TAMKANG JOURNAL OF MATHEMATICS
Volume 44, Number 1, 83-89, Spring 2013
doi:10.5556/j.tkjm.44.2013.1008

Available online at http://journals.math.tku.edu.tw/

A SUBCLASS OF CLOSE-TO-CONVEX FUNCTIONS

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Abstract. In this paper, we introduce and investigate an interesting subclass $\mathscr{J}_\alpha(h)$ of analytic and close-to-convex function in the open unit disk D. several coefficient inequalities, growth, and distortion theorem for this class are proved. The various results presented here would generalize many know results.

1. Introduction

Let $\mathbb{R} = (-\infty, +\infty)$ be the set of real numbers, $\mathbb{C}$ be the set of complex numbers and

$$\mathbb{N} = \{1, 2, 3, \ldots\}$$

be the set of positive integer. We also let $\mathscr{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open disk

$$D = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

We denote by $\mathscr{S}$ the subclass of the analytic function class $\mathscr{A}$ consisting of all functions in $\mathscr{A}$ which are also univalent in D.

A function $f(z) \in \mathscr{A}$ is said to be convex functions of order α if and only if

$$\Re(1 + \frac{zf''(z)}{f'(z)}) > \alpha \quad (|z| < 1, 0 \le \alpha < 1).$$

We denote by $\mathscr{K}(\alpha)$ the subclass of $\mathscr{A}$ consisting of all convex functions of order α. If $f \in \mathscr{A}$ and there exists a function $g \in \mathscr{S}^*$, such that

$$\Re\Big(\frac{zf'(z)}{g(z)}\Big) > 0 \quad (|z| < 1, 0 \le \alpha < 1),$$

Corresponding author: Zheng-lv Zhang.
2010 *Mathematics Subject Classification*. Primary 30C45; Secondary 34-99.
Key words and phrases. Convex functions of order α, close-to-convex, starlike function, subordination principle.
This work was supported the Jiangxi Provincial Natural Science Foundation of China (Grant No. 2010GZS0096), and Natural Science Foundation of Department of Education of Jiangxi Province, China (Grant No. GJJ 09149).

then f is said to be a close-to-convex function. Let $\mathscr{C}$ denote the set of all close-to-convex functions.

For two functions f and g analytic in D, we say that the function $f(z)$ is subordinate to $g(z)$ in D and write

$$f(z) \prec g(z) \quad (z \in D)$$

if there exists a Schwarz function $w(z)$, analytic in D with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \quad (z \in D)$$

such that

$$f(z) = g(w(z)) \quad (z \in D).$$

In particular, if the function g is univalent in D, the above subordination is equivalent to

$$f(0) = g(0) \quad \text{and} \quad f(D) \subset g(D).$$

In many earlier investigations, various interesting subclasses of the analytic function class $\mathscr{A}$ and the univalent function class $\mathscr{S}$ have been studied from a number of different viewpoints. We choose to recall here the investigation by H.R.Abdel-Gawad and D. K. Thomas [1], B. S. Mehrok and G. Singh [2], H. M. Srivastava et al. [3], X. Q. Hua et al. [4]. Selearaj.C[5]. In particular, H. R. Abdel-Gawad and D. K. Thomas [1] introduced a subclass $\mathscr{J}$ of analytic functions, which is indeed a subclass of close-to-convex functions.

Definition 1 ((see [1]). Let the function $f(z)$ be analytic in D and defined by (1). We say that $f \in \mathscr{J}$ if there exists a function $g \in \mathscr{K}$ such that:

$$\Re(\frac{zf'(z)}{g(z)}) > 0 \quad (z \in D).$$

Recently, B. S. Mehrok and G. Singh extend Definition 1 by introducing the following subclass of analytic functions.

