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ON 1-VERTEX BIMAGIC VERTEX LABELING

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Abstract. A 1-vertex magic vertex labeling of a graph G with p vertices is defined as a bijection f from the vertices to the integers $1, 2, \ldots, p$ with the property that there is a constant k such that at any vertex x, $\sum_{y \in N(x)} f(y) = k$, where $N(x)$ is the set of vertices adjacent to x. In this paper we introduce 1-vertex bimagic vertex labeling of a graph G and obtain the necessary condition for a graph to be 1-vertex bimagic. We exhibit the same type of labeling for some class of graphs and give some general results.

1. Introduction

A labeling of a graph is assigning labels to the vertices, edges or both vertices and edges. In most applications labels are positive (or nonnegative) integers, though in general real numbers could be used. For various types of graph labeling one can refer the survey of graph labeling by J.A. Gallian [3]. In 1963, Sedláĉek [7] introduced the magic labeling for a graph $G = G(V, E)$ which is defined as a bijection f from E to a set of positive integers such that

(i) $f(e_i) \neq f(e_j)$ for all distinct $e_i, e_j \in E$, and

(ii) $\sum_{e \in N_E(x)} f(e)$ is the same for every $x \in V$, where $N_E(x)$ is the set of edges incident to x.

MacDougall, Miller, Slamin and Wallis [4] introduced the notion of a vertex-magic total labeling in 1999. For a graph $G(V, E)$ a bijective mapping f from $V \cup E$ to the set $\{1, 2, \ldots, |V \cup E|\}$ is a vertex-magic total labeling if there is a constant k, called the magic constant, such that for every vertex v, $f(v) + \sum f(vu) = k$ where the sum is over all vertices u adjacent to v (some authors use the term "vertex-magic" for this concept). In [1, 2], edge bimagic total labeling was introduced by J. Baskar Babujee. A graph $G(p, q)$ with p vertices and q edges is called edge bimagic total if there exists a bijection $f : V \cup E \to \{1, 2, \ldots, p+q\}$ such that for any edge $uv \in E$, we have two constants k_1 and k_2 with $f(u) + f(v) + f(uv) = k_1$ or k_2.

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In [5] Mirka Miller et al. defined the 1-vertex-magic vertex labeling of a graph with p vertices defined as a bijection f taking the vertices to the integers $1,2,\dots,p$ with the property that there is a constant k such that at any vertex x, $\sum_{y\in N(x)} f(y)=k$, where $N(x)$ is the set of vertices adjacent to x (that is, distance 1 from x).

A necessary condition for the existence of a 1-vertex magic vertex labeling is given in [5] as follows:
If f is 1-vertex magic vertex labeling then $\sum_{x\in V} d(x)f(x)=kp$ where $d(x)$ is the degree of vertex x.

A 1-vertex-magic vertex labeling is the same as sigma labeling or Σ-labeling defined by Vilfred [11]. It becomes interesting when we arrive with 1-vertex magic vertex labeling summing to exactly two distinct constants say k_1 or k_2. This motivates us to work in 1-vertex bimagic vertex labeling. This paper is organized as follows. In section 2, we introduce a 1-vertex bimagic vertex labeling and which classes of graphs are and are not 1-vertex bimagic. In section 3, we prove a 1-vertex bimagic vertex labeling for complete symmetric multipartite graph. In section 4, we prove some general results on regular or bi-regular1-vertex bimagic-graphs. In section 5, we conclude this paper with an open problem.

2. 1-Vertex Bimagic Vertex Labeling

Definition 2.1. A bijective labeling $f: V(G)\to\{1,2,\dots,p\}$ is called a 1-vertex bimagic vertex labeling if for each vertex $u\in V(G)$, the sum of all $f(v)$ such that v is adjacent to u is either k_1 or k_2 (i.e) for all $u\in V(G)$, $\sum_{uv\in E(G)} f(v)=k_1$ or k_2 Where k_1 and k_2 are distinct constants. A graph which has a 1-vertex bimagic vertex labeling is called a 1-vertex bimagic graph.

Definition 2.2. A bijective labeling $f: V(G)\to\{1,3,\dots,2p-1\}$ is called a odd 1-vertex bimagic vertex labeling if for each vertex $u\in V(G)$, the sum of all $f(v)$ such that v is adjacent to u is either k_1 or k_2 (i.e) for all $u\in V(G)$, $\sum_{uv\in E(G)} f(v)=k_1$ or k_2 Where k_1 and k_2 are distinct constants.

Definition 2.3. A bijective labeling $f: V(G)\to\{0,2,\dots,2(p-1)\}$ is called a even 1-vertex bimagic vertex labeling if for each vertex $u\in V(G)$, the sum of all $f(v)$ such that v is adjacent to u is either k_1 or k_2 (i.e) for all $u\in V(G)$, $\sum_{uv\in E(G)} f(v)=k_1$ or k_2 Where k_1 and k_2 are distinct constants.

Theorem 2.1. *A necessary condition for the existence of a 1-vertex bimagic vertex labeling f of a graph G is*

$$\sum_{x\in V} d(x)f(x)=k_1p_1+k_2p_2 \tag{2.1}$$

where $d(x)$ is the degree of vertex x and p_1, p_2 are the number of vertices with common count k_1 and k_2 respectively.

