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L_p−WINTERNIZ PROBLEM ON FIREY PROJECTION OF CONVEX BODIES

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Abstract. For $p \geq 1$, Lutwak, Yang and Zhang introduced the concept of p-projection body, and Lutwak introduced the concept of L_p- affine surface area of convex body. In this paper, we develop the Minkowski-Funk transform approach in the L_p-Brunn-Minkowski theory. We consider the question of whether $\Pi_p K \subseteq \Pi_p L$ implies $\Omega_p(K) \leq \Omega_p(L)$, where $\Pi_p K$ and $\Omega_p K$ denotes the $p-$projection body of convex body K and the L_p-affine surface area of convex body K, respectively. We also formulate and solve a generalized L_p-Winterniz problem for Firey projections.

1. Introduction

Let $\mathscr{K}^n$ denote the set of convex bodies (compact, convex subsets with nonempty interiors) in $\mathbb{R}^n$. For the set of convex bodies containing the origin in their interiors and the set of origin-symmetric convex bodies in $\mathbb{R}^n$, we write $\mathscr{K}_o^n$ and $\mathscr{K}_c^n$, respectively. Denote by $\mathrm{vol}_n(K)$ the n-dimensional volume of body K. Let B^n is a standard unit ball in $\mathbb{R}^n$ with $n-$dimensional Lebesgue measure $\omega_n := \mathrm{vol}_n(B^n) = \pi^{n/2}/\Gamma(1+n/2)$, for surface S^{n-1} of B^n, denote $\sigma_{n-1} := |S^{n-1}| = 2\pi^{n/2}/\Gamma(n/2)$.

If $K \in \mathscr{K}^n$, its support function, $h_K(\cdot) = h(K,\cdot) : \mathbb{R}^n \to (0,\infty)$, is defined by

$$h(K,x) = \max\{x \cdot y : y \in K\},\ x \in \mathbb{R}^n,$$

where $x \cdot y$ denotes the standard inner product of x and y in $\mathbb{R}^n$.

If K is a compact star-shaped (about the origin) in $\mathbb{R}^n$, its radial function, $\rho_K(\cdot) = \rho(K,\cdot) : \mathbb{R}^n \setminus \{0\} \to [0,+\infty)$, is defined by

$$\rho(K,x) = \max\{\lambda \geq 0 : \lambda x \in K\}, x \in \mathbb{R}^n \setminus \{0\}.$$

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If $\rho(K,u)$ is positive and continuous, then K will be called a star body (about the origin). Let $\mathscr{S}_o^n$ denote the set of star bodies (about the origin) in $\mathbb{R}^n$. Two star bodied K and L are said to be dilates (of one another) if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$.

For $K \in \mathscr{K}_o^n$, the polar body, K^*, of K is defined by

$$K^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1, y \in K\}.$$

Obviously, we have $\rho(K^*,\cdot) = 1/h(K,\cdot)$.

The projection body was introduced at the turn of the previous century by Minkowski. For $K \in \mathscr{K}^n$, the projection body, ΠK, of K is centrally symmetric convex body whose support function is given by (see [3, 20])

$$h(\Pi K,\theta) := \mathrm{vol}_{n-1}(K|\theta^{\perp}) = \frac{1}{2}\int_{S^{n-1}} |\theta \cdot u| dS(K,u), \text{ for all } \theta \in S^{n-1},$$

where vol_{n-1} denotes $(n-1)$-dimensional volume, $K|\theta^{\perp}$ denotes the image of the orthogonal projection of K onto the codimensional 1 subspace orthogonal to θ, and $S(K,\cdot)$ is the surface area measure.

