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FOLDING ON THE CHAOTIC CARTESIAN PRODUCT OF MANIFOLDS AND THEIR FUNDAMENTAL GROUP

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Abstract. In this paper we introduce the chaotic fundamental group of foldings of the chaotic Cartesian product of manifolds into itself. Also the fundamental group of the limit of foldings of the chaotic Cartesian product of manifolds into itself are deduced. The effect of folding on the wedge sum of chaotic manifolds and their chaotic fundamental group will be achieved. Some types of conditional foldings restricted on the elements of a free chaotic group and their chaotic fundamental groups are presented. Theorems governing these relations are obtained.

1. Introduction

Chaos theory is the branch of mathematics for the study of processes that seem so complex that at first they do not appear to be governed by any known laws or principles, but which actually have an underlying order that can be described by vector calculus and its associated geometry. Examples of chaotic processes include a stream of rising smoke that breaks down and becomes trubulent, water flowing in a stream or crashing at the bottom of a waterfall, electroencephalographic activity of the brain, changes in animal populations, fluctuations on the stock exchange, and the weather (either local or global). All of these phenomena involve the interaction of several elements and the pattern of their changes over time.

The rate of change of each of the variables or elements involved depends on the other variables, and the rules of the rate of change must be nonlinear for the chaotic temporal patterns to occur. When basic processes of systems are connected interactively, they are called "dynamical systems", which is the parent branch of mathematics of which chaos theory is a sub discipline.

Classical chaos theory deals with a calculus of infinite duration and resolution which, of course, may or may not exist in the actual world, but is beyond the resolution of our knowledge of the actual world. Thus, in the mathematical models of chaos one encounters "sensitivity to initial conditions" where even the smallest difference in initial conditions can lead to a large difference in position later on within a chaotic attractor. Therefore, since our knowledge of initial conditions is never exact but bound to inexact observation, our prediction into the future is limited, more so the further into the future we try to predict. Until recently, it was presumed that chaotic systems, like classical linear systems, tended toward stable equilibrium

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(fixed point) or period attractors and that the erratic behavior found in actuality resulted from unidetified variables not yet detected [12].

The folding of a manifold was, firstly introduced by Robertson 1977 [13]. More studies on the folding of many types of manifolds were studied in [3, 4, 5, 6, 14]. The unfolding of a manifold introduced in [2]. Some application of the folding of a manifold discussed in [1]. The fundamental groups of some types of a manifold are discussed in [8, 9, 11].

2. Definitions

1- The set of homotopy classes of loops based at the point $x_\circ$ with the product operation $[f][g] = [f \cdot g]$ is called the fundamental group and denoted by $\pi_1(X, x_\circ)$[9].

2- Let M and N be two manifolds of dimension m and n respectively. A map $f : M \to N$ is said to be an isometric folding of M into N if for every piecewise geodesic path $\gamma : I \to M$ the induced path $f \circ \gamma : I \to N$ is piecewise geodesic and of the same length as γ [13]. If f does not preserve length it is called topological folding [7].

3- Let M and N be two manifolds of the same dimension. A map $g : M \to N$ is said to be unfolding of M into N if every piecewise geodesic path $\gamma : I \to M$, the induced path $g \circ \gamma : I \to N$ is piecewise geodesic with length greater than γ [2].

4- Given spaces X and Y with chosen points $x_0 \in X$ and $y_0 \in Y$, then the wedge sum $X \vee Y$ is the quotient of the disjoint union $X \cup Y$ obtained identifying x_0 and y_0 to a single point [9].

3. Main results

Aiming to our study we will introduce the following definitions.

Definition 1. The chaotic space is a space carries many physical characters each one of them homeomorphic to the original one either with fixed point or without fixed point as in Figure 1.

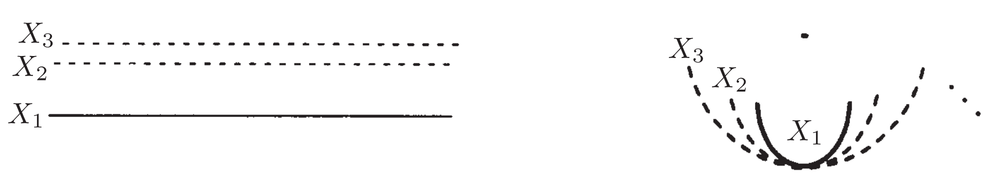

Figure 1.

For types of chaotic space we have two cases.
Case 1: The chaotic space wih homogeneous properties to each homeomorphic space and we denoted this type by $\overset{\text{ch}}{X}$ and $\overset{\text{ch}}{X} = (X_0, X_1, \ldots, X_\infty)$.

