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NORMAL CRITERION AND SHARED VALUE BY DERIVATIVES OF MEROMORPHIC FUNCTIONS

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Abstract. Let $\mathscr{F}$ be a family of meromorphic functions in a plane domain D. If for every function $f \in \mathscr{F}$, all of whose zeros have,at least,multiplicity l and poles have, at least,multiplicity p, and for each pair functions f and g in $\mathscr{F}$, $f^{(k)}$ and $g^{(k)}$ share 1 in D, where k, l, and p are three positive integer satisfying $\frac{k+1}{l} + \frac{1}{p} \leq 1$, then $\mathscr{F}$ is normal.

1. Introduction and The Main Result

W. K. Hayman[3] proved the following well-known result.

Theorem A. *Let f be a non-constant in the complex plane C, and k be a fixed positive integer. Then either f or $f^{(k)} - 1$ has at least one zero. Moreover, if f is transcendental, f or $f^{(k)} - 1$ has infinitely many zeros*

Corresponding to Theorem A , Gu[2] proved the following famous result that is called *Gu's Criterion*, which is one of Hayman's [4] conjecture.

Theorem B. *Let k be a positive integer and $\mathscr{F}$ be a family of zero-free meromorphic functions in a complex domain D such that $f^{(k)} \neq 1$ for each $f \in \mathscr{F}$, then $\mathscr{F}$ is normal.*

Theorems B have been extensively and deeply studied. They focus on relaxing the conditions in Theorem B to that f or $f^{(k)} - 1$ has a zero.

Y. Wang and M. Fang [9] obtained the following result.

Theorem C.(See [9], Corollary 3.) *Suppose that n, k are two positive integers, $n \geq k+2$. Let $\mathscr{F}$ be a family of meromorphic function in plane domain D. If every function $f(z) \in \mathscr{F}$ has a zero of multiplicity n at least, and $f^{(k)} \neq 1$, then $\mathscr{F}$ is normal in D.*

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Recently, M. J. Chang [5] generalizes Theorem B with only allowing that $f^{(k)}-1$ have zeros but restrcting their numbers. Y.T. Li[6] prove that the conclusion of Theorem B is also true under the condition that f and $f'-1$ may have a zero.

By the ideas of shared values, M.Fang and L.Zalcman[1], J. M. Qi and T.Y. Zhu[8] discuss the case that $f^{(k)}$ and $g^{(k)}$ share a in D and obtained

Theorem D.(see [1]) *Suppose that k is a positive integer and $a \neq 0$ is a finite complex numbers. Let $\mathscr{F}$ be a family of meromorphic function defined in a domain D. If for each pair of functions $f, g \in \mathscr{F}$, f and g share 0, $f^{(k)}$ and $g^{(k)}$ share a IM in D, and the zeros of f are of multiplicity $\geq k+2$, then $\mathscr{F}$ is normal in D.*

Theorem E. (for the case $k \geq 2$, see [8], for the case $k=1$, see [7])*Let $\mathscr{F}$ be a family of meromorphic function defined in a domain D. Let k be a positive integer and $a \neq 0$ is a finite complex numbers. If, for each $f \in \mathscr{F}$, all zeros of $f(z)$ are of multiplicity $\geq k+2$ and all zeros of $f^{(k)}$ are of multiplicity 2 at least and if, $f^{(k)}$ and $g^{(k)}$ share a IM in D for each pair of functions $f, g \in \mathscr{F}$, then $\mathscr{F}$ is normal in D.*

In this paper, we obtain our main result as follow.

Theorem 1.1. *Let p, l, and k be three positive integers, $\frac{k+1}{l}+\frac{1}{p} \leq 1$, and $\mathscr{F}$ be a meromorphic function family in a plane domain D, all of whose zeros have,at least,multiplicity l and poles have,at least, multiplicity p. If for each pair functions f and g in $\mathscr{F}$, $f^{(k)}$ and $g^{(k)}$ share 1 in D, then $\mathscr{F}$ is normal in D.*

With Theorem 1.1, we immediately deduce the following corollary.