Proof. Let $V = V_1 \cup V_2$ and $|V_1| = p_1$, $|V_2| = p_2 = p - p_1$ where p_1 and p_2 the number of vertices with common count k_1 and k_2 respectively. Consider L.H.S of (2.1),

$$\sum_{x\in V} d(x)f(x) = \sum_{x\in V_1} d(x)f(x) + \sum_{x\in V_2} d(x)f(x) \tag{2.2}$$

Consider the two sub graphs G_1 and G_2 of G with vertex sets V_1 and V_2 respectively. Applying the necessary condition for the existence of a 1-vertex magic vertex labeling given in [5] mentioned in introduction of our paper, (2.2) becomes

$$\sum_{x\in V} d(x)f(x) = k_1 p_1 + k_2 p_2.$$

□

Theorem 2.2. *If G has a 1-vertex magic vertex labeling and $G \neq C_4$, then $G + K_1$ admits a 1-vertex bimagic vertex labeling.*

Proof. If G is cycle C_4 then $C_4 + K_1$ is 1-vertex magic. Let $G(p,q)$ has a 1-vertex magic vertex labeling then there exist a function $f : V \to \{1,2,\dots,p\}$ such that for every vertex u, $\sum_{uv\in E(G)} f(v) = r$. Now we define the new graph called $G_1 = G + K_1$ with vertex set $V_1(G_1) = V(G) \cup \{x\}$ and $E_1(G_1) = E(G) \cup \{xv_i : 1 \le i\ leqp\}$. Consider the bijective function $g : V_1 \to \{1,2,\dots,p,p+1\}$ defined by

$g(v_i) = f(v_i)$; $1 \le i \le p$ or $v_i \in V(G)$

$g(x) = p+1$.

Since the graph G is already 1-vertex magic vertex labeling, for every $u \in V_1(G)$,

$$\begin{aligned}\sum_{uv\in E_1(G_1)} g(v) &= \sum_{uv\in E(G)} f(v) + g(x)\\ &= r + p + 1 = k_1(\text{say})\end{aligned}$$

For the newly added vertex x,

$$\sum_{v\in N(x)} g(v) = \sum_{i=1}^{p} g(v_i) = \sum_{i=1}^{p} f(v_i) = 1+2+3+\cdots+p = \frac{p(p+1)}{2} = k_2(\text{say})$$

This proves that $G + K_1$ admits a 1-vertex bimagic vertex labeling. □

Theorem 2.3. *The path graph P_n, $n > 3$ is not a 1-vertex bimagic.*

Proof. If $n \le 3$, P_n is a 1-vertex bimagic. Consider the path P_n, $n > 3$ with vertex set $\{v_1, v_2, \dots, v_n\}$ and edge set $\{v_i v_{i+1} : 1 \le i \le n-1\}$. If we arrange the labels $\{a_1, a_2, \dots, a_n\}$ for the vertices

$\{v_1, v_2, \dots, v_n\}$ by permutation, the common count of v_1 is $a_2 = k_1$ (say) and v_n is $a_{n-1} = k_2$ (say). For all other vertices from $\{v_2, \dots, v_{n-1}\}$ the common count is $\{a_1 + a_3, a_2 + a_4, \dots, a_{n-2} + a_n\}$ respectively. Since the function f is bijective, (all the vertices having distinct labeling) these $n-2$ counts cannot be either k_1 or k_2. Hence the path graph P_n is not a 1-vertex bimagic. □

Theorem 2.4. *The cycle graph C_n is a 1-vertex bimagic if and only if $n = 4$.*

Proof. Consider the cycle C_n $(n \geq 5)$ with vertex set $\{v_1, v_2, \dots, v_n\}$ and edge set $\{v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{v_n v_1\}$. If we arrange the labels $\{a_1, a_2, \dots, a_n\}$ for the vertices $\{v_1, v_2, \dots, v_n\}$ by permutation, the common count of v_2 is $a_1 + a_3 = k_1$ (say) and v_n is $a_{n-1} + a_1 = k_2$ (say). For all other vertices $\{v_1, v_3, \dots, v_{n-1}\}$ the common count is $\{a_1 + a_n, a_2 + a_4, \dots, a_{n-2} + a_n\}$ respectively. Since the function f is bijective, (all the vertices having distinct labeling) these $n-2$ counts cannot be either k_1 or k_2. Hence the cycle C_n $(n \geq 5)$ is not a 1-vertex bimagic.

Conversely, Consider the cycle C_n with $n = 4$. The vertices of the cycle C_4 are v_1, v_2, v_3, v_4. Let the function $f : V(G) \to \{1, 2, 3, 4\}$ be defined as $f(v_i) = i$. Consider the common count of the labels of v_1 and v_3, we have $\sum_{v_3 u \in E(G)} f(u) = f(v_2) + f(v_4) = 6$ and for the common count of the labels of v_2 and v_4, we have $\sum_{v_3 u \in E(G)} f(u) = f(v_1) + f(v_3) = 4$. Thus for the cycle C_4, we have two constants 6 and 4. Hence the cycle C_n is a 1-vertex bimagic when $n = 4$. □

Theorem 2.5. *The complete bipartite graph $K_{m.n}$ has a 1-vertex bimagic vertex labeling.*

Proof. Let $G(V, E)$ be a complete bipartite graph with vertex set $V(G) = \{v_1, v_2, \dots, v_m, u_1, u_2, \dots, u_n\}$ and edge set be $E(G) = \{v_i u_j : 1 \leq i \leq m, 1 \leq i \leq n\}$. Define the function $f : V(G) \dots \{1, 2, \dots, m+n\}$ as follows