A convex body K is said to have a curvature function $f(K,\cdot) : S^{n-1} \to \mathbb{R}$, if its surface area measure $S(K,\cdot)$ is absolutely continuous with respect to Lebesgue measure S on S^{n-1} and

$$\frac{dS(K,\cdot)}{dS} = f(K,\cdot) \in L^1(S^{n-1}).$$

Let $\mathscr{F}^n$ denote the set of all bodies in $\mathscr{K}^n$ that has a positive continuous curvature function. If K is an infinitely smooth body with positive curvature, then $f(K,\theta)$ is the reciprocal of the Gauss curvature at the boundary point with unit normal θ, see [20, p.419]. Abusing notations, we will also denote by $f(K,\cdot)$ the extension of $f(K,\cdot)$ to $\mathbb{R}^n$ as a homogeneous function of degree $-n-1$.

For a convex body K in $\mathbb{R}^n$ with positive curvature $f(K,\cdot)$, the classical affine surface area, $\Omega(K)$, of K is defined by (see [7, 8, 9, 16])

$$\Omega(K) = \int_{S^{n-1}} f(K,u)^{\frac{n}{n+1}} dS(u).$$

In [4], Lutwak studied the following problems:

Winterniz problem for projection body (see [1]). Let K and L be two origin-symmetric convex bodies in $\mathbb{R}^n$, and both of them have a positive continuous curvature function, and suppose that

$$\Pi K \subset \Pi L,$$

Does it follow that

$$\Omega(K) \le \Omega(L)?$$

In order to study these problems, Lutwak defines a class specific set for elliptic convex bodies(see [8]):

$$\mathcal{W}^n = \{K \in \mathcal{F}^n : \exists Z \in \mathcal{Z}^n \text{ with } f(K,\cdot) = h(Z,\cdot)^{-n-1}\},$$

where $\mathcal{Z}^n$ is the set of projection bodies. And he proved that if $L \in \mathcal{W}^n$, then the condition $\Pi K \subseteq \Pi L$ implies $\Omega(K) \le \Omega(L)$, while for $K \notin \mathcal{W}^n$ this is not necessarily true.

The main purpose of this paper is to give an answer of L_p−Winterniz problems by innovative methods of generalized cosine transform. To this end, we will use concept of a p−projection body, introduced by Lutwak [9, 10]. For each $K \in \mathcal{K}_o^n$ and real $p \ge 1$, then the p-projection body, $\Pi_p K$, of K is an origin-symmetric convex body whose support function is given by

$$h(\Pi_p K, x)^p = \frac{1}{2n} \int_{S^{n-1}} |x \cdot u|^p \, dS_p(K,u), \quad x \in \mathbb{R}^n. \tag{1}$$

Here $S_p(K,\cdot)$ is the L_p−surface area measure. A convex body M is called a p−projection body if there is a convex body K such that $M = \Pi_p K$. We say that the support function $h(\Pi_p K,\cdot)$ of $\Pi_p K$ defines L_p−Firey projection of a body K.

A convex body $K \in \mathcal{K}_o^n$ is said to have a L_p−curvature function (see [9]) $f_p(K,\cdot) : S^{n-1} \to \mathbb{R}$, if its L_p−surface area measure $S_p(K,\cdot)$ is absolutely continuous with respect to spherical Lebesgue measure S, and

$$\frac{dS_p(K,\cdot)}{dS} = f_p(K,\cdot). \tag{2}$$

Let $\mathcal{F}_o^n, \mathcal{F}_c^n$ denote the set of bodies in $\mathcal{K}_o^n, \mathcal{K}_c^n$, respectively, and both of them have a positive continuous curvature function.

Lutwak [9] showed the L_p−affine surface area as follow: For $K \in \mathcal{F}_o^n$, the L_p−affine surface area, $\Omega_p(K)$, of K is defined by

$$\Omega_p(K) = \int_{S^{n-1}} f_p(K,u)^{\frac{n}{n+p}} \, dS(u). \tag{3}$$

L_p−Winterniz problem will be expressed as follows:

L_p−**Winterniz problem for Firey projection body**. Consider two origin-symmetric convex bodies K and L in $\mathbb{R}^n$, and both of them have a positive continuous L_p−curvature function. Fix $p \ge 1$ and suppose that

$$\Pi_p K \subset \Pi_p L.$$

Does it follow that

$$\Omega_p(K) \le \Omega_p(L)?$$

In the case $p = 1$, the problem is just the Winterniz's problem. In this paper, we give the L_p-form of Winterniz problems and study its general answer. Our main result is the following two Theorems.

