\tau, T]$, and $k \geq 1$ be such that $t \in [t_k, t_{k+1})$. Let also $x \in \Omega_t$ be such that $d(t,x) \leq -\beta$. Notice that it follows from Proposition 2.5 that there is $C > 0$ such that $d_{\mathcal{H}}(\Gamma_t, \Gamma_{t_{k-1}}) \leq C(t - t_{k-1}) \leq 2A_1 C\sqrt{\varepsilon}$. Recall also that $\alpha > 0$ was chosen such that (20) holds, so that $d_{\mathcal{H}}(\Gamma_t, \Gamma^\alpha_t) \leq \frac{\beta}{2}$, for all $\tau \leq t \leq T+1$. As a result

$$d_{\mathcal{H}}(\Gamma_t, \Gamma_{t^\alpha_{k-1}}) \leq 2A_1 C\sqrt{\varepsilon} + \frac{\beta}{2} < \beta,$$

for $\varepsilon > 0$ small enough. Since $d(t,x) \leq -\beta$, this enforces $x \in \Omega^\alpha_{t_{k-1}}$ and, by (37), we get that $u^\varepsilon(t,x) \geq 1-\eta$. This concludes the proof of the lower estimates on the motion of the layers of $u^\varepsilon(t,x)$. □

Proof of claim (36). Recall that (see Proposition 4.3) Γ^α_t is the zero level set of $v^\alpha(t,\cdot)$, where

$$\partial_t v^\alpha + |\nabla v^\alpha| \left(c^* \left(\frac{\nabla v^\alpha}{|\nabla v^\alpha|}) \right) - \alpha \right) \geq 0. \tag{38}$$

To prove the claim (36), consider any $x \in \Omega^\alpha_{t_1} \setminus \Omega^\alpha_{t_0}$, and let us prove that $x \in D$. First, there exists some $x_0 \in \Gamma^\alpha_{t_0}$ such that $|x - x_0| = dist(x, \Gamma^\alpha_{t_0})$ and, by convexity, such an x_0 is unique. Moreover,

$$n = \frac{x - x_0}{|x - x_0|} = \frac{\nabla v^\alpha(t_0, x_0)}{|\nabla v^\alpha(t_0, x_0)|}$$

is, by construction, the unit outer normal of $\Gamma^\alpha_{t_0}$ at point x_0 (the first equality follows from the choice of x_0, and the second from the definition of Γ^α_t as the zero level set of $v^\alpha(t,\cdot)$).

In order to prove that $x \in D$, it only remains to check the inequality

$$|(x - x_0) \cdot n| = |x - x_0| \leq A_1 \left(c^*(n) - \frac{\alpha}{2} \right) \sqrt{\varepsilon}.$$

Note that, by convexity of v^α,

$$v^\alpha(t_0, x) \geq v^\alpha(t_0, x_0) + \nabla v^\alpha(t_0, x_0) \cdot (x - x_0),$$

and also that, thanks to the smoothness of v^α,

$$v^\alpha(t_1,x)-v^\alpha(t_0,x)\geq \partial_t v^\alpha(t_0,x)(t_1-t_0)-K|t_1-t_0|^2,$$

where K is a positive constant (recall that $\alpha>0$ has been fixed). Since $x\in\Omega^\alpha_{t_1}$ we have $v^\alpha(t_1,x)<0$ and since $x_0\in\Gamma^\alpha_{t_0}$ we have $v^\alpha(t_0,x_0)=0$. As a result, up to increasing K if necessary,

$$\begin{aligned}0&\geq v^\alpha(t_1,x)-v^\alpha(t_0,x_0)\\&\geq \nabla v^\alpha(t_0,x_0)\cdot(x-x_0)+\partial_t v^\alpha(t_0,x)(t_1-t_0)-K|t_1-t_0|^2\\&\geq \nabla v^\alpha(t_0,x_0)\cdot(x-x_0)+(\partial_t v^\alpha(t_0,x_0)-K|x-x_0|)(t_1-t_0)-K|t_1-t_0|^2.\end{aligned}$$

Using (38), we deduce that

$$0\geq|\nabla v^\alpha(t_0,x_0)|\times|x-x_0|-|\nabla v^\alpha(t_0,x_0)|\left(c^*(n)-\alpha\right)(t_1-t_0)-K[|t_1-t_0|\times|x-x_0|+|t_1-t_0|^2].$$

Recalling that ∇v^α does not cancel on Γ^α_t, we can infer by compactness that

$$\rho:=\inf_{0\leq t\leq T}\inf_{x\in\Gamma^\alpha_t}|\nabla v^\alpha(t,x)|>0.$$

Recalling also that $t_1-t_0=A_1\sqrt{\varepsilon}$, it follows from the above that

$$\begin{aligned}|x-x_0|&\leq\frac{|\nabla v^\alpha(t_0,x_0)|}{|\nabla v^\alpha(t_0,x_0)|-KA_1\sqrt{\varepsilon}}(c^*(n)-\alpha)A_1\sqrt{\varepsilon}+\frac{KA_1^2\varepsilon}{\rho-KA_1\sqrt{\varepsilon}}\\&\leq\left(c^*(n)-\frac{\alpha}{2}\right)A_1\sqrt{\varepsilon},\end{aligned}$$

provided $\varepsilon>0$ is small enough. As announced, $x\in D$ and the claim is proved. □

7. Control of the layers from above

In this section, we prove the upper estimate on the motion of level sets of the solutions $u^\varepsilon(t,x)$, namely statement (ii) of Theorem 3.1.

