SUBORDINATIONS FOR MULTIVALENT ANALYTIC FUNCTIONS ASSOCIATED WITH WRIGHT GENERALIZED HYPERGEOMETRIC FUNCTION

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Abstract. Recently M. K. Aouf and T. M. Seoudy, (2011, *Integral Trans. Spec. Func.* **22**(6) (2011), 423–430) have introduced families of analytic functions associated with the Dziok–Srivastava operator. In this paper we use the Dziok–Raina operator to consider classes of multivalent analytic functions. It is connected with Wright generalized hypergeometric function, see J. Dziok and R. K. Raina (2004, *Demonstratio Math.*, **37**(3) 533-542). In this paper we present a new result which extends some of the earlier results and give other properties of these classes. We have made use of differential subordinations and properties of convolution in geometric function theory.

1. Introduction

Let Λ_p denote the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} c_k z^k, \ (p \in \mathbb{N} := \{1, 2, 3, \ldots\}), \tag{1.1}$$

which are analytic in the open unit disk

$$\mathbb{U} := \{z : z \in \mathbb{C} \ \text{ and } \ |z| < 1\}.$$

For analytic functions $f \in \Lambda_p$, given by (1.1) and $g \in \Lambda_p$ given by

$$g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k, (p \in \mathbb{N}) \tag{1.2}$$

the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) = z^p + \sum_{k=p+1}^{\infty} c_k b_k z^k = (g * f)(z).$$

Let $a_1, A_1, \ldots, a_q, A_q$ and $b_1, B_1, \ldots, b_s, B_s$ $(q, s \in \mathbb{N})$ be positive real parameters such that

$$1 + \sum_{k=1}^{s} B_k \geq \sum_{k=1}^{q} A_k.$$

The Wright generalized hypergeometric function [17]

$${}_q\Psi_s[(a_1, A_1), \ldots, (a_q, A_q); (b_1, B_1), \ldots, (b_s, B_s); z] := {}_q\Psi_s[(a_m, A_m)_{1,q}; (b_m, B_m)_{1,s}; z]$$

is defined by

$${}_q\Psi_s[(a_k, A_k)_{1,q}; (b_k, B_k)_{1,s}; z] := \sum_{m=0}^{\infty} \left\{ \frac{\Pi_{m=1}^{q} \Gamma(a_m + nA_m)}{\Pi_{m=1}^{s} \Gamma(b_m + nB_m)} \right\} \frac{z^m}{m!}, \ (z \in \mathbb{U}). \tag{1.3}$$

If $A_m = 1$ $(m = 1, \ldots, q)$ and $B_m = 1$ $(m = 1, \ldots, s)$, we have the following relationship

$$\omega_q\Psi_s[(a_m, 1)_{1,q}; (b_m, 1)_{1,s}; z] := {}_qF_s(a_1, \ldots, a_q; b_1, \ldots, b_s; z),$$

where ${}_qF_s(a_1, \ldots, a_q; b_1, \ldots, b_s; z)$ is the generalized hypergeometric function (see for details [1, 4, 5] and any many others) and

$$\omega = \left\{\Pi_{m=1}^{q} \Gamma(a_m)\right\}^{-1} \left\{\Pi_{m=1}^{s} \Gamma(b_m)\right\}. \tag{1.4}$$

The Wright generalized hypergeometric functions (1.3) have been recently involved in the geometric function theory, see , as well as: [2, 3, 5, 6, 8, 12, 13]. Using the Wright generalized hypergeometric functions we introduce the linear operator

$$\Theta_p[(a_m, A_m)_{1,q}; (b_m, B_m)_{1,s}] : \Lambda_p \mapsto \Lambda_p$$

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defined by the convolution

$$\Theta_p[(a_m, A_m)_{1,q}; (b_m, B_m)_{1,s}]f(z) = \omega \left\{z^p {}_q\Psi_s[(a_m, A_m)_{1,q}; (b_m, B_m)_{1,s}; z]\right\} * f(z).$$