If $m \leq n$,

$f(v_i) = i$ for $1 \leq i \leq m$,

$f(u_j) = m + j$ for $1 \leq j \leq n$

If $m > n$,

$f(v_i) = i$ for $1 \leq i \leq n$,

$f(u_j) = n + j$ for $1 \leq j \leq m$

If $m \leq n$, for any vertex $v_i \in V(G)$,

$$\sum_{v_i u \in E(G)} f(u) = \sum_{j=1}^{n} u_j = \sum_{j=1}^{n} m + j = nm + (1 + 2 + \cdots + n) = nm + \frac{n(n+1)}{2} = k_1$$

For any vertex $u_j \in V(G)$,

$$\sum_{u_i u \in E(G)} f(u) = \sum_{i=1}^{m} v_i = \sum_{i=1}^{m} i = 1 + 2 + \cdots + m = \frac{m(m+1)}{2} = k_2$$

Thus for any vertex $v \in V(G)$ we have two constants k_1 or k_2.

Similarly for $m > n$, we get the two constants k_1 and k_2 vice versa. Which proves that complete bipartite graph $K_{m.n}$ has a 1-vertex-bimagic vertex labeling. □

Definition 2.4. A graph $G^o(P_n)$ is obtained from a path P_n by introducing new edges between any two vertices if they are at odd distance.

$G^o(P_n)$ has n vertices and $\left(\frac{n^2-1}{4}\right)$ edges if n is odd and $\left(\frac{n^2}{4}\right)$ edges if n is even.

Also $G^o(P_n)$ is $\left(\frac{n-1}{2}, \frac{n+1}{2}\right)$ bi-regular when n is odd and $\left(\frac{n}{2}\right)$ regular when n is even.

The 1-vertex-bimagic vertex labeling for $G^o(P_9)$ and $G^o(P_{10})$ is shown below.

Example 2.1.

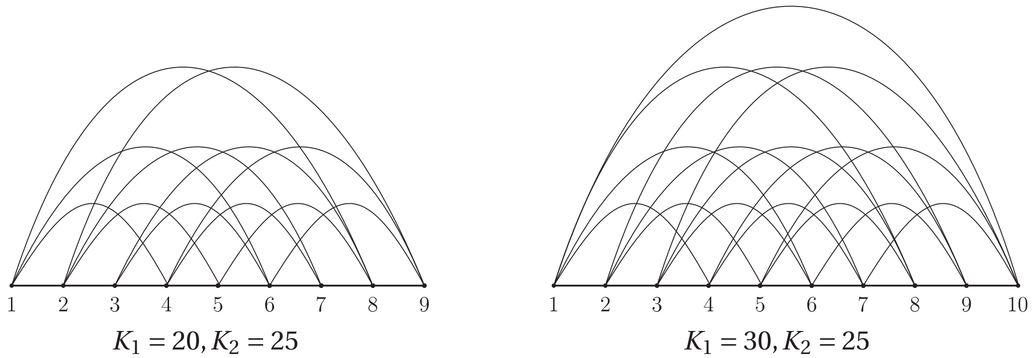

Figure 1: 1-vertex bimagic vertex labeling for $G^o(P_9)$ and $G^o(P_{10})$.

Theorem 2.6. *The graph $G^o(P_n)$, $n \geq 3$ has 1-vertex bimagic vertex labeling.*

Proof. Consider the graph $G^o(P_n)$. We prove this theorem in two cases

Case 1: When n is odd

The vertex set and edge set be defined as $V(G) = \{v_1, v_2, \ldots, v_n\}$. As per the definition of $G^o(P_n)$ we construct the edge set with respect to the index of the vertices as follows $E(G) = \left\{v_i v_{2j} : 1 \leq j \leq \frac{n-1}{2} \text{ and for } i \text{ is odd}\right\} \cup \left\{v_i v_{2j-1} : 1 \leq j \leq \frac{n+1}{2} \text{ and for} i \text{ is even}\right\}$. Define a bijective function $f : V(G) \to \{1, 2, \ldots, n\}$ as follows

$f(v_i) = i$ for $1 \leq i \leq n$.

For any vertex $v_i \in V(G)$, if i is odd

$$\sum_{v_i u \in E(G)} f(u) = \sum_{j=1}^{\frac{n-1}{2}} f(v_{2j}) = \sum_{j=1}^{\frac{n-1}{2}} 2j = 2\sum_{j=1}^{\frac{n-1}{2}} j$$
$$= 2\left(1 + 2 + \cdots + \frac{n-1}{2}\right)$$

$$= \frac{n-1}{2}\left(\frac{n+1}{2}\right) = k_1(\text{say})$$

For any vertex $v_i \in V(G)$, if i is even

$$\begin{aligned}\sum_{v_i u \in E(G)} f(u) &= \sum_{j=1}^{\frac{n-1}{2}} f(v_{2j-1}) = \sum_{j=1}^{\frac{n-1}{2}} (2j-1) = 2\sum_{j=1}^{\frac{n-1}{2}} j\left(\frac{n+1}{2}\right) \\ &= 2\left(1+2+\cdots+\frac{n+1}{2}\right) - \left(\frac{n+1}{2}\right) \\ &= \frac{n+1}{2}\left(\frac{n+3}{2}\right) - \left(\frac{n+1}{2}\right) \\ &= \left(\frac{n+1}{2}\right)^2 = k_2(\text{say})\end{aligned}$$

Thus for any vertex $v \in V(G)$ we have two constants k_1 or k_2.