To do so, we are going to construct a family of planar supersolutions (indexed by $y\in\Gamma_0$) for (P^ε), whose envelop is close to the zero level sets of $v(t,\cdot)$, that is Γ_t in virtue of Proposition 2.5. Then, for the sake of clarity, rather than using the uniform upper spreading speed (7), we instead use some kind of uniform asymptotics of the monostable minimal waves — which is proved in [4] and actually implies (7).

Lemma 7.1 (Uniform asymptotics for critical waves, [4])**.** *Let $u^*(t,x;n)=U^*(x\cdot n-c^*(n)t,x;n)$ be a family of increasing in time pulsating traveling waves of (1), with minimal speed $c^*(n)$ in each direction $n\in\mathbb{S}^{N-1}$, shifted so that $U^*(0,0;n)=\frac{1}{2}$.*

Then, the asymptotics $U^*(-\infty,x;n)=1$, $U^*(\infty,x;n)=0$ (which are uniform with respect to $x\in\mathbb{R}^N$) are uniform with respect to $n\in\mathbb{S}^{N-1}$.

Proof of statement (ii) of Theorem 3.1. Let $0 < T$ and a small $\beta > 0$ be given. For any $n \in \mathbb{S}^{N-1}$, denote by $U^*(z, x; n)$ a monostable pulsating front with minimal speed $c^*(n)$ in the direction n, shifted so that $U^*(0, 0; n) = \frac{1}{2}$.

Thanks to $\|g\|_\infty < 1$ (see Assumption 1.2) and the above lemma, we can select some $K > 0$ large enough so that

$$U^*(z, x; n) \geq \|g\|_\infty, \quad \forall z \leq -K, \forall x \in \mathbb{R}^N, \forall n \in \mathbb{S}^{N-1}. \tag{39}$$

Then, for any $y \in \Gamma_0$ and denoting again by n_y the outward unit normal of Γ_0 at point y, we define

$$\overline{u}(t, x) := U^*\left(\frac{(x - y) \cdot n_y - c^*(n_y)t}{\varepsilon} - K, \frac{x}{\varepsilon}, n_y\right).$$

From equation (5) for the traveling front, we deduce that $\overline{u}(t, x)$ solves the parabolic equation in (P^ε). We also have $u^\varepsilon(0, x) = g(x) \leq \overline{u}(0, x)$: indeed, for $x \notin \Omega_0$ we have $g(x) = 0$, whereas for $x \in \Omega_0$ we have $(x - y) \cdot n_y \leq 0$ by convexity and (39) gives the desired ordering. The comparison principle then implies $u^\varepsilon(t, x) \leq \overline{u}(t, x)$. As a result

$$0 \leq u^\varepsilon(t, x) \leq \inf_{y \in \Gamma_0} U^*\left(\frac{(x - y) \cdot n_y - c^*(n_y)t}{\varepsilon} - K, \frac{x}{\varepsilon}; n_y\right). \tag{40}$$

We recall that $d(t, \cdot)$ denotes the signed distance to the set Γ_t, which is chosen to be negative in Ω_t and positive in $\mathbb{R}^N \setminus (\Gamma_t \cup \Omega_t)$. Let us now prove that there is some $\theta > 0$ such that, for any $t \in [0, T]$ and any x such that $d(t, x) \geq \beta$, then

$$\exists y \in \Gamma_0, \quad (x - y) \cdot n_y - c^*(n_y)t \geq \theta\beta. \tag{41}$$

Assume by contradiction that there are some sequences $(t_k)_{k \geq 1}$, $(x_k)_{k \geq 1}$ as above such that

$$\forall y \in \Gamma_0, \quad (x_k - y) \cdot n_y - c^*(n_y)t_k \leq \frac{\beta}{k}.$$

This enforces the sequence $(x_k)_{k \geq 1}$ to be bounded so that, after extraction of some subsequences, we are equipped with some $t_\infty \in [0, T]$, some x_∞ with $d(t_\infty, x_\infty) \geq \beta > 0$, such that

$$\forall y \in \Gamma_0, \quad (x_\infty - y) \cdot n_y - c^*(n_y)t_\infty \leq 0.$$

Thus $v(t_\infty, x_\infty) \leq 0$, which contradicts $d(t_\infty, x_\infty) \geq \beta$.

Let us now choose any $t \in [0, T]$, any x such that $d(t, x) \geq \beta$. In view of (41), we can select some $y_0 \in \Gamma_0$ such that $(x - y_0) \cdot n_{y_0} - c^*(n_{y_0})t \geq \theta\beta$. Then, using (40) and the monotonicity of the pulsating traveling wave $U^*(z, x; n)$ with respect to its first variable, we get

$$0 \leq u^\varepsilon(t, x) \leq U^*\left(\frac{\theta\beta}{\varepsilon} - K, \frac{x}{\varepsilon}; n_{y_0}\right) \leq \sup_{n \in \mathbb{S}^{N-1}} \sup_{X \in \mathbb{R}^N} U^*\left(\frac{\theta\beta}{\varepsilon} - K, X; n\right).$$

Thanks to Lemma 7.1, this implies that $\sup_{0 \leq t \leq T} \sup_{\{x : d(t,x) \geq \beta\}} |u^\varepsilon(t, x)| \to 0$ as $\varepsilon \to 0$, which concludes the proof of Theorem 3.1. $\square$

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