Theorem 1.1. *Winterniz monotonicity problem for projections bodied has a affirmative answer if and only if $p = 1$ and $n \le 2$.*

Theorem 1.2. *L_p−Winterniz monotonicity problem for L_p−Firey projections has a negative answer if and only if $p > 1$ and $n \ge 2$.*

2. The Brunn-Minkowski Theory Background

2.1. The L_p-mixed volume

Firey [11] extended the concept of Minkowski linear combination. For $p \ge 1$, $K, L \in \mathscr{K}_o^n$ and $\alpha, \beta > 0$, the Firey L_p-combination $\alpha K +_p \beta L \in \mathscr{K}_o^n$ is defined by

$$h(\alpha K +_p \beta L, \cdot)^p = \alpha h(K, \cdot)^p + \beta h(L, \cdot)^p.$$

where " $\cdot$ " in $\varepsilon \cdot L$ denotes the Firey scalar multiplication. For $p = 1$, $K +_p \varepsilon \cdot L$ is just the Minkowski linear combination of K and L.

Lutwak (see [11]) showed that the Firey L_p-combination lead to a Brunn-Minkowski theory for $p \ge 1$. He introduced the notion of L_p-mixed volume as follows: For $K, L \in \mathscr{K}_o^n$ and $p \ge 1$, the L_p-mixed volume of K and L, $V_p(K, L)$, is defined by

$$\frac{n}{p} V_p(K, L) = \lim_{\varepsilon \to 0} \frac{V(K +_p \varepsilon L) - V(K)}{\varepsilon}.$$

Lutwak (see [11]) further proved that for each $K \in \mathscr{K}_o^n$, there exists a positive Borel measure $S_p(K, \cdot)$ on S^{n-1} so that

$$V_p(K, L) = \frac{1}{n} \int_{S^{n-1}} h(L, u)^p \, dS_p(K, u),$$

for all $L \in \mathscr{K}_o^n$. It turns out that the measure $S_p(K, \cdot)$ is absolutely continuous with respect to $S(K, \cdot)$, and has the Radon-Nikodym derivative

$$\frac{dS_p(K, \cdot)}{dS(K, \cdot)} = h^{1-p}(K, \cdot).$$

If $S_p(K, \cdot)$ is absolutely continuous with respect to spherical Lebesgue measure S, we have eq.(2).

From (2), we have

$$V_p(K,L)=\frac{1}{n}\int_{S^{n-1}} h(L,u)^p f_p(K,u)\,du,$$

for all $L\in\mathscr{K}_o^n$. In particular,

$$\mathrm{vol}_n(K)=\frac{1}{n}\int_{S^{n-1}} h(K,u)^p f_p(K,u)\,du.$$

If a convex body K has the curvature functions, then

$$f_p(K,\cdot)=h(K,\cdot)^{1-p}f(K,\cdot).$$

Lutwak also proved a generalization of the classical Minkowski theorem, which states that given $p>0, p\neq n$, and a continuous even function $g:S^{n-1}\to\mathbb{R}^+$, there exists a unique convex body K such that $f_p(K,\cdot)=g$.

2.2. The L_p−mixed affine surface area

Lutwak [9] showed the L_p−affine surface area as follows: For $K\in\mathscr{F}_o^n$, the L_p−affine surface area, $\Omega_p(K)$, of K is defined by

$$\Omega_p(K)=\int_{S^{n-1}} f_p(K,u)^{\frac{n}{n+p}}\,dS(u). \tag{4}$$

In [9], Lutwak gave an L_p−extension of Leichtweiβ's definition (see [15]) of extended affine surface area as follows: For $p\geq 1, K\in\mathscr{K}_o^n$. define $\Omega_p(K)$ by

$$n^{-\frac{p}{n}}\Omega_p(K)^{\frac{n+p}{n}}=\inf\{nV_p(K,Q^*)V(Q)^{\frac{p}{n}}:Q\in\mathscr{S}_o^n\}. \tag{5}$$

When $p=1$, the subscript will often be suppressed.