In particular, the operator

$$\Theta[(a_m, A_m)_{1,q}; (b_m, B_m)_{1,s}] = \Theta_1[(a_m, A_m)_{1,q}; (b_m, B_m)_{1,s}] \tag{1.5}$$

was investigated by Dziok and Raina [6]. We observe that for a function f defined by (1.1) we have

$$\Theta_p[(a_m, A_m)_{1,q}; (b_m, B_m)_{1,s}]f(z) = z^p + \sum_{k=p+1}^{\infty} \Omega_k[a_i; b_i] c_k z^k, \tag{1.6}$$

where

$$\Omega_k[a_i; b_i] = \omega \frac{\Gamma(a_1 + A_1(k-p)) \ldots \Gamma(a_q + A_q(k-p))}{\Gamma(b_1 + B_1(k-p)) \ldots \Gamma(b_s + B_s(k-p))(k-p)!} \tag{1.7}$$

and ω is given by (1.4). For our convenience, we write

$$\Theta_p[a_i]f := \Theta_p[(a_m, A_m)_{1,q}; (b_m, B_m)_{1,s}]f.$$

Let f and g be analytic in $\mathbb{U}$. Then we say that the function g is subordinate to f if there exists an analytic function w in $\mathbb{U}$ such that

$$w(0) = 0, \quad |w(z)| < 1 \quad (z \in \mathbb{U}) \quad \text{and} \quad g(z) = f(w(z)).$$

For this subordination, the symbol $g(z) \prec f(z)$ is used. In case f is univalent in $\mathbb{U}$, the subordination $g \prec f$ is equivalent to

$$g(0) = f(0) \quad \text{and} \quad g(\mathbb{U}) \subset f(\mathbb{U}).$$

Now we define two subclasses $\mathcal{S}_p[A, B]$ and $\mathcal{K}_p[A, B]$ of the class Λ_p, for $-1 \le B < A \le 1$, $p \in \mathbb{N}$, as follows:

$$\mathcal{S}_p[A, B] = \left\{ f \in \Lambda_p : \frac{zf'(z)}{f(z)} \prec p\frac{1 + Az}{1 + Bz}, \ (z \in \mathbb{U}) \right\}, \tag{1.8}$$

$$\mathcal{K}_p[A, B] = \left\{ f \in \Lambda_p : 1 + \frac{zf''(z)}{f'(z)} \prec p\frac{1 + Az}{1 + Bz}, \ (z \in \mathbb{U}) \right\}. \tag{1.9}$$

Clearly

$$zf' \in \mathcal{S}_p[A, B] \Leftrightarrow f \in \mathcal{K}_p[A, B]. \tag{1.10}$$

We note that for special choices of the parameters we obtain the classes of p-valent starlike of order α and of p-valent convex of order α

$$\mathcal{S}_1[1 - 2\alpha, -1] = \mathcal{S}^*(\alpha) \ , \ \mathcal{K}_1[1 - 2\alpha, -1] = \mathcal{K}(\alpha),$$

$$\mathcal{S}_p[1 - 2\alpha, -1] = \mathcal{S}_p^*(\alpha) \ , \ \mathcal{K}_p[1 - 2\alpha, -1] = \mathcal{K}_p(\alpha),$$

where $0 \le \alpha < 1$. Next, using the operator $\Theta_p[a_i]$, we introduce the following classes of analytic functions for $q, s \in \mathbb{N}$ and $-1 \le B < A \le 1$

$$\mathcal{S}^*_{p,q,s}[a_1; A, B] = \{f \in \Lambda_p : \Theta_p[a_i]f \in \mathcal{S}_p[A, B]\} \tag{1.11}$$

and

$$\mathcal{K}_{p,q,s}[a_i; A, B] = \{f \in \Lambda_p : \Theta_p[a_i]f \in \mathcal{K}_p[A, B]\}. \tag{1.12}$$

We also note that

$$zf' \in \mathcal{S}^*_{p,q,s}[a_i; A, B] \Leftrightarrow f \in \mathcal{K}_{p,q,s}[a_i; A, B].$$

If $A_m = 1$ $(m = 1, \ldots, q)$ and $B_m = 1$ $(m = 1, \ldots, s)$, then the operator $\Theta_p[a_i]f$ becomes the operator $\mathcal{H}_p$:

$$\mathcal{H}_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s)f = z^p {}_qF_s(a_1, \ldots, a_q; b_1, \ldots, b_s; z) * f,$$

and the class $\mathcal{S}^*_{p,q,s}[a_1; A, B]$ becomes the class $V_p(\alpha_i; A, B)$ considered in [14], see also [11, 15, 16].