Case 2: When n is even

The vertex set and edge set be defined as $V(G) = \{v_1, v_2, \dots, v_n\}$. As per the definition of $G^o(P_n)$ we construct the edge set with respect to the index of the vertices as follows $E(G) = \left\{v_i v_{2j} : 1 \le j \le \frac{n}{2} \text{ and for } i \text{ is odd}\right\} \cup \left\{v_i v_{2j-1} : 1 \le j \le \frac{n}{2} \text{ and for } i \text{ is even}\right\}$. Define a bijective function $f : V(G) \to \{1, 2, \dots, n\}$ as follows

$f(v_i) = i$ for $1 \le i \le n$.

For any vertex $v_i \in V(G)$, for i is odd

$$\begin{aligned}\sum_{v_i u \in E(G)} f(u) &= \sum_{j=1}^{\frac{n}{2}} f(v_{2j}) = \sum_{j=1}^{\frac{n}{2}} 2j = 2\sum_{j=1}^{\frac{n}{2}} j \\ &= 2\left(1+2+\cdots+\frac{n}{2}\right) \\ &= \frac{n}{2}\left(\frac{n}{2}+1\right) = k_1(\text{say})\end{aligned}$$

For any vertex $v_i \in V(G)$, for i is even

$$\begin{aligned}\sum_{v_i u \in E(G)} f(u) &= \sum_{j=1}^{\frac{n}{2}} f(v_{2j-1}) = \sum_{j=1}^{\frac{n}{2}} 2j-1 = 2\sum_{j=1}^{\frac{n}{2}} j - \left(\frac{n}{2}\right) \\ &= 2\left(1+2+\cdots+\frac{n}{2}\right) - \left(\frac{n}{2}\right) \\ &= \frac{n}{2}\left(\frac{n}{2}+1\right) - \left(\frac{n}{2}\right) \\ &= \left(\frac{n}{2}\right)^2 = k_2(\text{say})\end{aligned}$$

Thus for any vertex $v \in V(G)$ we have two constants k_1 or k_2.

Hence the graph $G^o(P_n)$ has 1-vertex-bimagic vertex labeling. □

3. 1-vertex bimagic vertex labeling for complete symmetric multipartite graph

Let $H_{n,p}$, $n > 1$ and $p > 1$ denote the complete symmetric multipartite graph with p parts, each of which contains n vertices. $H_{n,p}$ is a graph with p partition which has equal number of vertices and every vertex of G in each partition connected to all other vertices of the remaining $(p-1)$ partitions. The number of vertices and edges in $H_{n,p}$ is np and $\binom{p}{2}n^2$ respectively. Another way of describing $H_{n.p}$ is as follows: Consider a complete graph K_p with p vertices $\{x_1, x_2, x_3, \ldots, x_p\}$. By replacing every vertex in K_p with n pairs of vertices, each joined to all vertices corresponding to the neighbours of the original vertex of K_p, we obtain the graph $H_{n,p}$.

Example 3.2. 1-vertex bimagic vertex labeling for complete symmetric multipartite graphs $H_{6,6}$ and $H_{5,6}$ is given below

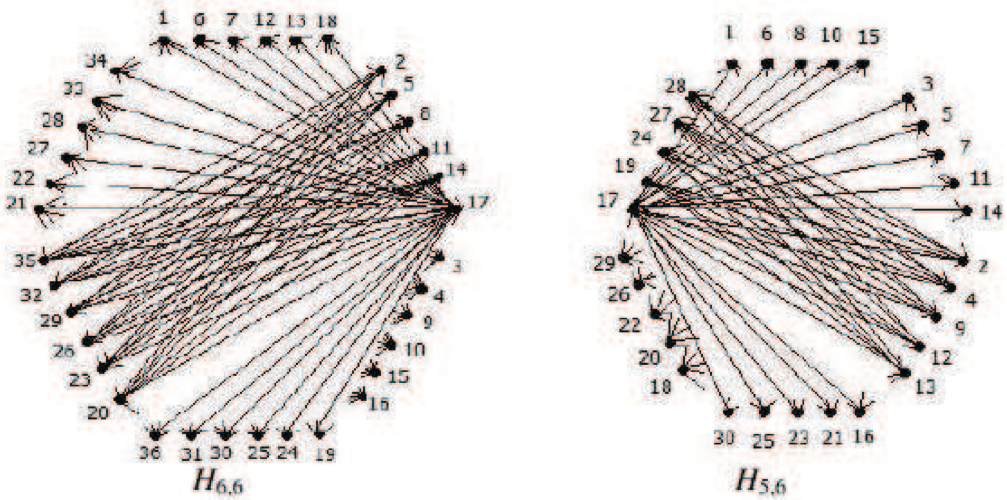

Figure 2: 1-vertex bimagic vertex labeling for complete symmetric multipartite graph

Theorem 3.7. *Let $n > 1$ and $p > 1$. $H_{n,p}$ has a 1-vertex bimagic vertex labeling, for*