Corollary 1.2. *Let $\mathscr{F}$ be a meromorphic function family in a plane domain D, all of whose zeros has,at least,multiplicity $2(k+1)$ and poles have,at least,multiplicity 2, and $a \neq 0$ be a finite complex number. If $f^{(k)}$ and $g^{(k)}$ share a in D for each pair functions f and g in $\mathscr{F}$, then $\mathscr{F}$ is normal in D, where k be positive integer.*

Corollary 1.3. *Let $\mathscr{F}$ be a meromorphic function family in a plane domain D, all of whose zeros have,at least,multiplicity $k+2$ and poles has at least multiplicity $k+2$, and $a \neq 0$ be a finite complex number. If $f^{(k)}$ and $g^{(k)}$ share a in D for each pair functions f and g in $\mathscr{F}$, then $\mathscr{F}$ is normal in D, where k be positive integer.*

With Corollary 1.3, we immediately the following result.

Corollary 1.4. *Let $\mathscr{F}$ be a meromorphic function family in a plane domain D, and $k, n \geq k+2$ be positive integer and $a \neq 0$ be a finite complex number. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share a in D for each pair functions f and g in $\mathscr{F}$, then $\mathscr{F}$ is normal in D.*

Thereby, our Theorem 1.1 improves the result due to Y. T. Li and Yongxing Gu[7].

2. Preliminary results

Lemma 2.1. ([10]) *Let $\mathscr{F}$ be a family of meromorphic functions on the unit disc Δ, all of whose zeroes have multiplicity p at least and poles have multiplicity q at least. Let α be a real number satisfying $-p < \alpha < q$. Then $\mathscr{F}$ is not normal at a point $z_0 \in \Delta$ if and only if there exist*

(i) points $z_n \in \Delta$, $z_n \to z_0$;
(ii) functions $f_n \in \mathscr{F}$; and
(iii) positive numbers $\rho_n \to 0$

such that $\rho_n^{\alpha} f_n(z_n + \rho_n \xi) = g_n(\xi) \to g(\xi)$ spherically uniformly on each compact subset of $\mathbb{C}$, where $g(\xi)$ is a nonconstant meromorphic function satisfying the zeros of $g(\xi)$ are of multiplicities p at least and the poles of $g(\xi)$ are of multiplicities q at least. Moreover, the order of $g(\xi)$ is not greater than 2.

Lemma 2.2. ([9]) *Let k be a positive integer, and $f(z)$ be a transcendental meromorphic function with finite orders in $\mathbb{C}$, whose zeros have multiplicity $k+2$ at least. Suppose that $f^{(k)} \neq 1$, then f is a constant.*

Lemma 2.3. ([4]) *Let $f(z)$ be a transcendental meromorphic function in $\mathbb{C}$. If all zeros of $f(z)$ have multiplicity 3 at least, for any positive integer k, then $f^{(k)}$ assumes non-zero finite value infinitely often.*

3. Auxiliary lemmas

Lemma 3.1. *Let p, l, and k be positive integers, $\frac{k+1}{l} + \frac{1}{p} \leq 1$, let $\varphi(z)$ be non-constant rational function, all of whose zeros has multiplicity l at least, and poles has multiplicity p at least, then $\varphi^{(k)}(z) - 1$ has two zeros at least.*

Proof. Suppose not, then $\varphi^{(k)}(z) - 1$ has one zeros at most. That is, either $\varphi^{(k)}(z) - 1$ is free-zero, or $\varphi^{(k)}(z) - 1$ has exactly one zeros. From Lemma 2.2, we have that $\varphi^{(k)}(z) - 1$ has only one zeros. Set

$$\varphi(z) = \frac{P(z)}{Q(z)} = A \frac{(z-\xi_1)^{n_1}(z-\xi_2)^{n_2}\cdots(z-\xi_s)^{n_s}}{(z-\xi_1)^{m_1}(z-\xi_2)^{m_2}\cdots(z-\xi_s)^{m_t}}, \tag{3.1}$$

where $P(z)$ and $Q(z)$ are co-prime polynomials, that is $(P(z),\ Q(z)) = 1$, $n_i \geq l (i = 1,2,\cdots,s)$, and $m_j \geq p (j = 1,2,\cdots,t)$. $n = \sum_{i=1}^{s} n_i$ and $m = \sum_{j=1}^{t} m_j$ are the degree of $\deg(P(z))$ and $P(z)$,respectively. Moreover, denote ω_P as a product of factors $z - \xi_i$ corresponding to distinct zeros $z = \xi_i$ of polynomials P, and d_P as a logithom derivative of $Q(z)$.