Case 2: The chaotic space with non homogeneous properties to each homeomorphic space and we denoted this type by $\overline{\overset{\text{ch}}{X}} = (\overset{\text{ch}_i}{X}_k,\ k = 0, 1, \ldots, \infty,\ i = 1, 2, \ldots, n)$ where n is the number of physical characters to each homeomorphic space $\overset{\text{ch}_i}{X}_k$ and $(\overset{\text{ch}_0}{X}_k, \overset{\text{ch}_1}{X}_k, \ldots, \overset{\text{ch}_n}{X}_k)$ represents all physical characters to k-homeomorphic space $\overset{\text{ch}_i}{X}_k$.

Definition 2. The chaotic loop is a geometric loop carries many other loops which are homotopic to each others, as in Figure 2 and the chaotic $\alpha^{\text{ch}} = (a_{n_0}, a_{n_1}, \ldots, \alpha_{n_\infty})$ can be represented as $\overset{\text{ch}}{\alpha}$ and the chaotic loop $\overset{\text{ch}}{x} = (x_{n_0}, x_{n_1}, \ldots, x_{n_\infty})$ represented as $\overset{\text{ch}}{x}$ base point.
For types of chaotic loops we have two cases: Case 1: All loops are of the same physical character $\alpha_{n_0}, \alpha_{n_1}, \ldots, \alpha_{n_\infty}$. Case 2: All loops represents different physical characters for example $\alpha_{n_0}, \alpha_{n_1}, \ldots, \alpha_{n_\infty}$. α_{n_0} represent magnetic field, α_{n_1} represent colors α_{n_2} represent electricity.

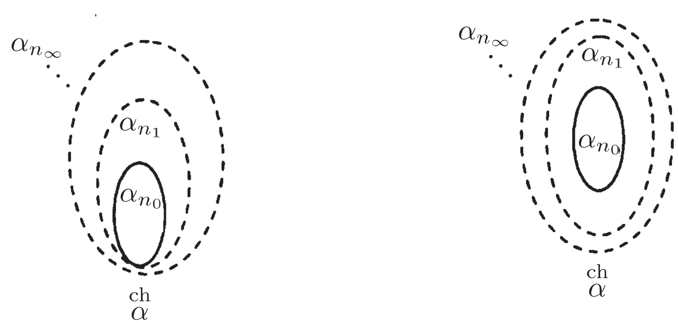

Figure 2.

Definition 3. The chaotic group $\overset{\text{ch}}{G}$ is a group with chaotic elements i.e. the chaotic elements g^{ch} can be represented as $g^{\text{ch}} = (g_{n_0}, g_{n_1}, \ldots, g_{n_\infty})$.

Definition 4. The chaotic fundamental group in chaotic space $\overset{\text{ch}}{X}$ at the chaotic base point $\overset{\text{ch}}{x}$ is the set of homotopy classes of chaotic loops with the product operation $[f][g] = [f \cdot g]$ and denoted by $\overset{\text{ch}}{\pi}_1(\overset{\text{ch}}{X}, \overset{\text{ch}}{x})$. Also, the chaotic fundamental group in the chaotic space $\overline{\overset{\text{ch}}{X}}$ depends on the base point to each physical character of $\overset{\text{ch}_i}{X}_k$ and so we can represented as $\pi_1(\overset{\text{ch}_i}{X}_k, \overset{\text{ch}_i}{x}_k)$, $k = 0, 1, \ldots, \infty$, $i = 1, 2, \ldots, n$.

Example 1. $\overset{\text{ch}}{\pi}_1(\overset{\text{ch}}{S^1}, \overset{\text{ch}}{x}) \approx \overset{\text{ch}}{Z}$, $\overset{\text{ch}}{\pi}_1(\overset{\text{ch}}{S^n}, \overset{\text{ch}}{x}) \approx \overset{\text{ch}}{0}$ (*chaotic identity group*) for $n \geq 2$, $\overset{\text{ch}}{\pi}_1(\overset{\text{ch}}{R^n}, \overset{\text{ch}}{x}) \approx \overset{\text{ch}}{0}$, $n \geq 1$ for homogenous space. Now, for non homogenous space $\pi_1((\overset{\text{ch}_i}{S^1_k}, \overset{\text{ch}_i}{x}_k)$ is either identity group or isomorphic to Z and $\overset{\text{ch}_i}{\pi}_1(\overset{\text{ch}_i}{R^n_k}, \overset{\text{ch}_i}{x}_k)$ is a free group of rank n for some n or identity group for $k = 0,1,\dots,\infty$, $i = 1,2,\dots,n$) which depends on the number of the holes for each physicals characters.