The definition of Blaschke L_p−combination for convex bodies was given by Lutwak (see [11]). For $K,L\in\mathscr{K}_o^n, p\geq 1, \lambda,\mu\geq 0$ (not both zero), the Blaschke L_p−combination, $\lambda K\check{+}_p\mu L\in\mathscr{K}_o^n$, of K and L is defined by

$$dS_p(\lambda K\check{+}_p\mu L,\cdot)=\lambda dS_p(K,\cdot)+\mu dS_p(L,\cdot). \tag{6}$$

From (6) and (2), it is obvious that

$$f_p(\lambda K\check{+}_p\mu L,\cdot)=\lambda f_p(K,\cdot)+\mu f_p(L,\cdot). \tag{7}$$

For $p\geq 1$, the L_p−mixed affine surface area of $K,L\in\mathscr{F}_o^n,\Omega_{-p}(K,L)$, can be defined by

$$\Omega_{-p}(K,L)=\frac{n}{n+p}\lim_{\varepsilon\to 0^+}\frac{\Omega_p(L\check{+}_p\varepsilon K)-\Omega_p(L)}{\varepsilon}. \tag{8}$$

More accurately, we have the following:

Proposition 2.1. *For $p \geq 1$, the L_p-mixed affine surface area of $K, L \in \mathscr{F}_o^n$, $\Omega_{-p}(K,L)$, has the following integral representation:*

$$\Omega_{-p}(K,L) = \int_{S^{n-1}} f_p(K,u) f_p(L,u)^{-\frac{p}{n+p}} dS(u). \tag{9}$$

Proof. From (4), (7) and (8), we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0^+} \frac{\Omega_p(L \check{+}_p \varepsilon K) - \Omega_p(L)}{\varepsilon} \\
&= \lim_{\varepsilon\to 0^+} \frac{\int_{S^{n-1}} \left(f_p(L \check{+}_p \varepsilon K, u)^{\frac{n}{n+p}} - f_p(L,u)^{\frac{n}{n+p}} \right) dS(u)}{\varepsilon} \\
&= \lim_{\varepsilon\to 0^+} \frac{\int_{S^{n-1}} \left[\left(f_p(L,u) + \varepsilon f_p(K,u) \right)^{\frac{n}{n+p}} - f_p(L,u)^{\frac{n}{n+p}} \right] dS(u)}{\varepsilon} \\
&= \frac{n+p}{n} \int_{S^{n-1}} f_p(K,u) f_p(L,u)^{-\frac{p}{n+p}} dS(u).
\end{aligned}$$

This completes the proof.

Clearly, from (9) and (4) it follows that for $p \geq 1$ and $K \in \mathscr{F}_o^n$,

$$\Omega_{-p}(K,K) = \Omega_p(K). \tag{10}$$

Since for any $K \in \mathscr{K}_o^n$, the L_p-surface area measure, $S_p(K,\cdot)$, is well-defined, we can give a natural extension of eq.(9) of the L_p-mixed affine surface area Ω_{-p} from $\mathscr{F}_o^n \times \mathscr{F}_o^n$ to $\mathscr{K}_o^n \times \mathscr{F}_o^n$. Specifically, for $K \in \mathscr{K}_o^n$ and $L \in \mathscr{F}_o^n$,let

$$\Omega_{-p}(K,L) = \int_{S^{n-1}} f_p(L,u)^{-\frac{p}{n+p}} dS_p(K,u). \tag{11}$$

It is well-known that for $K \in \mathscr{F}_o^n, dS_p(K,\cdot) = f_p(K,\cdot) dS(\cdot)$. Thus (11) boils down to (9) for $K \in \mathscr{F}_o^n$. Note that the case $p = 1$ was studies by Lutwak in [12].