2. Main results

We assume throughout this section that $0 \le \theta < 2\pi$, $-1 \le B < A \le 1$ and $\Omega_k[a_i]$ is defined by (1.7).

Theorem 2.1. *Suppose that the function f is defined by (1.1). Then f is in the class $\mathcal{S}_p[A,B]$ if and only if*

$$\frac{1}{z^p}\left[f(z) * \frac{z^p - Dz^{p+1}}{(1-z)^2}\right] \ne 0 \tag{2.1}$$

for all $z \in \mathbb{U}$ and $0 \le \theta < 2\pi$, where

$$D = 1 + \frac{e^{-i\theta} + B}{p(A-B)}. \tag{2.2}$$

Proof. First, suppose f is in the class $\mathcal{S}_p[A,B]$. Then by definition of subordination it is equivalent to

$$\frac{zf'(z)}{f(z)} \ne p\frac{1+Ae^{i\theta}}{1+Be^{i\theta}}$$

for all $z \in \mathbb{U}$ and $0 \le \theta < 2\pi$. Since,

$$\frac{z^p}{(1-z)^2} = z^p + \sum_{k=p+1}^{\infty}(k-p+1)z^k, \text{ and } \frac{z^{p+1}}{(1-z)^2} = \sum_{k=p+1}^{\infty}(k-p)z^k \tag{2.3}$$

we have

$$\begin{aligned}
&\frac{1}{z^p}\left[f(z) * \frac{z^p - Dz^{p+1}}{(1-z)^2}\right] \\
=\ &\frac{1}{z^p}\left[f(z) * \{z^p + \sum_{k=p+1}^{\infty} z^k + (1-D)\sum_{k=p+1}^{\infty} kz^k + p(D-1)\sum_{k=p+1}^{\infty} z^k\}\right] \\
=\ &\frac{1}{z^p}\left[f(z) + (1-D)\sum_{k=p+1}^{\infty} kc_k z^k + p(D-1)\sum_{k=p+1}^{\infty} c_k z^k\right] \\
=\ &\frac{1}{z^p}\left[f(z) + (1-D)[zf'(z) - pz^p] + p(D-1)[f(z) - z^p]\right] \\
=\ &\frac{1}{z^p}\left[f(z)\left\{\frac{zf'(z)}{f(z)}(1-D) + (1-p+pD)\right\}\right] \\
\ne\ &\frac{1}{z^p}f(z)\left[-\frac{e^{-i\theta}+A}{A-B} + \frac{e^{-i\theta}+A}{A-B}\right] \\
=\ &0
\end{aligned}$$

which proves the necessary part.
Again, if the condition (2.1) hold, then because

$$\frac{1}{z^p}\left[f(z) * \frac{z^p - Dz^{p+1}}{(1-z)^2}\right] = \frac{1}{z^p}\left[f(z)\left\{\frac{zf'(z)}{f(z)}(1-D) + (1-p+pD)\right\}\right], \tag{2.4}$$

we can write

$$\frac{1}{z^p}\left[f(z)\left\{\frac{zf'(z)}{f(z)}(1-D) + (1-p+pD)\right\}\right] \ne 0$$

for all $z \in \mathbb{U}$ and $0 \le \theta < 2\pi$. Then we easily obtain the required result as

$$\frac{zf'(z)}{f(z)} \ne p\frac{1+Ae^{i\theta}}{1+Be^{i\theta}}$$

for all $z \in \mathbb{U}$ and $0 \le \theta < 2\pi$, which proves that $f \in \mathcal{S}_p[A,B]$. □

Theorem 2.2. *Suppose that the function f is defined by (1.1). Then f is in the class $\mathcal{K}_p[A,B]$ if and only if*

$$\frac{1}{z^p}\left[f(z) * \frac{pz^p - \{p + D(p+1) - 2)\}z^{p+1} + D(p-1)z^{p+2}}{(1-z)^3}\right] \ne 0 \tag{2.5}$$

for all $z \in \mathbb{U}$ and $0 \le \theta < 2\pi$, where D is given by (2.2).