Theorem 1. *If* $\overset{\text{ch}}{M}_1, \overset{\text{ch}}{M}_2, \dots, \overset{\text{ch}}{M}_n$ *are path connected chaotic manifolds and* F *is a folding from* $\overset{n}{\underset{i=1}{\vee}} \overset{\text{ch}_i}{M}_i$ *into itself then there is an induced folding* $\overline{F}$ *of* $\overset{n}{\underset{i=1}{*}} \pi_1((\overset{\text{ch}}{M}_i)$ *into itself which reduce the degree of* $\overset{n}{\underset{i=1}{*}} \pi_1((\overset{\text{ch}}{M}_i)$.

Proof. Let $F : \overset{n}{\underset{i=1}{\vee}} \overset{\text{ch}}{M}_i \longrightarrow \overset{n}{\underset{i=1}{\vee}} \overset{\text{ch}}{M}_i$ be folding on $\overset{n}{\underset{i=1}{\vee}} \overset{\text{ch}}{M}$ into itself, then $F : \overset{n}{\underset{i=1}{\vee}} \overset{\text{ch}}{M}_i \longrightarrow \overset{n}{\underset{i=1}{\vee}} \overset{\text{ch}}{M}_i$ has the following forms:
If $F(\overset{n}{\underset{i=1}{\vee}} \overset{\text{ch}}{M}_i) = \overset{\text{ch}}{M}_1 \vee \overset{\text{ch}}{M}_2 \vee \cdots \vee F(\overset{\text{ch}}{M}_s) \vee \cdots \vee \overset{\text{ch}}{M}_n$ for $s = 1,2,\dots,n$, then
$\overline{F}(\overset{n}{\underset{i=1}{*}} \pi_1(\overset{\text{ch}}{M}_i)) = \pi_1(F(\overset{n}{\underset{i=1}{\vee}} \overset{\text{ch}}{M}_i)) \approx \pi_1(\overset{\text{ch}}{M}_1) * \pi_1(\overset{\text{ch}}{M}_2) * \cdots * \pi_1(F(\overset{\text{ch}}{M}_s)) * \cdots * \pi_1(\overset{\text{ch}}{M}_n)$.
Since degree $(\pi_1(F(\overset{\text{ch}}{M}_s))) \le$ degree $(\pi_1(\overset{\text{ch}}{M}_s))$ it follows that $\overline{F}$ reduce the degree of $\overset{n}{\underset{i=1}{*}} \pi_1((\overset{\text{ch}}{M}_i)$.
Also, if $F(\overset{n}{\underset{i=1}{\vee}} \overset{\text{ch}}{M}_i) = \overset{\text{ch}}{M}_1 \vee \overset{\text{ch}}{M}_2 \vee \cdots \vee F(\overset{\text{ch}}{M}_s) \vee \cdots \vee F(\overset{\text{ch}}{M}_k) \vee \cdots \vee \overset{\text{ch}}{M}_n$
for $k = 1,2,\dots,n$, $s < k$.
Then $\overline{F}(\overset{n}{\underset{i=1}{*}} \pi_1((\overset{\text{ch}}{M}_i)) =$
$\pi_1(F(\overset{n}{\underset{i=1}{\vee}} \overset{\text{ch}}{M}_i)) \approx \pi_1(\overset{\text{ch}}{M}_1) * \pi_1(\overset{\text{ch}}{M}_2) * \cdots * \pi_1(F(\overset{\text{ch}}{M}_s)) * \cdots * \pi_1(F(\overset{\text{ch}}{M}_k)) * \cdots * \pi_1(\overset{\text{ch}}{M}_n)$ and so $\overline{F}$ reduce the degree of $\overset{n}{\underset{i=1}{*}} \pi_1(\overset{\text{ch}}{M}_i)$.
Moreover, by continuing this process if $F(\overset{n}{\underset{i=1}{\vee}} \overset{\text{ch}}{M}_i) = \overset{n}{\underset{i=1}{\vee}} F(\overset{\text{ch}}{M}_i)$.
Then $\overline{F}(\overset{n}{\underset{i=1}{*}} \pi_1(\overset{\text{ch}}{M}_i)) = \pi_1(F(\overset{n}{\underset{i=1}{\vee}} \overset{\text{ch}}{M}_i)) = \pi_1(\overset{n}{\underset{i=1}{\vee}} F(\overset{\text{ch}}{M}_i)) \approx \overset{n}{\underset{i=1}{*}} \pi_1(F(\overset{\text{ch}}{M}_i))$. Hence $\overline{F}$ reduce the degree of $\overset{n}{\underset{i=1}{*}} \pi_1((\overset{\text{ch}}{M}_i)$.

