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COMPLETELY MONOTONIC FUNCTION ASSOCIATED WITH THE GAMMA FUNCTIONS AND PROOF OF WALLIS' INEQUALITY

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Abstract. We prove: (i) A logarithmically completely monotonic function is completely monotonic. (ii) For $x > 0$ and $n = 0, 1, 2, \ldots$, then

$$(-1)^n \left(\ln \frac{x\Gamma(x)}{\sqrt{x+1/4}\,\Gamma(x+1/2)} \right)^{(n)} > 0.$$

(iii) For all natural numbers n, then

$$\frac{1}{\sqrt{\pi(n+4/\pi-1)}} \le \frac{(2n-1)!!}{(2n)!!} < \frac{1}{\sqrt{\pi(n+1/4)}}.$$

The constants $\frac{4}{\pi} - 1$ and $\frac{1}{4}$ are the best possible.

A function f is said to be completely monotonic on an interval I, if f has derivatives of all orders on I and satisfies

$$(-1)^n f^{(n)}(x) \ge 0 \quad (x \in I; n = 0, 1, 2, \ldots). \tag{1}$$

If the inequality (1) is strict, then f is said to be strictly completely monotonic on I. Completely monotonic functions have remarkable applications in different branches. For instance, they play a role in potential theory [2], probability theory [4, 7, 9], physics [6], numerical and asymptotic analysis [8, 14], and combinatorics [1]. A detailed collection of the most important properties of completely monotonic functions can be found in [13, Chapter IV], and in an abstract in [3].

A positive function f is said to be logarithmically completely monotonic on an interval I if its logarithm $\ln f$ satisfies

$$(-1)^n [\ln f(x)]^{(n)} \ge 0 \quad (x \in I; n = 1, 2, \ldots). \tag{2}$$

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If the inequality (2) is strict, then f is said to be strictly logarithmically completely monotonic, see [10].

The gamma function

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \quad (\mathrm{Re} z > 0)$$

is one of the most important function in analysis and its applications. The history and the development of this function are described in detail in [5]. The logarithm of the gamma function can be expressed [11, p.152] as

$$\ln \Gamma(z) = \int_0^\infty \left[e^{-t}(z-1) + \frac{e^{-zt} - e^{-t}}{1 - e^{-t}} \right] \frac{dt}{t} \quad (\mathrm{Re} z > 0). \tag{3}$$

In this paper, we obtain the following results.

Theorem 1. *If the function ϕ defined on an interval I is* (*strictly*) *completely monotonic, then* $\exp \phi$ *is also* (*strictly*) *completely monotonic on* I.

Proof. Since ϕ is completely monotonic on I, we have

$$(-1)^k \phi^{(k)}(x) \geq 0 \quad (x \in I; k = 0, 1, 2, \ldots).$$

It is clear that $\exp \phi(x) \geq 1$, $[\exp \phi(x)]' = \phi'(x) \exp \phi(x) \leq 0$ and $[\exp \phi(x)]'' = \{\phi''(x) + [\phi'(x)]^2\} \exp \phi(x) \geq 0$, that is, for $x \in I$ and $k = 0, 1, 2$, we have

$$(-1)^k [\exp \phi(x)]^{(k)} \geq 0. \tag{4}$$

Suppose (4) holds for all nonnegative integers $k \leq n$. By Leibnitz's formula, we have

$$\begin{aligned} &(-1)^{n+1} [\exp \phi(x)]^{(n+1)} = (-1)^{n+1} \{[\exp \phi(x)]'\}^{(n)} \\ &= (-1)^{n+1} [\phi'(x) \exp \phi(x)]^{(n)} = (-1)^{n+1} \sum_{i=0}^{n} \binom{n}{i} \phi^{(i+1)}(x) [\exp \phi(x)]^{(n-i)} \\ &= \sum_{i=0}^{n} \binom{n}{i} \left[(-1)^{i+1} \phi^{(i+1)}(x) \right] \left\{ (-1)^{n-i} [\exp \phi(x)]^{(n-i)} \right\} \geq 0. \end{aligned}$$

By induction, it is proved that the function $\exp \phi$ is completely monotonic on I.

In the proof of Theorem 1, we see that if the function ϕ is strictly completely monotonic on I, then $\exp \phi$ is also strictly completely monotonic on I. The proof is complete.