Proof. Choose

$$g(z)=\frac{z^p-Dz^{p+1}}{(1-z)^2}$$

and we note that

$$zg'(z)=\left[\frac{pz^p-\{p+D(p+1)-2)\}z^{p+1}+D(p-1)z^{p+2}}{(1-z)^3}\right]. \tag{2.6}$$

From the identity $zf'(z)*g(z)=f(z)*zg'(z)$ and the fact that

$$f\in\mathcal{K}_p[A,B]\Leftrightarrow zf'(z)\in\mathcal{S}_p[A,B]$$

we can say that $f\in\mathcal{K}_p[A,B]$ if and only if

$$\begin{aligned}&\frac{1}{z^p}\left[zf'(z)*g(z)\right]\neq 0,\ \text{(By Theorem 2.1)}\\ \Leftrightarrow\quad&\frac{1}{z^p}\left[f(z)*zg'(z)\right]\neq 0\end{aligned}$$

which, on using (2.6) gives the required result (2.5). □

Theorem 2.3. *Suppose that the function f is defined by* (1.1). *Then a necessary and sufficient condition for the function f to be in the class $\mathcal{S}^*_{p,q,s}[a_i;A,B]$ is that*

$$1-\sum_{k=p+1}^{\infty}\left[\frac{(k-p)e^{-i\theta}-Ap+B(k-p+1)}{p(A-B)}\right]\Omega_k[a_i;b_i]c_kz^{k-p}\neq 0 \tag{2.7}$$

for all $z\in\mathbb{U}$ and $0\le\theta<2\pi$.

Proof. From Theorem 2.1, we find that $f\in\mathcal{S}^*_p[a_i;A,B]$ if and only if

$$\frac{1}{z^p}\left[\Theta_p[a_i]f(z)*\frac{z^p-Dz^{p+1}}{(1-z)^2}\right]\neq 0 \tag{2.8}$$

for all $z\in\mathbb{U}$ and $0\le\theta<2\pi$. Using relations (2.3) and after a long calculations with the help of (1.6), we can write from (2.7) that

$$\begin{aligned}&1+\sum_{k=p+1}^{\infty}\left[1+(p-k)(D-1)\right]\Omega_k[a_i;b_i]c_kz^{k-p}\neq 0\\ \Leftrightarrow\quad&1-\sum_{k=p+1}^{\infty}\left[\frac{(k-p)e^{-i\theta}-Ap+B(k-p+1)}{p(A-B)}\right]\Omega_k[a_i;b_i]c_kz^{k-p}\neq 0\end{aligned}$$

This proves Theorem 2.3. □

Theorem 2.4. *Suppose that the function f is defined by* (1.1). *Then a necessary and sufficient condition for the function f to be in the class $\mathcal{K}_{p,q,s}[a_i;A,B]$ is that*

$$1-\sum_{k=p+1}^{\infty}k\frac{(k-p)e^{-i\theta}-Ap+Bk}{p(A-B)}\Omega_k[a_i;b_i]c_kz^{k-p}\neq 0 \tag{2.9}$$

for all $z\in\mathbb{U}$ and $0\le\theta<2\pi$.

Proof. From Theorem 2.2, we find that $f\in\mathcal{K}_{p,q,s}[a_i;A,B]$ if and only if

$$\frac{1}{z^p}\left[\Theta_p[a_i]f(z)*\frac{pz^p-\{p+D(p+1)-2)\}z^{p+1}+D(p-1)z^{p+2}}{(1-z)^3}\right]\neq 0 \tag{2.10}$$

for all $z\in\mathbb{U}$ and $0\le\theta<2\pi$. Now, it can be easily shown that

$$\frac{z^p}{(1-z)^3}=z^p+\sum_{k=p+1}^{\infty}\frac{(k-p+1)(k-p+2)}{2}z^k, \tag{2.11}$$

$$\frac{z^{p+1}}{(1-z)^3}=\sum_{k=p+1}^{\infty}\frac{(k-p)(k-p+1)}{2}z^k, \tag{2.12}$$

$$\frac{z^{p+2}}{(1-z)^3}=\sum_{k=p+1}^{\infty}\frac{(k-p-1)(k-p)}{2}z^k. \tag{2.13}$$