Theorem 2. *For every* $k \le n$, *there is a folding* F_k *of* $\overset{n}{\underset{i=1}{\vee}} \overset{\text{ch}}{S^1_i}$ *into itself which induces a folding* $\overline{F_k}$ *of* $\overset{n}{\underset{i=1}{*}} \pi_1(\overset{\text{ch}}{S^1_i})$ *into itself such that* $\overline{F}_k(\overset{n}{\underset{i=1}{*}} \pi_1(\overset{\text{ch}}{S^1_i}))$ *is a free chaotic group of rank* $n - k$.

Proof. Let $F_1 : \overset{n}{\underset{i=1}{\vee}} \overset{\text{ch}}{S^1_i} \longrightarrow \overset{n}{\underset{i=1}{\vee}} \overset{\text{ch}}{S^1_i}$ be folding such that
$F_1(\overset{n}{\underset{i=1}{\vee}} \overset{\text{ch}}{S^1_i}) = \overset{\text{ch}}{S^1_1} \vee \overset{\text{ch}}{S^1_2} \vee \cdots \vee F_1(\overset{\text{ch}}{S^1_t}) \vee \cdots \vee \overset{\text{ch}}{S^1_n}$, for $t = 1,2,\dots,n$ and $F_1(\overset{\text{ch}}{S^1_t}) \ne \overset{\text{ch}}{S^1_t}$ folding with singularity as in Figure 1 then consider the induced folding

$\overline{F_1} : \overset{n}{\underset{i=1}{*}} \pi_1(\overset{\text{ch}}{S_i^1}) \longrightarrow \overset{n}{\underset{i=1}{*}} \pi_1(\overset{\text{ch}}{S_i^1})$ such that $\overline{F}_1(\overset{n}{\underset{i=1}{*}} \pi_1(\overset{\text{ch}}{S_i^1})) = \pi_1(F_1(\overset{n}{\underset{i=1}{\vee}} \overset{\text{ch}}{S_i^1}))$ and so $\overline{F}_1(\overset{n}{\underset{i=1}{*}} \pi_1(\overset{\text{ch}}{S_i^1})) \approx \pi_1(\overset{\text{ch}}{S_1^1}) * \pi_1(\overset{\text{ch}}{S_2^1}) * \cdots * \pi_1(F_1(\overset{\text{ch}}{S_t^1})) * \cdots * \pi_1(\overset{\text{ch}}{S_n^1})$. Since $\pi_1(F_1(\overset{\text{ch}}{S_n^1})) = \overset{\text{ch}}{0}$ and $\pi_1(\overset{\text{ch}}{S_i^1}) \approx \overset{\text{ch}}{Z}$, it follows that $\overline{F}_1(\overset{n}{\underset{i=1}{*}} \pi_1(\overset{\text{ch}}{S_i^1}))$ is a free chaotic group of rank $n-1$. Also, let $F_2 : \overset{n}{\underset{i=1}{\vee}} \overset{\text{ch}}{S_i^1} \longrightarrow \overset{n}{\underset{i=1}{\vee}} \overset{\text{ch}}{S_i^1}$ be folding such that

$F_2(\overset{n}{\underset{i=1}{\vee}} \overset{\text{ch}}{S_i^1}) = \overset{\text{ch}}{S_1^1} \vee \overset{\text{ch}}{S_2^1} \vee \cdots \vee F_2(\overset{\text{ch}}{S_s^1}) \vee \cdots \vee F_2(\overset{\text{ch}}{S_t^1}) \cdots \vee \overset{\text{ch}}{S_n^1}$, for $s, t = 1, 2, \dots, n$, $s < t$, and $F_2(\overset{\text{ch}}{S_s^1}) \neq \overset{\text{ch}}{S_s^1}$, $F_2(\overset{\text{ch}}{S_s^1}) \neq \overset{\text{ch}}{S_t^1}$ then we can get the induced folding