(i) *any p, when n is even*

(ii) *even p, when n is odd.*

Proof. Consider a graph $H_{n,p}$. We prove this theorem in two cases.
Case 1: For any p, when n is even
Let x_{ij} be the vertices of $H_{n,p}$, $1 \le i \le n$; $1 \le j \le p$. Label the vertices in the following way. Let

$$m = \begin{cases} \frac{p}{2} & \text{if } p \equiv 0 \bmod 2 \\ \frac{p+1}{2} & \text{otherwise} \end{cases}$$

For $1 \le j \le m$, $f(x_{i,j}) = \begin{cases} j + (i-1)m & \text{if } i \text{ is odd} \\ m - j + 1 + (i-1)m & \text{if } i \text{ is even} \end{cases}$

For $m+1 \le j \le p$, $f(x_{i,j}) = \begin{cases} nm + (j-m) + (i-1)m & \text{if } i \text{ is odd} \\ m(n+2) - j + 1 + (i-1)m & \text{if } i \text{ is even} \end{cases}$

For $1 \le j \le m$, the sum of the labels of the n vertices in each set j is $\frac{n}{2}(mn+1) = S_1$ say. For $m+1 \le j \le p$, the sum of the labels of the n vertices in each set j is $\frac{n}{2}(n(m+p)+1) = S_2$ say.

Sub case (i): When p is even

For any vertex $x \in V(G)$ in the interval $1 \le j \le m$

$$\sum_{xy \in E(G)} f(y) = (m-1)S_1 + mS_2 = k_1$$

and for any vertex $x \in V(G)$ in the interval $m+1 \le j \le p$

$$\sum_{xy \in E(G)} f(y) = mS_1 + (m-1)S_2 = k_2$$

Sub case (ii): When p is odd

For any vertex $x \in V(G)$ in the interval $1 \le j \le m$

$$\sum_{xy \in E(G)} f(y) = (m-1)S_1 + (m-1)S_2 = k_1$$

and for any vertex $x \in V(G)$ in the interval $m+1 \le j \le p$

$$\sum_{xy \in E(G)} f(y) = mS_1 + (m-2)S_2 = k_2$$

Thus for any vertex $x \in V(G)$ we have two constants k_1 or k_2 for n is even and p is even or odd.

Case 2: If p is even and n is odd then we can prove this case in two sub cases. Let $m = \frac{p}{2}$.

Sub Case (i): when m is odd then $2t+1 = m$. The labeling is defined below

$$f(x_{i,j}) = \begin{cases} 2j-1, & 1 \le j \le t+1 \text{ and } i = 1, \\ 2(j-t-1), & t+2 \le j \le 2t+1 \text{ and } i = 1, \\ 4t+3-j, & 1 \le j \le 2t+1 \text{ and } i = 2, \\ 5t+4-j, & 1 \le j \le t+1 \text{ and } i = 3, \\ 7t+5-j, & t+2 \le j \le 2t+1 \text{ and } i = 3, \\ j+(i-1)(2t+1), & 1 \le j \le 2t+1 \text{ and } i > 3, i \text{ even}, \\ 2t+2-j+(i-1)(2t+1), & 1 \le j \le 2t+1 \text{ and } i > 3, i \text{ odd}. \end{cases}$$

and

$$f(x_{i,2t+j+1}) = \begin{cases} n(2t+1)+2j-1, & 1 \le j \le t+1 \text{ and } i=1, \\ n(2t+1)+2(j-t-1), & t+2 \le j \le 2t+1 \text{ and } i=1, \\ n(2t+1)+4t+3-j, & 1 \le j \le 2t+1 \text{ and } i=2, \\ n(2t+1)+5t+4-j, & 1 \le j \le t+1 \text{ and } i=3, \\ n(2t+1)+7t+5-j, & t+2 \le j \le 2t+1 \text{ and } i=3, \\ n(2t+1)+j+(i-1)(2t+1), & 1 \le j \le 2t+1 \text{ and } i>3, i \text{ even}, \\ n(2t+1)+2t+2-j+(i-1)(2t+1), & 1 \le j \le 2t+1 \text{ and } i>3, i \text{ odd}. \end{cases}$$

For every j in the interval $1 \le j \le m$, we split the calculation for $j=1$ to $t+1$ and $j=t+2$ to $2t+1$ as follows
for $j=1$ to $t+1$, $f(x_{1,j})+f(x_{2,j})+f(x_{3,j}) = 2j-1+4t+3-j+5t+4-j = 9t+6$
for $j=t+2$ to $2t+1$, $f(x_{1,j})+f(x_{2,j})+f(x_{3,j}) = 2(j-t-1)+4t+3-j+7t+5-j = 9t+6$
and for $i>1$,
$f(x_{2i,j})+f(x_{2i+1,j}) = j+(2i-1)(2t+1)+2t+2-j+(2i+1-1)(2t+1) = (2t+2)+(4i-1)(2t+1)$.
Thus the sum of the labels of n vertices in each set j is
$= f(x_{1,j})+f(x_{2,j})+f(x_{3,j})+\sum_{i=2}^{\frac{n-1}{2}}[f(x_{2i,j})+f(x_{2i+1,j})]$
$= (9t+6)+(n-3)(t+1)+\left(\frac{n^2-n-6}{2}\right)(2t+1) = S_1$, say.
For every j in the interval $m+1 \le j \le p$, we split the calculation for $j=1$ to $t+1$ and $j=t+2$ to $2t+1$ as follows
for $j=1$ to $t+1$,