$$\omega_P = (z-\xi_1)(z-\xi_2)\cdots(z-\xi_s),$$

$$
\begin{aligned}
d_P &= \frac{P'}{P} = \frac{n_1}{z-\xi_1} + \frac{n_2}{z-\xi_2} + \cdots + \frac{n_s}{z-\xi_s},\\
\omega_Q &= (z-z_1)(z-z_2)\cdots(z-z_s),\\
d_Q &= \frac{Q'}{Q} = \frac{m_1}{z-z_1} + \frac{m_2}{z-z_2} + \cdots + \frac{m_s}{z-z_s}.
\end{aligned}
$$

By mathematical induction, we have

$$
Q^{(k)} = \begin{cases} Q\left[(d_Q)^k + \sum_{p=2}^{k-1} \Psi_p(d_Q, d_Q', \cdots, d_Q^{(k-1)}) + d_Q^{(k-1)}\right], & k \le m,\\ 0, & k > m. \end{cases} \tag{3.2}
$$

where Ψ_p is a homogeneous differential polynomials with order p and weight k about d_Q, and

$$
\Psi_p(d_Q, d_Q', \cdots, d_Q^{(k-1)}) = \sum_j a_{pj} (d_Q)^{t_{pj0}} (d_Q')^{t_{pj1}} \cdots (d_Q^{(k-1)})^{t_{pj,k-1}},\ k \le m, \tag{3.3}
$$

satisfying $t_{pj0} + t_{pj1} + \cdots + t_{pj,k-1} = p$, and $t_{pj0} + 2t_{pj1} + \cdots + k t_{pj,k-1} = k$.

Set

$$
g_k(\psi_1, \psi_2, \cdots, \psi_k) = \psi_1^k + \sum_{p=2}^{k-1} \Psi_p(\psi_1, \psi_2, \cdots, \psi_k) + \psi_k, k \le m, \tag{3.4}
$$

where $\psi_1 = \omega_Q d_Q,\ \psi_2 = \omega_Q^2 d_Q',\ \cdots,\ \psi_k = \omega_Q^k d_Q^{(k-1)}$.

From (3.2),(3.3) and (3.4), we have

$$
Q^{(k)} = Q\omega_Q^{-k} \cdot g_k(\psi_1, \psi_2, \cdots, \psi_k), k \le m, \tag{3.5}
$$

Since $(z - z_j, \psi_r) = 1 (j = 1, 2, \cdots, t;\ r = 1, 2, \cdots, k)$, we have $(z - z_j, g_k(\psi_1, \psi_2, \cdots, \psi_k)) = 1$. Thus

$$
deg\left[g_k(\psi_1, \psi_2, \cdots, \psi_k)\right] = k(t-1), \tag{3.6}
$$

On the other hand, also by mathematical induction, we deduce that

$$
\left(\frac{1}{Q}\right)^{(k)} = \frac{(-1)^k k! (Q')^k}{Q^{k+1}} + \sum_{p=3}^{k} C_p \frac{H_{p-1}(Q', Q'', \cdots, Q^{(k-1)})}{Q^p} - \frac{Q^{(k)}}{Q^2}, \tag{3.7}
$$

where $H_p(Q', Q'', \cdots, Q^{(k-1)})$ is a homogeneous differential polynomials about Q with order p and weight k satisfying

$$
H_p(Q', Q'', \cdots, Q^{(k-1)}) = \sum_j b_{pj} (Q')^{n_{pj1}} (Q'')^{n_{pj2}} \cdots (Q^{(k-1)})^{n_{pj,k-1}}, \tag{3.8}
$$

where $k \le m$, $n_{pj1} + n_{pj2} + \cdots + n_{pj,k-1} = p$ and $n_{pj1} + 2n_{pj2} + \cdots + (k-1)n_{pj,k-1} = k$.

If $k > m$,

$$H_p(Q',Q'',\cdots,Q^{(k-1)}) = \sum_j b_{pj}(Q')^{n_{pj1}}(Q'')^{n_{pj2}}\cdots(Q^{(m-1)})^{n_{pj,m-1}}, \tag{3.9}$$

where $n_{pj1}+n_{pj2}+\cdots+n_{pj,m-1}=p$ and $n_{pj1}+2n_{pj2}+\cdots+(m-1)n_{pj,m-1}=k$. It would be looked as $n_{pj,m+1}=n_{pj,m+2}=\cdots=n_{pj,k-1}=0$ in (3.8).