$\overline{F_2} : \overset{n}{\underset{i=1}{*}} \pi_1(\overset{\text{ch}}{S_i^1}) \longrightarrow \overset{n}{\underset{i=1}{*}} \pi_1(\overset{\text{ch}}{S_i^1})$ such that $\overline{F}_2(\overset{n}{\underset{i=1}{*}} \pi_1(\overset{\text{ch}}{S_i^1}))$ is a free chaotic group of rank $n-2$. By continuing this process we obtain the folding $F_n : \overset{n}{\underset{i=1}{\vee}} \overset{\text{ch}}{S_i^1} \longrightarrow \overset{n}{\underset{i=1}{\vee}} \overset{\text{ch}}{S_i^1}$ such that $F_n(\overset{n}{\underset{i=1}{\vee}} \overset{\text{ch}}{S_i^1}) = \overset{n}{\underset{i=1}{\vee}} F_n(\overset{\text{ch}}{S_i^1})$ and $F_n(\overset{\text{ch}}{S_i^1}) \neq \overset{\text{ch}}{S_i^1}$ which induces

a folding $\overline{F_n} : \overset{n}{\underset{i=1}{*}} \pi_1(\overset{\text{ch}}{S_i^1}) \longrightarrow \overset{n}{\underset{i=1}{*}} \pi_1(\overset{\text{ch}}{S_i^1})$ such that $\overline{F}_n(\overset{n}{\underset{i=1}{*}} \pi_1(\overset{\text{ch}}{S_i^1}))$ is a free chaotic group of rank 0.

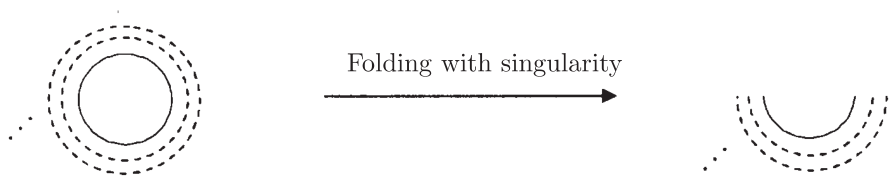

Figure 1.

Theorem 3. *Let D_n be the disjoint union of n discs on the chaotic sphere $\overset{\text{ch}}{S^2}$ and $\{F_m,\ m \in N\}$ be a sequence of conditional folding from $(S^2 - D_n^{\text{ch}})$ into itself then there is an induced folding $\overline{F}_m : \pi_1((S^2 - D_n^{\text{ch}})) \longrightarrow \pi_1((S^2 - D_n^{\text{ch}}))$ which depends on the conditional folding F_m such that $\pi_1(\lim_{m\to\infty} F_m((S^2 - D_n^{\text{ch}}))$ is a free chaotic group of rank $n-2$.*

Proof. Let D_n be the disjoint union of n discs on the chaotic sphere $\overset{\text{ch}}{S^2}$ then we can define a sequence of foldings $\overline{F}_m : (S^2 - D_n^{\text{ch}}) \longrightarrow (S^2 - D_n^{\text{ch}})$, $m = 1, 2, \dots$ such that $\lim_{m\to\infty} F_m((S^2 - D_n^{\text{ch}})) = (S^2 - D_n^{\text{ch}}) \vee (S^2 - D_k^{\text{ch}})$ where $k + h = n$ as in Figure 3

thus $\pi_1(\lim_{m\to\infty} F_m((S^2 - D_n^{\text{ch}})) \approx \pi_1((S^2 - D_h^{\text{ch}})) * \pi_1((S^2 - D_k^{\text{ch}}))$

and so $\pi_1(\lim_{m\to\infty} F_m((S^2 - D_n^{\text{ch}}))) \approx \underbrace{\overset{\text{ch}}{Z} * \overset{\text{ch}}{Z} * \cdots * \overset{\text{ch}}{Z}}_{h-1} * \underbrace{\overset{\text{ch}}{Z} * \overset{\text{ch}}{Z} * \cdots * \overset{\text{ch}}{Z}}_{k-1}$.

Hence, $\pi_1(\lim_{m\to\infty} F_m((S^2 - \overset{\text{ch}}{D_n})))\approx \underbrace{\overset{\text{ch}}{Z} * \overset{\text{ch}}{Z} * \cdots * \overset{\text{ch}}{Z}}_{h+k-2}$. Therefore,

$\pi_1(\lim_{m\to\infty} F_m((S^2 - \overset{\text{ch}}{D_n})))$ is a free chaotic group of rank $n-2$.