Using (2.11)–(2.13) in (2.10) and noting that $\Theta_p[a_i]f(z) * z^p = z^p$, we can say that (2.10) is equivalent to

$$\begin{aligned} p &+ \frac{1}{2}\sum_{k=p+1}^{\infty}(k-p+1)[p(k-p+2)-(k-p)\{(p-2)+D(p+1)\} \\ &+ D((k-p)(p-1)]\Omega_k[a_i;b_i]c_k z^{k-p} \neq 0 \\ &\Leftrightarrow\ p-\sum_{k=p+1}^{\infty}k\frac{(k-p)e^{-i\theta}-Ap+Bk}{p(A-B)}\Omega_k[a_i;b_i]c_k z^{k-p}\neq 0 \\ &\Leftrightarrow\ 1-\sum_{k=p+1}^{\infty}k\frac{(k-p)e^{-i\theta}-Ap+Bk}{p(A-B)}\Omega_k[a_i;b_i]c_k z^{k-p}\neq 0 \end{aligned}$$

Thus, the proof of Theorem 2.4 is completed. □

Theorem 2.5. *If the function f is defined by* (1.1) *and it belongs to $\mathcal{S}^*_{p,q,s}[a_i;A,B]$, then*

$$\sum_{k=p+1}^{\infty}[(k-p)+|Ap-B(k-p+1)|]\Omega_k[a_i;b_i]|c_k|\leq p(A-B). \tag{2.14}$$

Proof. Since

$$\begin{aligned} &\left|1-\sum_{k=p+1}^{\infty}\left[\frac{(k-p)e^{-i\theta}-Ap+B(k-p+1)}{p(A-B)}\right]\Omega_k[a_i;b_i]c_k z^{k-p}\right| \\ > &\ 1-\sum_{k=p+1}^{\infty}\left|\frac{(k-p)e^{-i\theta}-Ap+B(k-p+1)}{p(A-B)}\right|\Omega_k[a_i;b_i]\,|c_k| \end{aligned}$$

and

$$\begin{aligned} &\left|\frac{(k-p)e^{-i\theta}-Ap+B(k-p+1)}{p(A-B)}\right| \\ = &\ \frac{|(k-p)e^{-i\theta}-Ap+B(k-p+1)|}{p(A-B)} \\ \leq &\ \frac{(k-p)+|Ap-B(k-p+1)|}{p(A-B)} \end{aligned}$$

the results follows from Theorem 2.3. □

In the same way, we can also prove the following theorem.

Theorem 2.6. *If the function f is defined by* (1.1) *and it belongs to $\mathcal{K}_{p,q,s}[a_i;A,B]$, then*

$$\sum_{k=p+1}^{\infty}k\{(k-p)+|Ap-Bk|\}\Omega_k[a_i;b_i]|c_k|\leq p(A-B). \tag{2.15}$$

We will need the following Lemma on Briot–Bouquet differential subordination.

Lemma 2.7. [9] *Let h be convex univalent in $\mathbb{U}$, with $\mathfrak{Re}[(\beta h(z)+\gamma)]\geq 0$. If q is analytic in $\mathbb{U}$, with $q(0)=h(0)$, then*

$$q(z)+\frac{zq'(z)}{\beta q(z)+\gamma}\prec h(z)\Rightarrow q(z)\prec h(z).$$

Corollary 2.8. *If q is an analytic function in $\mathbb{U}$, $q(0)=p$ and*

$$q(z)+\frac{zq'(z)}{q(z)+\gamma}\prec p\frac{1+Az}{1+Bz}\ ,\qquad \mathfrak{Re}\left[\gamma+p\frac{1+A}{1+B}\right]\geq 0,$$

then

$$q(z)\prec p\frac{1+Az}{1+Bz}.$$

Theorem 2.9. *Suppose that $i\in\{1,\ldots,q\}$ and*

$$\frac{a_i}{A_i}\geq p\frac{A-B}{1+B}.$$

Then for $m\in\mathbb{N}$ we have

$$\mathcal{K}_p[a_i+m;A,B]\subset\mathcal{K}_p[a_i;A,B].$$

Proof. It is clear that it is sufficient to prove this theorem for $m = 1$. Let a function f belong to the class $\mathcal{K}_p[a_i+1; A, B]$, then from (1.12), we can write

$$1 + \frac{z[\Theta_p[a_i+1]f(z)]''}{[\Theta_p[a_i+1]f(z)]'} \prec p\frac{1+Az}{1+Bz} \quad (z \in \mathbb{U}). \tag{2.16}$$