Theorem 2. *For $x > 0$ and $n = 0, 1, 2, \ldots$, then*

$$(-1)^n \left(\ln \frac{x \Gamma(x)}{\sqrt{x + 1/4}\, \Gamma(x + 1/2)} \right)^{(n)} > 0.$$

Proof. Using (3) and the representation

$$\ln x = \int_0^\infty \frac{e^{-t} - e^{-xt}}{t} dt \quad (x > 0),$$

we conclude that

$$\begin{aligned} &\ln \frac{x\Gamma(x)}{\sqrt{x+1/4}\,\Gamma(x+1/2)} \\ &= \ln\Gamma(x) - \ln\Gamma(x+1/2) + \ln x - \frac{1}{2}\ln(x+1/4) \\ &= \frac{1}{2}\int_0^\infty \frac{e^{t/4} + e^{-t/4} - 2}{1+e^{t/2}} \cdot \frac{e^{-xt}}{t} dt > 0 \quad (x > 0). \end{aligned} \tag{5}$$

From (5) we conclude that

$$(-1)^n \left(\ln \frac{x\Gamma(x)}{\sqrt{x+1/4}\,\Gamma(x+1/2)} \right)^{(n)} = \frac{1}{2}\int_0^\infty \frac{e^{t/4} + e^{-t/4} - 2}{1+e^{t/2}} \cdot \frac{e^{-xt}}{t^{1-n}} dt > 0$$

for $x > 0$ and $n = 0, 1, 2 \ldots$. The proof is complete.

Remark. From (5) we get

$$\frac{x\Gamma(x)}{\sqrt{x+1/4}\,\Gamma(x+1/2)} > 1 \quad (x > 0). \tag{6}$$

In fact, using the asymptotic expansion (see [8])

$$x^{b-a}\frac{\Gamma(x+a)}{\Gamma(x+b)} = 1 + \frac{(a-b)(a+b-1)}{2x} + O(x^{-2}) \quad (x \to \infty), \tag{7}$$

we conclude that

$$\lim_{x\to\infty} \frac{x\Gamma(x)}{\sqrt{x+1/4}\,\Gamma(x+1/2)} = 1.$$

By Theorem 1, the function $f(x) = \frac{x\Gamma(x)}{\sqrt{x+1/4}\,\Gamma(x+1/2)} - 1$ is strictly completely monotonic on $(0, \infty)$.

As an application of (6), we prove the following Wallis' inequality [12].

Theorem 3. *For all natural numbers n, then*

$$\frac{1}{\sqrt{\pi(n+4/\pi-1)}} \le \frac{(2n-1)!!}{(2n)!!} < \frac{1}{\sqrt{\pi(n+1/4)}}, \tag{8}$$

where $(2n)!! = \prod_{k=1}^n (2k)$ and $(2n-1)!! = \prod_{k=1}^n (2k-1)$. The constants $\frac{4}{\pi} - 1$ and $\frac{1}{4}$ are the best possible.

Proof. First, we show that the sequence

$$Q_n = \left[\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})}\right]^2 - n \quad (n = 1, 2, \ldots)$$

is strictly decreasing. It is sufficient to show that $Q_{n+1} < Q_n$, which is equivalent to

$$\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})} < \frac{2n+1}{\sqrt{4n+3}}. \tag{9}$$

Take in (6) $x = n + \frac{1}{2}$, (9) holds clearly.

Now, we prove (8). Since

$$\Gamma(n+1) = n!, \quad \Gamma\left(n+\frac{1}{2}\right) = \frac{(2n-1)!!}{2^n}\sqrt{\pi}, \quad 2^n n! = (2n)!!,$$

the inequality (8) is equivalent to

$$\frac{1}{4} < Q_n = \left[\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})}\right]^2 - n \le \frac{4}{\pi} - 1.$$

From the monotonicity of the sequence Q_n, it follows that

$$\lim_{n\to\infty} Q_n < Q_n \le Q_1 = \frac{4}{\pi} - 1.$$

Using the asymptotic expansion (7) we conclude from

$$Q_n = n\left[n^{-\frac{1}{2}}\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})} - 1\right]\left[n^{-\frac{1}{2}}\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})} + 1\right]$$

that

$$\lim_{n\to\infty} Q_n = \frac{1}{4}.$$

Thus, the inequality (8) follows. The proof is complete.

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