By (3.5), we have

$$\begin{aligned} H_p(Q',Q'',\cdots,Q^{(k-1)}) &= \sum_j b_{pj}(Q')^{n_{pj1}}(Q'')^{n_{pj2}}\cdots(Q^{(k-1)})^{n_{pj,k-1}}, \\ &= \sum_j b_{pj}(Q\omega_Q^{-1}g_1)^{n_{pj1}}(Q\omega_Q^{-2}g_2)^{n_{pj2}}\cdots(Q\omega_Q^{-k}g_{k-1})^{n_{pj,k-1}}, \\ &= Q^p\omega_Q^{-k}\sum_j b_{pj}(g_1)^{n_{pj1}}(g_2)^{n_{pj2}}\cdots(g_{k-1})^{n_{pj,k-1}} \end{aligned} \tag{3.10}$$

Thereby, from (3.7) and (3.10), it follows

$$Q^{k+1}\left(\frac{1}{Q}\right)^{(k)} = Q^k\omega_Q^{-k}\cdot\Phi_k(\psi_1,\psi_2,\cdots,\psi_k), \tag{3.11}$$

when $k \le m$

$$\Phi_k(\psi_1,\cdots,\psi_k) = (-1)^k k!(\psi_1)^k + \sum_{p=3}^{k} C_p Q^{k-p+1}H_{p-1}(\psi_1,\cdots,\psi_k) - Q^{k-1}\psi_k, \tag{3.12}$$

and $k > m$,

$$\Phi_k(\psi_1,\cdots,\psi_k) = (-1)^k k!(\psi_1)^k + \sum_{p=3}^{k} C_p Q^{k-p+1}H_{p-1}(\psi_1,\cdots,\psi_m), \tag{3.13}$$

and furthermore $deg[\Phi_k(\psi_1,\psi_2,\cdots,\psi_k)] = k(t-1)$.

Similarly, set $\phi_1=\omega_P d_P$, $\phi_2=\omega_P^2 d_P'$, $\cdots$, $\phi_k=\omega_P^k d_P^{(k-1)}$, and

$$f_k(\phi_1,\phi_2,\cdots,\phi_k) = \phi_1^k + \sum_{p=2}^{k}\Psi_p(\phi_1,\phi_2,\cdots,\phi_k) + \phi_k, \tag{3.14}$$

So from (3.5), we have

$$P^{(k)} = P\omega_P^{-k}\cdot f_k(\phi_1,\phi_2,\cdots,\phi_k), \tag{3.15}$$

Similarly,

$$deg\left[f_k(\psi_1,\psi_2,\cdots,\psi_k)\right] = k(s-1). \tag{3.16}$$

Thus, $(z-\xi_i,\phi_r)=1(i=1,2,\cdots,s;\ r=1,2,\cdots,k)$, and we also have $(z-\xi_i,f_k(\phi_1,\phi_2,\cdots,\phi_k))=1$

Therefore, it deduce that

$$\left(\frac{P}{Q}\right)^{(k)} = \sum_{r=0}^{k}\binom{k}{r}\left(\frac{1}{Q}\right)^{(r)}P^{(k-r)},$$
$$= \frac{\sum_{r=0}^{k}\binom{k}{r}\left[Q^{r+1}\left(\frac{1}{Q}\right)^{(r)}\right]\cdot Q^{k-r}\cdot P^{(k-r)}}{Q^{k+1}} \equiv \frac{R(z)}{Q^{k+1}}, \tag{3.17}$$
$$= \frac{\sum_{r=0}^{k}\binom{k}{r}\left[(Q\omega_Q^{-1})^r\Phi_r\right]\cdot Q^{k-r}\cdot P\omega_P^{-(k-r)}f_{k-r}}{Q^{k+1}},$$
$$= \frac{P\omega_P^{-k}}{Q\omega_Q^k}\sum_{r=0}^{k}\binom{k}{r}\omega_Q^{k-r}\Phi_r\cdot\omega_P^r f_{k-r},$$

Set

$$h_k(z) = \sum_{r=0}^{k}\binom{k}{r}\omega_Q^{k-r}\Phi_r\cdot\omega_P^r f_{k-r}, \tag{3.18}$$

we have $(z-\xi_i, h(z)) = 1(i=1,2,\cdots,s)$, $(z-z_j, h(z)) = 1(j=1,2,\cdots,t)$, $deg[h_k(z)] \le k(s+t-1)$, and

$$\left(\frac{P}{Q}\right)^{(k)} = \frac{P\omega_P^{-k}}{Q\omega_Q^k}h_k(z). \tag{3.19}$$