$$\begin{aligned} & f(x_{1,2t+j+1})+f(x_{2,2t+j+1})+f(x_{3,2t+j+1}) \\ & \quad = n(2t+1)+2j-1+n(2t+1)+4t+3-j+n(2t+1)+5t+4-j \\ & \quad = 3n(2t+1)+9t+6 \end{aligned}$$

for $j=t+2$ to $2t+1$,

$$\begin{aligned} & f(x_{1,2t+j+1})+f(x_{2,2t+j+1})+f(x_{3,2t+j+1}) \\ & \quad = n(2t+1)+2(j-t-t1)+n(2t+1)+4t+3-j+n(2t+1)+7t+5-j \\ & \quad = 2n(2t+1)+9t+6 \end{aligned}$$

and for $i>1$,

$$\begin{aligned} f(x_{2i,2t+j+1})+f(x_{2i+1,2t+j+1}) &= n(2t+1)+j+(2i-1)(2t+1)+n(2t+1)+2t+2-j+(2i)(2t+1) \\ &= 2n(2t+1)+2t+2+(4i-1)(2t+1). \end{aligned}$$

Thus the sum of the labels of n vertices in each set j is

$= f(x_{1,2t+j+1}) + f(x_{2,2t+j+1}) + f(x_{3,2t+j+1}) + \sum_{i=2}^{\frac{n-1}{2}} [f(x_{2i,2t+j+1}) + f(x_{2i+1,2t+j+1})]$

$= [3n(2t+1)+(9t+6)]+(n-3)n(2t+1)+(n-3)(t+1)+\left(\frac{n^2-n-6}{2}\right)(2t+1) = S_2$ say. For any vertex $x \in V(G)$ in the interval $1 \le j \le m$

$$\sum_{xy\in E(G)} f(y) = (m-1)S_1 + mS_2 = k_1$$

And for any vertex $x \in V(G)$ in the interval $m+1 \le j \le p$

$$\sum_{xy\in E(G)} f(y) = mS_1 + (m-1)S_2 = k_2$$

Thus in this case, for any vertex $x \in V(G)$ we have two constants k_1 or k_2.

Sub Case (ii): when m is even then $2t+1 = m+1$. The labeling is defined below

The labeling of $f(x_{i,j})$ is same as given in above sub case (i) and the labeling of $f(x_{i,2t+j+1})$ is shown below

$$f(x_{i,2t+j+1}) = \begin{cases} n(2t+1)+2j-1, & 1 \le j \le t+1 \text{ and } i=1, \\ n(2t+1)+2(j-t), & t+1 \le j \le 2t-1 \text{ and } i=1, \\ n(2t+1)+4(t-1)+3-j, & 1 \le j \le 2t-1 \text{ and } i=2, \\ n(2t+1)+5(t-1)+4-j, & 1 \le j \le t+1 \text{ and } i=3, \\ n(2t+1)+7(t-1)+5-j, & t+1 \le j \le 2t-1 \text{ and } i=3, \\ n(2t+1)+j+(i-1)(2t-1), & 1 \le j \le 2t-1 \text{ and } i>3, i \text{ even}, \\ n(2t+1)+2(t-1)+2-j+(i-1)(2t-1), & 1 \le j \le 2t-1 \text{ and } i>3, i \text{ odd}. \end{cases}$$

For every j in the interval $1 \le j \le m+1$, we split the calculation for $j=1$ to $t+1$ and $j=t+2$ to $2t+1$ as follows

for $j=1$ to $t+1$, $f(x_{1,j}) + f(x_{2,j}) + f(x_{3,j}) = 2j-1+4t+3-j+5t+4-j = 9t+6$

for $j=t+2$ to $2t+1$, $f(x_{1,j}) + f(x_{2,j}) + f(x_{3,j}) = 2(j-t-1)+4t+3-j+7t+5-j = 9t+6$

and for $i>1$, $f(x_{2i,j}) + f(x_{2i+1,j}) = j+(2i-1)(2t+1)+2t+2-j+(2i+1-1)(2t+1)$

$= (2t+2)+(4i-1)(2t+1)$

Thus the sum of the labels of n vertices in each set j is

$= f(x_{1,j}) + f(x_{2,j}) + f(x_{3,j}) + \sum_{i=2}^{\frac{n-1}{2}} [f(x_{2i,j}) + f(x_{2i+1,j})]$

$= (9t+6)+(n-3)(t+1)+\left(\frac{n^2-n-6}{2}\right)(2t+1) = S_1$ say.

For every j in the interval $m+2 \le j \le p$, we split the calculation for $j=1$ to t and $j=t+1$ to $2t-1$ as follows

for $j=1$ to t,

$$f(x_{1,2t+j+1}) + f(x_{2,2t+j+1}) + f(x_{3,2t+j+1})$$

$$= n(2t+1)+2j-1+n(2t+1)+4(t-1)+3-j+n(2t+1)+5(t-1)+4-j$$
$$= 3n(2t+1)+9(t-1)+6$$

for $j = t+2$ to $2t-1$,

$$f(x_{1,2t+j+1})+f(x_{2,2t+j+1})+f(x_{3,2t+j+1})$$
$$= n(2t+1)+2(j-t)+n(2t+1)+4(t-1)+3-j+n(2t+1)+7(t-1)+5-j$$
$$= 3n(2t+1)+9(t-1)+6$$

and for $i > 1$

$$f(x_{2i,2t+j+1})+f(x_{2i+1,2t+j+1})$$
$$= n(2t+1)+j+(i-1)(2t-1)+n(2t+1)+2(t-1)+2-j+(i-1)(2t-1)$$
$$= 2n(2t+1)+2(t-1)+2+(4i-1)(2t-1)$$