Definition 2 ((see [2]). Let the function $f(z)$ be analytic in D and defined by (1). We say that $f \in \mathscr{J}(A, B)$ if there exists a function $g \in \mathscr{K}$, satisfying the following condition:

$$\Re(\frac{zf'(z)}{g(z)}) \prec \frac{1+Az}{1+Bz} \quad (z \in D, -1 \leq B < A \leq 1).$$

Definition 3. Let $h : D \longrightarrow \mathbb{C}$ be a convex function such that

$$h(0) = 1 \quad \text{and} \quad h(\bar{z}) = \overline{h(z)}(z \in D, \Re(h(z)) > 0).$$

Suppose also that the function h satisfies the following conditions for $r \in (0,1)$

$$\begin{cases} \min_{|z|=r} |h(z)| = \min\{h(r), h(-r)\} & (0<r<1) \\ \max_{|z|=r} |h(z)| = \max\{h(r), h(-r)\} & (0<r<1). \end{cases} \tag{2}$$

We denote by $\mathcal{J}_\alpha(h)$ the class of functions given by

$$\mathcal{J}_\alpha(h) = \left\{ f : f \in \mathcal{A} \quad \text{and} \quad \frac{zf'(z)}{g(z)} \in h(D) \quad (z \in D, g \in \mathcal{K}(\alpha)) \right\}.$$

Remark 1. There are many choices of the function h which would provide interesting subclass of analytic functions. For example, if we set

$$h(z) = \frac{1+Az}{1+Bz} \quad (z \in D, -1 \le B < A \le 1),$$

then it is easily verified that h is a convex function in D and satisfies the hypotheses of Definition 3. If $\mathcal{J}_\alpha(h)$, then

$$\frac{zf'(z)}{g(z)} \in h(D) \quad (z \in D)$$

for some $g \in \mathcal{K}(\alpha)$.

Let

$$p(z) = \frac{zf'(z)}{g(z)} \quad (z \in D),$$

and we have

$$p(0) = h(0) = 1 and p(z) \in h(D) \quad (z \in D).$$

According the principle of subordination, we obtain

$$\frac{zf'(z)}{g(z)} \prec \frac{1+Az}{1+Bz},$$

if $\alpha = 0$ then, $f \in \mathcal{J}(A,B)$ (see Definition 2).

2. Preliminaries

In order to prove the desired results, we first recall the following lemma.

Lemma 1 (see [6]). *Let the function $h(z)$ given by*

$$h(z) = \sum_{n=1}^{\infty} h_n z^n$$

be convex in D. Suppose also that the function $f(z)$ given by

$$f(z) = \sum_{n=1}^{\infty} a_n z^n$$

is holomorphic in D. If $f(z) \prec h(z)$, $(z \in D)$, then

$$|a_n| \le |h_1| \quad (n \in \mathbb{N}).$$

Theorem 1. *Let the function $f(z)$ be analytic in D and defined by (1). If $f \in \mathcal{J}_\alpha(h)$, then*

$$|a_n| \le \frac{|h'(0)| + \frac{1}{n!}\prod_{k=2}^{n}(k-2\alpha)}{n} + \frac{|h'(0)|}{n}\sum_{k=2}^{n-1}\frac{1}{k!}\prod_{j=2}^{k}(j-2\alpha).$$

Proof. Because $f \in \mathcal{J}_\alpha(h)$, we have

$$\frac{zf'(z)}{g(z)} \in h(D) \quad (z \in D), \tag{3}$$

where $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{K}(\alpha)$. Next, by setting

$$p(z) = \frac{zf'(z)}{g(z)} \quad (z \in D), \tag{4}$$

we deduce from (3) that

$$p(0) = h(0) = 1 \quad \text{and} \quad p(z) \in h(D) \quad (z \in D).$$

Therefore, we have

$$p(z) \prec h(z) \quad (z \in D).$$

According to Lemma 1, we thus deduce that

$$|p_n| = \left|\frac{p^{(n)}(0)}{n!}\right| \le |h'(0)| (n \in \mathbb{N}).$$

On the other hand, we readily find from (4) that

$$zf'(z) = g(z)p(z) \quad (z \in D). \tag{5}$$

Further, by letting

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots \quad (z \in D). \tag{6}$$