Using Hölder's inequality, we can easily obtain the following inequality : If $p \geq 1$, and $K \in \mathscr{K}_o^n, L \in \mathscr{F}_c^n$, then

$$\Omega_{-p}(K,L)^n \geq \Omega_p(K)^{n+p} \Omega_p(L)^{-p}. \tag{12}$$

If $n \neq p > 1$ and $K, L \in \mathscr{F}_o^n$, then equality holds in (12) if and only if K and L are dilates. If $p = 1$, $K \in \mathscr{K}^n$ and $L \in \mathscr{F}_c^n$, then (12) equality hold if and only if K and L are homothetic.

2.3. L_p-curvature image

Lutwak (see [9]) showed the notion of L_p-curvature image as follows: For each $K \in \mathscr{F}_o^n$ and real $p \geq 1$, define $\Lambda_p K \in \mathscr{S}_o^n$ be a star body (about the origin) in $\mathbb{R}^n$, the L_p-curvature image of K, by

$$f_p(K,\cdot) = \frac{\omega_n}{\mathrm{vol}_n(\Lambda_p K)} \rho(\Lambda_p K,\cdot)^{n+p}. \tag{13}$$

Note that for $p = 1$, this definition differs from the definition of classical curvature image (see [8, 12, 13]).

For the L_p-curvature image and L_p-affine surface area, we have the following result: If $K \in \mathscr{F}_o^n$, $p \geq 1$, then

$$\mathrm{vol}_n(\Lambda_p K)^{\frac{p}{n+p}} = \frac{1}{n}\omega_n^{-\frac{n}{n+p}}\Omega_p(K). \tag{14}$$

3. Analytic Families of The Generalized Cosine Transforms

3.1. Basic integral transforms

In the following, $\mathbb{N}^+ = \{1,2,\ldots\}$ is the set of all non-zero natural numbers, $\mathbb{N} = \mathbb{N}^+ \cup \{0\}$. $C(S^{n-1})$ and $C_e(S^{n-1})$ denote the space of continuous functions on S^{n-1} and the space of even continuous functions on S^{n-1}, respectively. And the subset of $C_e(S^{n-1})$ that contains the infinitely differentiable functions will be denoted by $C_e^\infty(S^{n-1})$. $\mathscr{D}(S^{n-1})$ is the subspace of $C_e^\infty(S^{n-1})$ equipped with the standard topology, and $\mathscr{D}'(S^{n-1})$ stands for the corresponding dual space of distributions. The subspaces of even test functions (distribution) are denoted by $\mathscr{D}_e(S^{n-1})$ ($\mathscr{D}'_e(S^{n-1})$). We write $\mathscr{M}(S^{n-1})$ for the spaces of finite Borel measures on S^{n-1}. $\mathscr{M}_+(S^{n-1})$ are the relevant spaces of non-negative measures. $\mathscr{M}_{e+}(S^{n-1})$ denotes the space of even measures $\mu \in \mathscr{M}_+(S^{n-1})$.

The Minkowski-Funk transform is as follows:

$$(Mf)(u) = \int_{\{\theta:\ \theta\cdot u=0\}} f(\theta)d_u\theta, \quad u \in S^{n-1}, \tag{15}$$

which integrates a function f over great circles of codimension 1. This transform is a member of the analytic family[17]:

$$(M^\alpha f)(u) = \gamma_n(\alpha)\int_{S^{n-1}} f(\theta)|\theta\cdot u|^{\alpha-1}d\theta, \tag{16}$$

$$\gamma_n(\alpha) = \frac{\sigma_{n-1}\Gamma((1-\alpha)/2)}{2\pi^{(n-1)/2}\Gamma(\alpha/2)}, \quad Re\alpha > 0,\ \ \alpha \neq 1,3,5,\ldots;$$

$$(\widetilde{M}^\alpha f)(u) = \int_{S^{n-1}} f(\theta)|\theta\cdot u|^{\alpha-1}d\theta, \quad \alpha = 1,3,5,\ldots. \tag{17}$$