Set

$$\tau_1 = \deg(P\omega_P^{-k}h_k(z)),$$
$$\tau_2 = \deg(Q\omega_Q^k),$$
$$N = \max(\tau_1,\tau_2),$$

then

$$\left(\frac{P}{Q}\right)^{(k)} - 1 = A\frac{(z-z_0)^N}{Q\omega_Q^k} \tag{3.20}$$
$$\left(\frac{P}{Q}\right)^{(k+1)} = A\frac{(z-z_0)^{N-1}g(z)}{Q\omega_Q^{k+1}} \tag{3.21}$$

where $g(z) = N\omega_Q - (z-z_0)(g_1 + kg_0)$, and $g_0 = \omega_Q d_{\omega_Q} = \omega_Q(\frac{1}{z-z_1} + \frac{1}{z-z_2} + \cdots + \frac{1}{z-z_t})$.

On the other hand, from (3.19) we have

$$\left(\frac{P}{Q}\right)^{(k+1)} = \frac{P\omega_P^{-k-1}}{Q\omega_Q^{k+1}}h_{k+1}(z), \quad deg[h_{k+1}] \le (k+1)(s+t-1). \tag{3.22}$$

In the following, we distinguish three cases.

Case 1. $n < m$. We have

$$\tau_1 - \tau_2 = n - m - k(s+t) + \deg(h_k) < 0.$$

Then, $N = \tau_2 = m + kt$ and $deg[g(z)] = t$. From (3.21) and (3.22), we have $N - 1 \leq deg[h_{k+1}]$ and $n - s(k+1) \leq deg[g(z)]$. That is, $m \leq (k+1)s + t - k$ and $n \leq (k+1)s + t$. Since $\varphi(z)$ has a zero of multiplicity l at least and a pole of multiplicity p at least, we have $pt \leq m \leq (k+1)s + t - k$ and $ls \leq n \leq (k+1)s + t$. Thus, $\frac{k+1}{l} + \frac{1}{p} > 1$, which contradicts $\frac{k+1}{l} + \frac{1}{p} \leq 1$.

Case 2. $n = m$. We have $\tau_1 < \tau_2$. Thus, $N = m + kt$ and $deg[g(z)] \leq t - 1$. Similar to Case 1, it deduce that $\frac{k+1}{l} + \frac{1}{p} > 1$, a contradiction.

Case 3. $n > m$. We divide it into two subcase.

Subcase 3.1 $n \geq m + k$. Set

$$\frac{P(z)}{Q(z)} = P_{n-m}(z) + \frac{R_\tau}{Q(z)},$$

then

$$\left[\frac{P(z)}{Q(z)}\right]^{(k)} - 1 = P_{n-m}^{(k)}(z) + \left[\frac{R_\tau(z)}{Q(z)}\right]^{(k)} - 1 = P_{n-m}^{(k)}(z) + \frac{R(z)}{Q^{k+1}(z)} - 1$$
$$= \frac{P_{n-m}^{(k)}(z)Q^{k+1}(z) + R(z) - Q^{k+1}(z)}{Q^{k+1}(z)}.$$

Since $deg[R(z)] = r + k(m-1) < m(k+1) \leq deg[P_{n-m}^{(k)}(z)Q^{k+1}(z)]$, by (3.20) we obtain $N = n + k(t-1)$. From (3.21) and (3.22), we get $n \leq (k+1)s + t$. Since $n \geq ls$ and $n \geq pt + k$, we can obtain $\frac{k+1}{l} + \frac{1}{p} > 1$, a contradiction.

Subcase 3.2. $n < m + k$. We also distinguish two subcases. Write

$$\frac{P(z)}{Q(z)} = P_{n-m}(z) + \frac{R_\tau}{Q(z)}.$$

Then

$$\left[\frac{P(z)}{Q(z)}\right]^{(k)} - 1 = P_{n-m}^{(k)}(z) + \left[\frac{R_\tau(z)}{Q(z)}\right]^{(k)} - 1$$
$$= P_{n-m}^{(k)}(z) + \frac{R(z)}{Q^{k+1}(z)} - 1 = \frac{R(z) - Q^{k+1}(z)}{Q^{k+1}(z)}.$$