From (5), (6), and comparing the coefficients of two sides of the equation, we deduce that

$$na_n = p_{n-1} + b_2 p_{n-2} + \cdots + b_{n-1} p_1 + b_n \quad (n \in \mathbb{N}).$$

Since $g(z) \in \mathcal{K}(\alpha)$, it follows that $|b_n| \le \frac{1}{n!}\prod_{k=2}^{n}(k-2\alpha)(see[7])$. So, we from Lemma 1 obtain that

$$|na_n| \le |h'(0)| + |h'(0)|\sum_{k=2}^{n-1}\frac{1}{k!}\prod_{j=2}^{k}(j-2\alpha) + \frac{1}{n!}\prod_{k=2}^{n}(k-2\alpha) \quad (n \in \mathbb{N}).$$

Hence

$$|a_n| \le \frac{|h'(0)| + \frac{1}{n!}\prod_{k=2}^{n}(k-2\alpha)}{n} + \frac{|h'(0)|}{n}\sum_{k=2}^{n-1}\frac{1}{k!}\prod_{j=2}^{k}(j-2\alpha).$$

This completes the proof of Theorem 1. □

In view of Remark 1, if we set

$$h(z) = \frac{1+Az}{1+Bz} \quad (z \in D, -1 \le B < A \le 1),$$

$\alpha = 0$ in Theorem 1, we have the following coefficient bounds for functions belonging to the class $\mathscr{J}(A,B)$, which we merely state here without proof.

Corollary 1. *Let the function $f(z)$ be analytic in D and defined by (1). If $f \in \mathscr{J}(A,B)$, then*

$$|a_n| \le \frac{1}{n} + \frac{(n-1)(A-B)}{n} (n \ge 2).$$

Remark 2. Corollary 1 was proven by B. S. Mehrok and G. Singh [2, Theorem 3.1]. However, by using Theorem 1, we are able to deduce this result as an easy consequence of Theorem 1.

Theorem 2. *Let the function $f(z)$ be analytic in D and defined by (1), and $f \in \mathscr{J}_\alpha(h)$ (1) If $\alpha \ne \frac{1}{2}$, then for $|z| = r, 0 < r < 1$, we have*

$$\min\{h(-r), h(r)\} \frac{(1+r)^{2\alpha-1} - 1}{(2\alpha-1)r} \le |f'(z)| \le \frac{1-(1-r)^{2\alpha-1}}{(2\alpha-1)r} \max\{h(-r), h(r)\} \tag{7}$$

and

$$\int_0^r \min\{h(-\tau), h(\tau)\} \frac{(1+\tau)^{2\alpha-1}-1}{(2\alpha-1)\tau} d\tau \le |f(z)| \le \int_0^r \frac{1-(1-\tau)^{2\alpha-1}}{(2\alpha-1)\tau} \max\{h(-\tau), h(\tau)\} d\tau. \tag{8}$$

(2) *If $\alpha = \frac{1}{2}$, then for $|z| = r, 0 < r < 1$, we have*

$$\min\{h(-r), h(r)\} \frac{\log(1+r)}{r} \le |f'(z)| \le -\frac{\log(1-r)}{r} \max\{h(-r), h(r)\} \tag{9}$$

and

$$\int_0^r \min\{h(-\tau), h(\tau)\} \frac{\log(1+\tau)}{\tau} d\tau \le |f(z)| \le \int_0^r -\frac{\log(1-\tau)}{\tau} \max\{h(-\tau), h(\tau)\} d\tau. \tag{10}$$

Proof. Since $f \in \mathscr{J}_\alpha(h)$, there exist a function $g \in \mathscr{K}(\alpha)$, such that

$$\frac{zf'(z)}{g(z)} \prec h(D) \quad (z \in D).$$

From Definition 3, we find that

$$\min\{h(-r), h(r)\} \le \left| \frac{zf'(z)}{g(z)} \right| \le \max\{h(-r), h(r)\} \quad (|z| = r, 0 \le r < 1). \tag{11}$$