Let $\{Y_{j,k}\}$ be an orthonormal basis of spherical harmonics on S^{n-1}. Here $j = 0,1,2,\ldots$, and $k = 1,2,\ldots,d_n(j)$, where $d_n(j)$ is the dimension of the subspace of spherical harmonics of degree j. Each function $\omega \in \mathscr{D}(S^{n-1})$ admits a decomposition $\omega = \sum_{j,k}\omega_{j,k}Y_{j,k}$ with the Fourier-Laplace coefficients $\omega_{j,k} = \int_{S^{n-1}}\omega(\theta)Y_{j,k}(\theta)d\theta$, which decay rapidly as $j \to \infty$. Each distribution $f \in \mathscr{D}'(S^{n-1})$ can be defined by $(f,\omega) = \sum_{j,k} f_{j,k}\omega_{j,k}$ where $f_{j,k} = (f, Y_{j,k})$ grow not faster than j^m for some integer m.

Analytic continuation of integrals (16) can be realized in spherical harmonics as

$$M^{\alpha} f = \sum_{j,k} m_{j,\alpha} f_{j,k} Y_{j,k},$$

where

$$m_{j,\alpha} = \begin{cases} (-1)^{j/2} \frac{\Gamma(j/2+(1-\alpha)/2)}{\Gamma(j/2+(n-1+\alpha)/2)}, & \text{if } j \text{ is even;} \\ 0, & \text{if } j \text{ is odd}, \end{cases}$$

see [18]. If $f \in \mathscr{D}'(S^{n-1})$, then $M^{\alpha} f$ is a distribution defined by

$$(M^{\alpha} f, \omega) = (f, M^{\alpha}\omega) = \sum_{j,k} m_{j,\alpha} f_{j,k} \omega_{j,k}, \ \ \omega \in \mathscr{D}(S^{n-1}); \ \ \alpha \neq 1,3,5,\dots. \tag{18}$$

Lemma 3.1 ([17]). *Let $\alpha, \beta \in \mathbb{C}; \alpha, \beta \neq 1,3,5,\dots$. If $\alpha + \beta = 2-n$ and $f \in \mathscr{D}_e(S^{n-1})$ (or $f \in \mathscr{D}'_e(S^{n-1})$), then*

$$M^{\alpha} M^{\beta} f = f. \tag{19}$$

If $\alpha, 2-n-\alpha \neq 1,3,5,\dots$, then M^{α} is an automorphism of the spaces $\mathscr{D}_e(S^{n-1})$ and $\mathscr{D}'_e(S^{n-1})$.

Using (16), (17) and (2), the formula (1) can be rewritten by

$$\big(M^{p+1} f_p(K,\cdot)\big)(u) = 2n\gamma_n(p+1)h(\Pi_p K, u)^p, \ \ \text{if } \ p \geq 1, p \neq 2,4,6\cdots; \tag{20}$$

$$\big(\widetilde{M}^{p+1} f_p(K,\cdot)\big)(u) = 2nh(\Pi_p K, u)^p, \ \ \text{if } \ p = 2,4,6\cdots, \tag{21}$$

where the constant

$$\gamma_n(p+1) = \frac{\sigma_{n-1}\Gamma(-p/2)}{2\pi^{(n-1)/2}\Gamma((1+p)/2)} = \frac{-2^{p-1}\sigma_{n-1}}{\pi^{(n-2)/2}\Gamma(1+p)\sin(\pi p/2)}$$

is positive for each $p \in (4k-2,\ 4k)$ and negative for each $p \in (4k,\ 4k+2)$, where $k \in \mathbb{N}$.

3.2. $\lambda-$intersection bodies and $(\mathbb{R}^n, ||\cdot||_K)$ isometric embedding L_p

Let λ be a real number,

$$s_{\lambda} = \begin{cases} 1, & \text{if } \lambda > 0, \ \lambda \neq n, n+2, n+4, \dots; \\ \Gamma(\lambda/2), & \text{if } \lambda < 0, \ \lambda \neq -2,-4,-6,-8,\dots. \end{cases}$$

The values $\lambda = 0, n, n+2, n+4, \dots$ will not be considered in the following, but values $\lambda = -2, -4, \dots$ will be included.