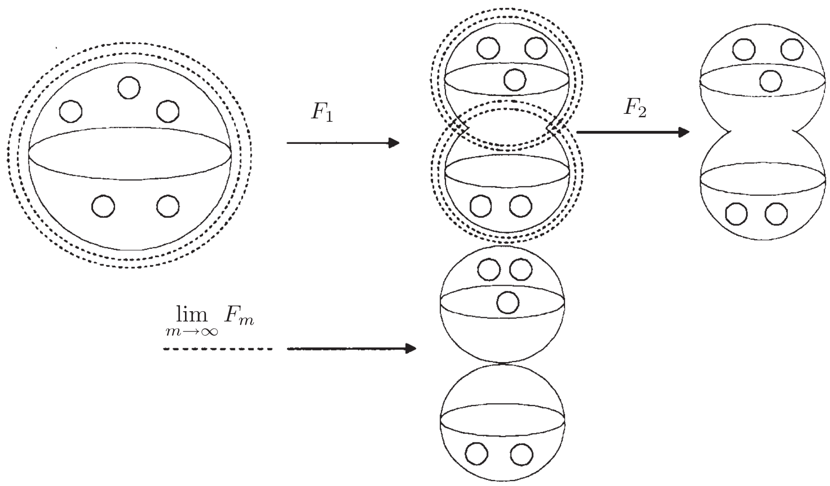

Figure 3.

Theorem 4. *If* $\overset{\text{ch}}{M_1}, \overset{\text{ch}}{M_2}, \ldots, \overset{\text{ch}}{M_n}$ *are path connected chaotic manifolds and* F *is a folding from* $\prod_{i=1}^{n} \overset{\text{ch}}{M_i}$ *into itself then there is an induced folding* $\overline{F}$ *of*

$\pi_1(\prod_{i=1}^{n} \overset{\text{ch}}{M_i})$ *into itself such that*

$\overline{F}(\pi_i(\prod_{i=1}^{n} \overset{\text{ch}}{M_i})) \approx \pi_1(\overset{\text{ch}}{M_1}) \times \pi_1(\overset{\text{ch}}{M_2}) \times \cdots \times \pi_1(F(\overset{\text{ch}}{M_s})) \times \cdots \times \pi_1(\overset{\text{ch}}{M_n})$ *for* $s = 1, 2, \ldots, n$

or $\approx \pi_1(\overset{\text{ch}}{M_1}) \times \pi_1(\overset{\text{ch}}{M_2}) \times \cdots \times \pi_1(F(\overset{\text{ch}}{M_s})) \times \cdots \times \pi_1(F(\overset{\text{ch}}{M_k})) \times \cdots \times \overset{\text{ch}}{M_n}$

for $s, k = 1, 2, \ldots, n$, $s < k$

$\vdots$

or $\approx \pi_1(\prod_{i=1}^{n} F(\overset{\text{ch}}{M_i}))$.

Proof. Let $F : \prod_{i=1}^{n} \overset{\text{ch}}{M_i} \longrightarrow \prod_{i=1}^{n} \overset{\text{ch}}{M_i}$ be folding from $\prod_{i=1}^{n} \overset{\text{ch}}{M_i}$ into itself, then F is continuous map. So we have the coordinate system of $\prod_{i=1}^{n} \overset{\text{ch}}{M_i}$ will be on the form $\{(\overset{\text{ch}}{U_{\alpha_1}} \times \overset{\text{ch}}{U_{\alpha_2}} \times \cdots \times \overset{\text{ch}}{U_{\alpha_n}}), (\overset{\text{ch}}{X_{\alpha_1}} \times \overset{\text{ch}}{X_{\alpha_2}} \times \cdots \times \overset{\text{ch}}{X_{\alpha_n}})\}$, where $\overset{\text{ch}}{X_{\alpha_i}}$ is an injective and bicontinous mapping from an open subset