Thus the sum of the labels of n vertices in each set j is

$$= f(x_{1,2t+j+1})+f(x_{2,2t+j+1})+f(x_{3,2t+j+1})+\sum_{i=2}^{\frac{n-1}{2}}[f(x_{2i,2t+j+1})+f(x_{2i+1,2t+j+1})]$$
$$= [3n(2t+1)+9(t-1)+6]+(n-3)n(2t+1)+(n-3)t+\left(\frac{n^2-n-6}{2}\right)(2t-1) = S_2 \text{ say.}$$

For any vertex $x \in V(G)$ in the interval $1 \le j \le m+1$

$$\sum_{xy\in E(G)} f(y) = mS_1+(m-1)S_2 = k_1$$

And for any vertex $x \in V(G)$ in the interval $m+2 \le j \le p$

$$\sum_{xy\in E(G)} f(y) = (m+1)S_1+(m-2)S_2 = k_2$$

Thus in this case, for any vertex $x \in V(G)$ we have two constants k_1 or k_2.
Hence using the above cases $H_{n,p}$ has a 1-vertex bimagic vertex labeling, if either n is even or p is even and n is odd. □

4. General results on regular and bi-regular graphs

All the graphs for which 1-vertex bimagic vertex labeling is done so far are either regular or bi-regular. But the Complete graph K_n is not a 1-vertex bimagic, since it clearly admits distinct neibourhood sum for every vertex. Also form the definition and the theorems proved above we observe that the vertex set of every 1-vertex bimagic biregular graphs are partitioned into two sets V_1 and V_2 in such a way that each set contains the vertices of common count k_1 (degree r_1) or k_2 (degree r_2) respectively. We use this observation to prove the following theorems.

Theorem 4.8. *A regular or bi-regular graph G has a 1-vertex-bimagic vertex labeling iff it has an odd 1-vertex-bimagic vertex labeling.*

Proof. Suppose that G is a 1-vertex bimagic graph with p vertices and q edges. Then there exists a function $h : V(G) \to \{1,2,\dots,p\}$ such that for every vertex u, $\sum_{uv\in E(G)} h(v) = k_1$ or k_2. Now we define $f : V(G) \to \{1,3,\dots,2p-1\}$ such that $f(v_i) = 2h(v_i)-1;\ 1 \le i \le p$. For every vertex $u \in V(G)$ we have

$$\sum_{uv\in E(G)} f(v) = \sum_{uv\in E(G)} [2h(v)-1] = 2\sum_{uv\in E(G)} h(v) - deg(u)$$
$$= 2(k_1 \text{ or } k_2) - deg(u)$$
$$= 2k_1 - r_1 \text{ or } 2k_2 - r_2 = s_1 \text{ or } s_2.$$

Then G has an odd 1-vertex-bimagic vertex labeling with common counts s_1 and s_2. Conversely, suppose that G is a graph with p vertices and q edges and $f : V(G) \to \{1,3,\dots,2p-1\}$ is an odd 1-vertex-bimagic vertex labeling with two common count s_1 and s_2. Then $h : V(G) \to \{1,2,\dots,p\}$ defined by $h(v_i) = \frac{1}{2}[f(v_i)+1]$ for $1 \le i \le p$.

For every vertex $u \in V(G)$ we have

$$\sum_{uv\in E(G)} h(v) = \sum_{uv\in E(G)} \frac{1}{2}[f(v)+1] = \frac{1}{2}\left(\sum_{uv\in E(G)} f(v) + \sum_{uv\in E(G)} 1\right)$$
$$= \frac{1}{2}(s_1 \text{ or } s_2) + \frac{1}{2}deg(u) = k_1 \text{ or } k_2$$

Hence 1-vertex-bimagic vertex labeling has two common counts $k_1 = \frac{1}{2}(s_1+r_1)$ and $k_2 = \frac{1}{2}(s_2+r_2)$. □

Theorem 4.9. *A regular or bi-regular graph G has a 1-vertex-bimagic vertex labeling iff it has an even 1-vertex-bimagic vertex labeling.*

Proof. Suppose that G is a 1-vertex-bimagic graph with p vertices and q edges. Then there exists a function $h : V(G) \to \{1,2,\dots,p\}$ such that for every vertex u, $\sum_{uv\in E(G)} h(v) = k_1$ or k_2. Now we define $g : V(G) \to \{0,2,\dots,2p-2\}$ such that $g(v_i) = 2h(v_i)-2;\ 1 \le i \le p$. For every vertex $u \in V(G)$ we have

$$\sum_{uv\in E(G)} g(v) = \sum_{uv\in E(G)} [2h(v)-2] = 2\sum_{uv\in E(G)} h(v) - 2deg(u) = 2((k_1 \text{ or } k_2) - deg(u))$$
$$= 2k_1 - r_1 \text{ or } 2k_2 - r_2 = s_1 \text{ or } s_2$$

Then G has an even 1-vertex bimagic vertex labeling with common counts s_1 and s_2.