Definition 3.2 ([17]). Let $\lambda < n, \lambda \neq 0$. An origin-symmetric star body K in $\mathbb{R}^n$ is said to be a $\lambda-$intersection body if there is a measure $\mu \in \mathscr{M}_{e+}(S^{n-1})$ such that $s_{\lambda}\rho_K^{\lambda} = M^{1-\lambda}\mu$ for $\lambda \neq -2l, l \in \mathbb{N}$, and $\rho_K^{-2l} = \widetilde{M}^{1+2l}\mu$ for $\lambda = -2l$.

We denote by $\mathscr{I}_\lambda^n$ the set of all $\lambda-$intersection bodies of origin-symmetric star bodies in $\mathbb{R}^n$.

Definition 3.3 ([17])**.** For a star body $K \in \mathscr{S}_o^n$, the quasi-normed space $(\mathbb{R}^n, ||\cdot||_K)$ is said to be isometrically embedded in $L_p, p > 0$, if there is a linear operator $T : \mathbb{R}^n \longrightarrow L_p([0,1])$ such that $||x||_K = ||Tx||_{L_p([0,1])}$.

Lemma 3.4 ([17])**.** *Let $p > -n, p \neq 0$. Then $(\mathbb{R}^n, ||\cdot||_K)$ embeds isometrically in L_p if and only if $K \in \mathscr{I}_{-p}^n$.*

Lemma 3.5. (see [4, Lecture 6.1]) *For $p > 0$, an $n-$dimensional space $(\mathbb{R}^n, ||\cdot||)$ embeds in L_p if and only if there exists a finite Borel measures $\mu \in \mathscr{M}(S^{n-1})$ such that for every $x \in \mathbb{R}^n$ satisfying*

$$||x||^p = \int_{S^{n-1}} |(x,\xi)|^p d\mu(\xi). \tag{22}$$

On the other hand, this can be considered as the definition of embedding in $L_p, -1 < p < 0$ (see [5]).

Lemma 3.6 ([6])**.** *Let L be an origin-symmetric star body in $\mathbb{R}^n$, $p \geq 1$, then following is equivalent:*

(1) *L is a $p-$projection body;*

(2) *$(\mathbb{R}^n, ||\cdot||_{L^*})$ is isometrically embedded to a subspace of L_p .*

Combining Lemma 3.4 and Lemma 3.6, we can get the following Lemma:

Lemma 3.7. *Let L be an origin-symmetric convex body in $\mathbb{R}^n$, $p \geq 1$, then the following is equivalent:*

(1) *$L \in \mathscr{I}_{-p}^n$;*

(2) *$(\mathbb{R}^n, ||\cdot||_L)$ is isometrically embedded to a subspace of L_p;*

(3) *L^* is a $p-$projection body.*

We remind the notation

$$\Lambda_0 = \{n, n+2, n+4, \ldots\} \cup \{0, -2, -4, \cdots\}.$$

We also need to use the following results in [17]:

Lemma 3.8. *For $\lambda \in \mathbb{R} \setminus \Lambda_0$, the following statements are equivalent:*

(1) *$K \in \mathscr{I}_\lambda^n$;*

(2) *The Fourier transform* $\left[s_\lambda \|\cdot\|_K^{-\lambda}\right]^\wedge$ *is a positive distribution on* $\mathbb{R}^n \setminus \{0\}$ *(for* $\lambda > 0$*, this can be replaced by* $\|\cdot\|_K^{-\lambda}$ *is a positive definite distribution on* $\mathbb{R}^n$ *);*

(3) $s_\lambda M^{1+\lambda-n}\rho_K^\lambda \in \mathscr{M}_{e+}(S^{n-1})$.