form $\overset{ch}{U}_{\alpha_i} \subseteq \overset{ch}{R^{n_i}} \longrightarrow \overset{ch}{M}_i$ for
$i = 1,2,\dots,n$ and $\{(\overset{ch}{U}_{\alpha_i}, \overset{ch}{X}_{\alpha_i})\}$ is the atlas of $\overset{ch}{M}_i$ for $i = 1,2,\dots,n$ then
$F : \prod_{i=1}^{n} \overset{ch}{M}_i \longrightarrow \prod_{i=1}^{n} \overset{ch}{M}_i$ has the following forms:
If $F(\prod_{i=1}^{n} \overset{ch}{M}_i) = F(\overset{ch}{U}_{\alpha_1} \times \overset{ch}{U}_{\alpha_2} \times \cdots \times \overset{ch}{U}_{\alpha_n}, \overset{ch}{X}_{\alpha_1} \times \overset{ch}{X}_{\alpha_2} \times \cdots \times \overset{ch}{X}_{\alpha_n})$
$= (\overset{ch}{U}_{\alpha_1} \times \overset{ch}{U}_{\alpha_2} \times \cdots \times \overset{ch}{U}_{\alpha_n}, F(\overset{ch}{U}_{\alpha_s}, \overset{ch}{X}_{\alpha_s}), \overset{ch}{X}_{\alpha_1} \times \overset{ch}{X}_{\alpha_2} \times \cdots \times \overset{ch}{X}_{\alpha_n})$
$= \overset{ch}{M}_1 \times \overset{ch}{M}_2 \times \cdots \times F(\overset{ch}{M}_s) \times \cdots \times \overset{ch}{M}_n$ for $s = 1,2,\dots,n$, then
$\overline{F}(\pi_1(\prod_{i=1}^{n} \overset{ch}{M}_i)) = \pi_1(F(\prod_{i=1}^{n} \overset{ch}{M}_i)) \approx \pi_1(\overset{ch}{M}_1) \times \pi_1(\overset{ch}{M}_2) \times \cdots \times \pi_1(F(\overset{ch}{M}_s)) \times \cdots \times \pi_1(\overset{ch}{M}_n)$
Also, if $F(\prod_{i=1}^{n} \overset{ch}{M}_i) = F(\overset{ch}{U}_{\alpha_1} \times \overset{ch}{U}_{\alpha_2} \times \cdots \times \overset{ch}{U}_{\alpha_n}, \overset{ch}{X}_{\alpha_1} \times \overset{ch}{X}_{\alpha_2} \times \cdots \times \overset{ch}{X}_{\alpha_n})$
$= (\overset{ch}{U}_{\alpha_1} \times \overset{ch}{U}_{\alpha_2} \times \cdots \times \overset{ch}{U}_{\alpha_n}, F(\overset{ch}{U}_{\alpha_s}, \overset{ch}{X}_{\alpha_s}), F(\overset{ch}{U}_{\alpha_k}, \overset{ch}{X}_{\alpha_k}), \overset{ch}{X}_{\alpha_1}, \overset{ch}{X}_{\alpha_2} \times \cdots \times \overset{ch}{X}_{\alpha_n})$
$= \overset{ch}{M}_1 \times \overset{ch}{M}_2 \times \cdots \times F(\overset{ch}{M}_s) \times F(\overset{ch}{M}_k) \cdots \times \overset{ch}{M}_n$ for $s, k = 1,2,\dots,n$, $s < k$,
then, $\overline{F}(\pi_1(\prod_{i=1}^{n} \overset{ch}{M}_i)) = \pi_i(F(\prod_{i=1}^{n} \overset{ch}{M}_i))$
$\approx \pi_1(\overset{ch}{M}_1) \times \pi_1(\overset{ch}{M}_2) \times \cdots \times \pi_1(F(\overset{ch}{M}_s)) \times \pi_1(F(\overset{ch}{M}_k)) \times \cdots \times \pi_1(\overset{ch}{M}_n)$.
Moreover, by continuing this process if
$F(\prod_{i=1}^{n} \overset{ch}{M}_i) = F(\overset{ch}{U}_{\alpha_1} \times \overset{ch}{U}_{\alpha_2} \times \cdots \times \overset{ch}{U}_{\alpha_n}, \overset{ch}{X}_{\alpha_1}, \overset{ch}{X}_{\alpha_2} \times \cdots \times \overset{ch}{X}_{\alpha_n})$
$= F(\overset{ch}{U}_{\alpha_1}, \overset{ch}{X}_{\alpha_1}), F(\overset{ch}{U}_{\alpha_2}, \overset{ch}{X}_{\alpha_2}), \dots, F(\overset{ch}{U}_{\alpha_n}, \overset{ch}{X}_{\alpha_n}) = F(\overset{ch}{M}_1) \times F(\overset{ch}{M}_2) \times \cdots \times F(\overset{ch}{M}_n)$,
then, $\overline{F}(\pi_1(\prod_{i=1}^{n} \overset{ch}{M}_i)) = \pi_1(F(\prod_{i=1}^{n} \overset{ch}{M}_i)) \approx \pi_1(F(\overset{ch}{M}_1)) \times \cdots \times \pi_1(F(\overset{ch}{M}_n)) = \pi_1(\prod_{i=1}^{n} F(\overset{ch}{M}_i))$.

Theorem 5. *If $\overset{ch}{M}_1, \overset{ch}{M}_2, \dots, \overset{ch}{M}_n$ are path connected manifolds which are homeomorphic to $\overset{ch}{S^1}$ and F is a folding such that $F(\prod_{i=1}^{n} \overset{ch}{M}_i) \neq \prod_{i=1}^{n} F(\overset{ch}{M}_i)$ then*

$$\pi_1\Big(\lim_{m\to\infty} F_m\Big(\prod_{i=1}^{n} \overset{ch}{M}_i\Big)\Big) \not\approx \prod_{i=1}^{n}\Big(\pi_1\Big(\lim_{m\to\infty}\Big(F_m(\overset{ch}{M}_i)\Big)\Big)\Big).$$