From (1.7) we get

$$\Omega_k[a_i+1; b_i] = \frac{a_i + A_i(k-p)}{a_i}\Omega_k[a_i; b_i],$$

thus through (1.6) we obtain

$$\begin{aligned}
\Theta_p[a_i+1]f(z) &= z^p + \sum_{k=p+1}^{\infty} \Omega_k[a_i+1; b_i]c_k z^k \\
&= z^p + \sum_{k=p+1}^{\infty} \frac{a_i + A_i(k-p)}{a_i}\Omega_k[a_i; b_i]c_k z^k \\
&= \left(1 - \frac{pA_i}{a_i}\right)\left[z^p + \sum_{k=p+1}^{\infty} \Omega_k[a_i; b_i]c_k z^k\right] \\
&\quad + \frac{A_i}{a_i}\left[pz^p + \sum_{k=p+1}^{\infty} \Omega_k[a_i; b_i]kc_k z^k\right] \\
&= \left(1 - \frac{pA_i}{a_i}\right)\Theta_p[a_i]f(z) + \frac{A_i}{a_i}z\left[\Theta_p[a_i]f(z)\right]'. \qquad (2.17)
\end{aligned}$$

Therefore, after some calculations we obtain from (2.17)

$$1 + \frac{z\left[\Theta_p[a_i+1]f(z)\right]''}{[\Theta_p[a_i+1]f(z)]'} = F(z) + \frac{zF'(z)}{F(z) + \frac{a_i}{A_i} - p}, \tag{2.18}$$

where

$$F(z) = 1 + \frac{z\left[\Theta_p[a_i]f(z)\right]''}{[\Theta_p[a_i]f(z)]'}$$

Thus from (2.16) the right side of (2.18) is subordinated to the function $h(z) = p\frac{1+Az}{1+Bz}$ which is convex and univalent in $\mathbb{U}$. Therefore, by Corollary 2.8

$$1 + \frac{z\left[\Theta_p(a_i)f(z)\right]''}{[\Theta_p(a_i)f(z)]'} \prec p\frac{1+Az}{1+Bz} \quad (z \in \mathbb{U}),$$

whenever $\mathfrak{Re}\left[p\frac{1+Az}{1+Bz} + \frac{a_i}{A_i} - p\right] > 0$ for $z \in \mathbb{U}$, which follows from the assumption that $\frac{a_i}{A_i} \geq p\frac{A-B}{1+B}$ and $-1 \leq B < A \leq 1$. □

Remarks :

(i) On substituting $p = 1$ and $A_i = B_j = 1$, where $i = 1, \ldots, q$ and $j = 1, \ldots, s$ in Theorems 2.1-2.6, we obtain known results given by Aouf and Seoudy [2, Theorems 1-6].

(ii) On substituting $p = m = 1$ and $A_i = B_j = 1$, where $i = 1, \ldots, q$ and $j = 1, \ldots, s$ in Theorems 2.9, we obtain known results given by Aouf and Seoudy [2, Theorem 8] but the proof is not convincing there.

Using Lemma 2.7 one can obtain a sufficient conditions for functions to be in the class $\mathcal{S}_p^*[a_i; A, B]$. This problem was considered earlier in [7] in a more general situation. This result is presented below, in current notation.

Theorem 2.10. [7] *Let $p \in \mathbb{N}$, $i \in \{1, \ldots, q\}$, $0 \leq B < A \leq 1$ and $\frac{a_i}{A_i} \geq p\frac{A-B}{1+B}$. If a function $f \in \Lambda_p$ satisfies the following inequality*

$$\left|\frac{\Theta_p[a_i+2]f(z)}{\Theta_p[a_i+1]f(z)} - 1\right| < \frac{p(A-B)(1+A_i)}{(1+B)(1+a_i)} \quad (z \in \mathbb{U}), \tag{2.19}$$

then f belongs to the class $\mathcal{S}_p^*[a_i; A, B]$.

To obtain next theorem which is in a way the sharp version of Theorem 2.10 we shall recall an another basic lemma in the theory of Briot–Bouquet differential subordinations.