Conversely, suppose that G is a graph with p vertices and q edges and $g: V(G) \to \{0,2,\ldots,2p-2\}$ is an even 1-vertex-bimagic vertex labeling with two common edge count s_1 and s_2. Then $h: V(G) \to \{1,2,\ldots,p\}$ defined by $h(v_i) = \frac{1}{2}[g(v_i)+2]$ for $1 \le i \le p$.
For every vertex $u \in V(G)$ we have

$$\sum_{uv \in E(G)} h(v) = \sum_{uv \in E(G)} \frac{1}{2}[g(v)+2] = \frac{1}{2}\left(\sum_{uv \in E(G)} g(v) + \sum_{uv \in E(G)} 2\right)$$
$$= \frac{1}{2}(s_1 \text{ or } s_2) + deg(u) = k_1 \text{ or } k_2$$

Hence 1-vertex-bimagic vertex labeling has two common counts $k_1 = \frac{1}{2}(s_1) + deg(u)$ and $k_2 = \frac{1}{2}(s_2) + deg(u)$. □

Theorem 4.10. *If H is a regular or bi-regular 1-vertex bimagic graph then $G = H_n^{mK_1}$ $(m > 1)$, has a 1-vertex bimagic vertex labeling.*

Proof. Consider a graph H is a 1-vertex bimagic with n vertices $\{x_1, x_2, x_3, \ldots, x_n\}$. By replacing every vertex in H with m isolated vertices, each joined to all vertices corresponding to the neighbours of the original vertex of H, we obtain the graph $G = H_n^{mK_1}$ which is a multipartite graph. For $1 \le i \le m$ and $1 \le j \le n$, Let x_{ij} be the vertices of G that replace x_j, $1 \le j \le n$ in H. Given that H has a 1-vertex bimagic vertex labeling, then there exist a function $f: V \to \{1,2,\ldots,n\}$ such that $\sum_{un \in E(H)} f(v) = k_1$ or k_2 where k_1 and k_2 are constants. Let V_1 be the set of vertices having constant value k_1, that is for every vertex $y \in V_1$, $\sum_{yx \in E(H)} f(x) = k_1$. Let V_2 be the set of vertices having constant value k_2, that is for every vertex $y \in V_2$, $\sum_{yx \in E(H)} f(x) = k_2$.

Label the vertices of G in the following way
$g(x_{i,j}) = f(x_j) + (i-1)n$ for $1 \le i \le m$ and $1 \le j \le n$.
For every j in the interval $1 \le j \le n$,

$$\sum_{i=1}^{m} g(x_{i,j}) = \sum_{i=1}^{m} [f(x_j) + (i-1)n] = \sum_{i=1}^{m} f(x_j) + \sum_{i=1}^{m} (i-1)n = mf(x_j) + \frac{nm(m-1)}{2}$$

Let U and W be the two partitions of the graph G in such a way that U contains the sets which is having the vertices of V_1 in H and W contains the sets which is having the vertices of V_2 in H.

For any vertex $u \in U$, the $deg(u) = r_1$ (say).

$$\sum_{ux_{i,j} \in E(G)} g(x_{i,j}) = \sum_{ux_{i,j} \in E(G)} \sum_{i=1}^{m} g(x_{i,j}) = \sum_{ux_{i,j} \in E(G)} \left(mf(x_j) + \frac{nm(m-1)}{2}\right)$$

$$= \sum_{ux_{i,j}\in E(G)} mf(x_j) + \sum_{ux_{i,j}\in E(G)} \left(\frac{nm(m-1)}{2}\right)$$
$$= mk_1 + \left(\frac{nm(m-1)}{2}\right) deg(u) = mk_1 + \left(\frac{nm(m-1)}{2}\right) r_1 = s_1(\text{say})$$

For any vertex $u \in W$, the $deg(u) = r_2$ (say).

$$\sum_{ux_{i,j}\in E(G)} g(x_{i,j}) = \sum_{ux_{i,j}\in E(G)} \sum_{i=1}^{m} g(x_{i,j}) = \sum_{ux_{i,j}\in E(G)} \left(mf(x_j) + \frac{nm(m-1)}{2}\right)$$
$$= \sum_{ux_{i,j}\in E(G)} mf(x_j) + \sum_{ux_{i,j}\in E(G)} \left(\frac{nm(m-1)}{2}\right)$$
$$= mk_2 + \left(\frac{nm(m-1)}{2}\right) deg(u) = mk_2 + \left(\frac{nm(m-1)}{2}\right) r_2 = s_2(\text{say})$$

Thus for any vertex $u \in V(G)$ we have two constants s_1 or s_2.
Hence $G = H_n^{mK_1}$ has a 1-vertex bimagic vertex labeling. □

5. Conclusion

Theorem 3.7 shows a 1-vertex bimagic vertex labeling for certain class of symmetric multipartite graphs and the Theorem 4.10 shows a 1-vertex bimagic vertex labeling for a class of multipartite graphs. We belive that complete multipartite graph and multipartite graph have 1-vertex bimagic vertex labeling along with some condition. So, we conclude this paper with the following open problems.

Open Problem 5.1

(i) Does there exist 1-vertex bimagic vertex labeling for other classes of multipartite graphs.

(ii) Are there any necessary and sufficient conditions for a complete multipartite graph to have a 1-vertex bimagic vertex labeling.

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