Proof. Take $\overset{ch}{M}_1 = \overset{ch}{S^1}$, $\overset{ch}{M}_2 = \overset{ch}{S^2}$, then $\overset{ch}{S^1} \times \overset{ch}{S^1} = \overset{ch}{T^1}$ chaotic torus since
$\lim_{m\to\infty} F_m(\overset{ch}{S^1})$ = chaotic point as in Figure 4, then
$\lim_{m\to\infty} F_m(\overset{ch}{S^1}) \times \lim_{m\to\infty} F_m(\overset{ch}{S^1})$ = chaotic point and so $\pi_1(\lim_{m\to\infty} F_m(\overset{ch}{S^1}) \times \lim_{m\to\infty} F_m(\overset{ch}{S^1})) = 0$.
Also, it follows from $F(\overset{ch}{S^1} \times \overset{ch}{S^1}) \neq F(\overset{ch}{S^1}) \times F(\overset{ch}{S^1})$ that $F(\overset{ch}{S^1} \times \overset{ch}{S^1}) = F(\overset{ch}{S^1}) \times \overset{ch}{S^1}$ or
$F(\overset{ch}{S^1} \times \overset{ch}{S^1}) = \overset{ch}{S^1} \times F(\overset{ch}{S^1})$ thus $\lim_{m\to\infty} F_m(\overset{ch}{S^1} \times \overset{ch}{S^1}) = \overset{ch}{S^1}$ so

$\pi_1(\lim_{m\to\infty} F_m(\overset{\text{ch}}{S^1} \times \overset{\text{ch}}{S^1})) = \pi_1(\overset{\text{ch}}{S^1}) \approx \overset{\text{ch}}{Z}$. Hence

$\pi_1(\lim_{m\to\infty} F_m(\overset{\text{ch}}{S^1} \times \overset{\text{ch}}{S^1})) \not\approx \pi_1(\lim_{m\to\infty} F_m(\overset{\text{ch}}{S^1})) \times \pi_1(\lim_{m\to\infty} F_m(\overset{\text{ch}}{S^1}))$.

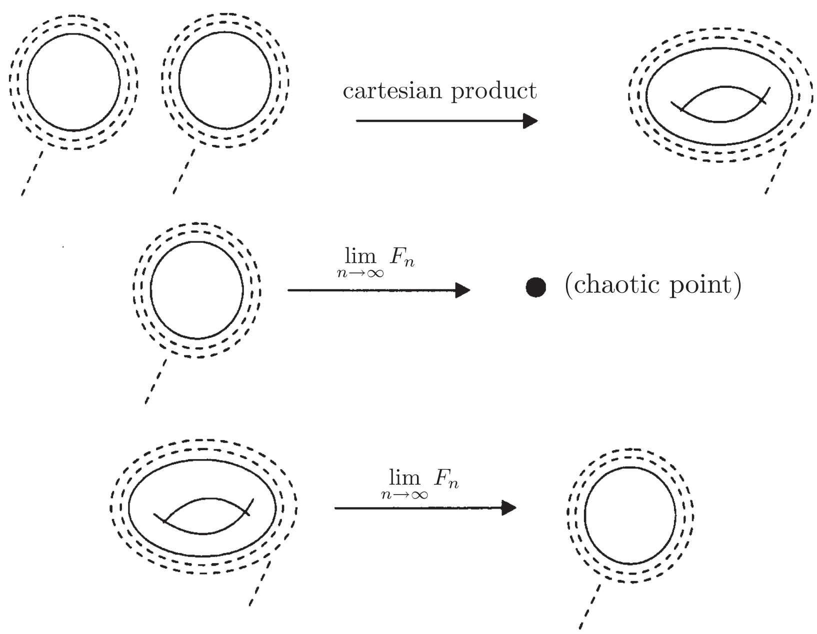

Figure 4.

Corollary 1. *If* $\overset{\text{ch}}{M}_1, \overset{\text{ch}}{M}_2, \dots, \overset{\text{ch}}{M}_n$ *are path connected manifolds and* F *is a folding such that* $F(\prod_{i=1}^{n} \overset{\text{ch}}{M}_i) = \prod_{i=1}^{n} F(\overset{\text{ch}}{M}_i)$ *then*

$$\pi_1\Big(\lim_{m\to\infty} F_m\Big(\prod_{i=1}^{n} \overset{\text{ch}}{M}_i\Big)\Big) \approx \prod_{i=1}^{n}\Big(\lim_{m\to\infty} \pi_1\Big(F_m(\overset{\text{ch}}{M}_i)\Big)\Big).$$

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