Lemma 2.11. [10] *Let $\beta, \gamma, \delta \in \mathbb{C}$ with $\mathfrak{Re}[\beta + \gamma] > |\beta\delta|$. If $s(z) = 1 + a_1 z + a_2 z^2 + \cdots$ satisfies*

$$s(z) + \frac{zs'(z)}{\beta s(z) + \gamma} \prec 1 + \delta z \quad (z \in \mathbb{U}),$$

then

$$s(z) \prec q(z) \prec 1 + \delta z \quad (z \in \mathbb{U}),$$

where

$$q(z) = \frac{1}{\beta g(z)} - \frac{\gamma}{\beta},$$

and

$$g(z) = \int_0^1 e^{\beta\delta(t-1)z} t^{\beta+\gamma-1} \mathrm{d}t = {}_1F_1(1, \beta+\gamma+1; -\beta\delta z)(\beta+\gamma)^{-1}.$$

Moreover, the function q is the best dominant in the sense that if $s \prec q_1$, then $q \prec q_1$.

Lemma 2.11 in a more general case one can be found in [10, p. 109].

Theorem 2.12. *Let $p \in \mathbb{N}$, $i \in \{1, \ldots, q\}$, $0 \le B < A \le 1$ and $\frac{a_i}{A_i} \ge p\frac{A-B}{1+B}$. If a function $f \in \Lambda_p$ satisfies the following inequality*

$$\left| \frac{\Theta_p[a_i+2]f(z)}{\Theta_p[a_i+1]f(z)} - 1 \right| < r = \frac{p(A-B)(1+A_i)}{(1+B)(1+a_i)} \quad (z \in \mathbb{U}), \tag{2.20}$$

then

$$\frac{[\Theta_p[a_i]f(z)]'}{\Theta_p[a_i]f(z)} \prec q(z) \prec p + \frac{(a_i+1)r}{A_i+1} z \quad (z \in \mathbb{U}), \tag{2.21}$$

where the function q is given by

$$q(z) = \left[\int_0^1 e^{\frac{rz(a_i+1)(t-1)}{1+A_i}} t^{\frac{a_i}{A_i}-1} \mathrm{d}t \right]^{-1} - \left(\frac{a_i}{A_i} - p \right) \tag{2.22}$$

and it is the best dominant.

Proof. By using (2.17) the inequality (2.19) becomes

$$\frac{z\,[\Theta_p[a_i+1]f(z)]'}{\Theta_p[a_i+1]f(z)} \prec p + \frac{a_i+1}{A_i+1} rz \quad (z \in \mathbb{U}), \tag{2.23}$$

where

$$r = \frac{p(A-B)(1+A_i)}{(1+B)(1+a_i)}.$$

From (2.23) trough (2.17) we have

$$\begin{aligned} \frac{z\,[\Theta_p[a_i+1]f(z)]'}{p\Theta_p[a_i+1]f(z)} &= \frac{z\,[\Theta_p[a_i]f(z)]'}{p\Theta_p[a_i]f(z)} + \frac{z\left[\frac{z[\Theta_p[a_i]f(z)]'}{p\Theta_p[a_i]f(z)}\right]'}{p\frac{z[\Theta_p[a_i]f(z)]'}{p\Theta_p[a_i]f(z)} + \frac{a_i}{A_i} - p} \\ &\prec 1 + \frac{r(1+a_i)}{p(1+A_i)} z. \end{aligned} \tag{2.24}$$

If we apply Lemma 2.11 to (2.24) with $\delta = \frac{r(1+a_i)}{p(1+A_i)}$, $\beta = p$, $\gamma = \frac{a_i}{A_i} - p$ and

$$s(z) = \frac{z\,[\Theta_p[a_i]f(z)]'}{p\Theta_p[a_i]f(z)}$$

we obtain

$$s(z) = \frac{z\,[\Theta_p[a_i]f(z)]'}{p\Theta_p[a_i]f(z)} \prec \frac{1}{p\int_0^1 e^{\frac{rz(a_i+1)(t-1)}{1+A_i}} t^{\frac{a_i}{A_i}-1} \mathrm{d}t} - \left(\frac{a_i}{pA_i} - 1 \right),$$

what gives (2.21) and the best dominant (2.22). $\square$

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