Again, $g(z) \in \mathcal{K}(\alpha)$. We have (see[8]) (1) If $\alpha \neq \frac{1}{2}$,

$$\frac{(1+r)^{2\alpha-1}-1}{2\alpha-1} \leq |\, g(z)\,| \leq \frac{1-(1-r)^{2\alpha-1}}{2\alpha-1} \quad (|z|=r, 0 \leq r < 1). \tag{12}$$

(2) If $\alpha = \frac{1}{2}$,

$$\log(1+r) \leq |g(z)| \leq -\log(1-r) \tag{13}$$

Combining (11), (12) and (13), we can get the inequalities (7) and (9). To prove the inequalities (8), let $z = re^{i\theta}(0 < r < 1)$. If ε denotes the closed line-segment in the complex ζ-plane from $\zeta = 0$ and $\zeta = z$, we have

$$f(z) = \int_{\varepsilon} f'(\zeta)d\zeta = \int_0^r f'(\tau e^{i\theta} e^{i\theta})d\tau \quad (|z|=r, 0 \leq r < 1).$$

Thus, by using the upper estimate in (7). If $\alpha \neq \frac{1}{2}$, we have

$$\begin{aligned} |\, f(z)\,| &= |\int_0^z f'(\zeta)d\zeta \,| \leq \int_0^r |f'(\tau e^{i\theta})| d\tau \\ &\leq \int_0^r \frac{1-(1-\tau)^{2\alpha-1}}{(2\alpha-1)\tau} \max\{h(-\tau), h(\tau)\} d\tau \quad (|z|=r, 0 \leq r < 1). \end{aligned}$$

To prove the lower bound of $f(z)$, it is sufficient to show that it holds true for the nearest point $f(z_0)$ from zero, where

$$|z| = r \quad (0 < r < 1).$$

Moreover. we have

$$|f(z)| \geq |f(z_0)| \quad (|z|=r, 0 \leq r < 1),$$

Since $f(z)$ is a close-to-convex function in the open unit disk D, it is univalent in D. We deduce that the original image of the closed line-segment ε_0 in the complex ζ-plane from $\zeta = 0$ and $\zeta = f(0)$ is a piece of arc Γ in the disk D_r given by

$$D_r = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| \leq r \quad (0 \leq r < 1)\}.$$

Hence. We have

$$\begin{aligned} |f(z_0)| &= \int_{f(\Gamma)} |dw| = \int_{\Gamma} |f'(z)||dz| \\ &\geq \int_0^r \min\{h(-\tau), h(\tau)\} \frac{(1+\tau)^{2\alpha-1}-1}{(2\alpha-1)\tau} d\tau \quad (|z|=r, 0 \leq r < 1). \end{aligned}$$

Similarly we can prove (10). This completes the proof of Theorem 2. □

In view of Review 1, if we set

$$h(z) = \frac{1+Az}{1+Bz} \quad (z \in D, -1 \leq B < A \leq 1)$$

$\alpha = 0$ in Theorem 2, we have the following the distortion and growth theorems for functions belonging to the class $\mathcal{J}(A,B)$, which we merely state here without proof.

Corollary 2. *If $f \in \mathscr{J}(A,B)$, then $|z| = r < 1, 0 < r < 1$, we have*

$$\frac{1-Ar}{(1+r)(1-Br)} \le |f'(z)| \le \frac{1+Ar}{(1-r)(1-Br)},$$

and

$$\int_0^r \frac{1-At}{(1+t)(1-Bt)} dt \le | f(z) | \le \int_0^r \frac{1+At}{(1-t)(1-Bt)} dt.$$

Remark 3. Corollary 2 was proven by B. S. Mehrok and G. Singh [2, Theorem 4.1]. However, by using Theorem 2, we are able to derive this result much more easily as consequence of Theorem 2.

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