Since $deg[R(z)] = r + k(m-1) < m(k+1)$, $deg[R(z) - Q^{k+1}(z)] = m(k+1)$. Therefore, from (3.20),we have $N = m + kt$. Applying (3.21) and (3.22), we get $m + kt - 1 \leq (k+1)(s+t-1)$ and $n \leq (k+1)s + t$. Noting $n \geq ls$ and $m \geq pt$, we can obtain $\frac{k+1}{l} + \frac{1}{p} > 1$, a contradiction. □

4. Proof of Theorem 1.1

Suppose that $\mathscr{F}$ is not normal in D, that is, there exists one point $z_0 \in D$ such that $\mathscr{F}$ is not normal at point z_0. Without loss of generality we assume that $z_0 = 0$. By Lemma 2.1, there exists a point sequence $\{z_n\}$, $z_n \in D$, $z_n \to z_0$, functions $f_n \in \mathscr{F}$, and positive numbers $\rho_n \to 0$ such that

$$\varphi_j(\xi) = \rho_j^{-k} f_j(z_j + \rho_j \xi) \Rightarrow \varphi(\xi), \tag{4.1}$$

spherically uniformly on each compact subset of $\mathbb{C}$, where $\varphi(\xi)$ is a nonconstant meromorphic function with order ≤ 2, all of whose poles are of multiplicities p at least and zeroes are of multiplicities l at least.

From (4.1), we have

$$\varphi_j^{(k)}(\xi) - 1 = f_j^{(k)}(z_j) - 1 \Rightarrow \varphi^{(k)}(\xi) - 1, \tag{4.2}$$

also locally uniformly with respect to the spherical metric.

Obviously, $\varphi^{(k)} \not\equiv 1$, since otherwise $\varphi(z) = P_k(z)$ is a polynomial with a degree k. Thus, all zeros of $\varphi(z)$ are of multiplicity k at most, a contradiction.

By Lemma 2.3, it follows that $\varphi^{(k)} - 1$ has a zero at least. Claim that $\varphi^{(k)} - 1$ has a unique zero $\xi = \xi_0$.

Indeed, if there exist two distinct zeros ξ_0 and ξ_0^*, $\xi_0 \ne \xi_0^*$, then we choose a positive number $\delta > 0$ such that a neighborhood $N(\xi_0, \delta) = \{\xi : |\xi - \xi_0| < \delta\}$ with radius δ of ξ_0 and a neighborhood $N(\xi_0^*, \delta) = \left\{\xi : |\xi - \xi_0^*| < \delta\right\}$ with radius δ of ξ_0^* having no intersection, $N(\xi_0, \delta) \bigcap N(\xi_0^*, \delta) = \phi$.

Since $\varphi^{(k)} \not\equiv 1$, from (4.2) and Hurwitz's theorem, there exists points $\xi_j \in N(\xi_0, \delta)$ and $\xi_j^* \in N(\xi_0^*, \delta)$ such that for sufficiently large j

$$f_j^{(k)}(z_j + \rho_j \xi_j) - 1 = 0,$$
$$f_j^{(k)}(z_j + \rho_j \xi_j^*) - 1 = 0.$$

According to the hypothesis that $f^{(k)}$ and $g^{(k)}$ share 1 in D for each pair of f and g in $\mathscr{F}$, we know that for any integer m

$$f_m^{(k)}(z_j + \rho_j \xi_j) - 1 = 0,$$
$$f_m^{(k)}(z_j + \rho_j \xi_j^*) - 1 = 0.$$

Fixing m, and taking $j \to +\infty$, we deduce that

$$f_m^{(k)}(0) - 1 = 0.$$

Since the isolation of zeros for $f_m^{(k)}(z)-1$, for sufficiently large j, we have

$$z_j+\rho_j\xi_j=0,\quad z_j+\rho_j\xi_j^*=0.$$

Thus,

$$\xi_j=-\frac{z_j}{\rho_j},\quad \xi_j^*=-\frac{z_j}{\rho_j}.$$

That is, $\xi_j=\xi_j^*\in N(\xi_0,\delta)\bigcap N(\xi_0^*,\delta)$, this contradicts that $N(\xi_0,\delta)\bigcap N(\xi_0^*,\delta)=\phi$. And claim is proved.

From Lemma 2.3, it follows that $\varphi(z)$ must be a rational function. Also by Lemma 3.1, we deduce that $\varphi(z)$ has two zeros at least. This contradicts with previous claim that $\varphi(z)$ only has a unique zeros.

This completes the proof of Theorem 1.1. □

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