4. Main results and its proofs

In order to prove Theorem 1.1 and Theorem 1.2 that we proposed in the introduction, the following two main Lemma are required.

Lemma 4.1. *Let* $p \geq 1$*, where* p *is not an even integer. Let* K *and* L *be two origin-symmetric convex bodies in* $\mathscr{F}_c^n$*, and let* $\Lambda_p L \in \mathscr{S}_o^n$ *be such that radial function* $\rho(\Lambda_p L, \cdot)$ *is infinitely smooth. Suppose also that the surface area measures of* K *and* L *are absolutely continuous. If* $\Gamma(-p/2)\left(M^{1-p-n} f_p(L,\cdot)^{-\frac{p}{n+p}}\right)(\theta) \in \mathscr{M}_{e+}(S^{n-1})$ *for all* $\theta \in S^{n-1}$*, and*

$$\gamma_n(1+p)^{-1}\left(M^{1+p} f_p(K,\cdot)\right)(\theta) \leq \gamma_n(1+p)^{-1}\left(M^{1+p} f_p(L,\cdot)\right)(\theta), \quad \theta \in S^{n-1},$$

then

$$\Omega_p(K) \leq \Omega_p(L).$$

Proof. By the conditions we have

$$\begin{aligned}
&\Gamma(-p/2)\gamma_n(1+p)^{-1}\int_{S^{n-1}} \left(M^{1+p} f_p(K,\cdot)\right)(\theta)\left(M^{1-p-n} f_p(L,\cdot)^{-\frac{p}{n+p}}\right)(\theta)\,d\theta \\
&\quad \leq \Gamma(-p/2)\gamma_n(1+p)^{-1}\int_{S^{n-1}} \left(M^{1+p} f_p(L,\cdot)\right)(\theta)\left(M^{1-p-n} f_p(L,\cdot)^{-\frac{p}{n+p}}\right)(\theta)\,d\theta. \qquad (23)
\end{aligned}$$

Using Lemma 3.1 in (23), we have

$$\begin{aligned}
&\Gamma(-p/2)\gamma_n(1+p)^{-1}\int_{S^{n-1}} f_p(K,u) f_p(L,u)^{-\frac{p}{n+p}}\,du \\
&\leq \Gamma(-p/2)\gamma_n(1+p)^{-1}\int_{S^{n-1}} f_p(L,u) f_p(L,u)^{-\frac{p}{n+p}}\,du. \qquad (24)
\end{aligned}$$

By formula (9) of the L_p-mixed affine surface area, we know that (24) is equivalent to

$$\Gamma(-p/2)\gamma_n(1+p)^{-1}\Omega_{-p}(K,L) \leq \Gamma(-p/2)\gamma_n(1+p)^{-1}\Omega_p(L). \qquad (25)$$

Note that $p \geq 1, \Gamma(-p/2)\gamma_n(1+p)^{-1}$ is positive all along, thus

$$\Omega_{-p}(K,L) \leq \Omega_p(L). \qquad (26)$$

Now we apply inequality (12), then

$$\Omega_p(L) \geq \Omega_{-p}(K,L) \geq \Omega_p(K)^{\frac{n+p}{n}}\Omega_p(L)^{-\frac{p}{n}},$$

this implies

$$\Omega_p(K) \leq \Omega_p(L).$$

Remark 4.2. From formula (13), Lemma 3.7 and Lemma 3.8, we know that for $p \geq 1$ and p is not an even integer the following statements are equivalent:

(1) $\Lambda_p L \in \mathscr{I}^n_{-p}$;

(2) $(\mathbb{R}^n, ||\cdot||_{\Lambda_p L})$ is isometrically embedded to a subspace of L_p;

(3) $\Gamma(-p/2)\big(M^{1-p-n} f_p(L,\cdot)^{-\frac{p}{n+p}}\big)(\theta) \in \mathscr{M}_{e+}(S^{n-1})$;

(4) $\Gamma(-p/2)\big(M^{1-p-n}\rho(\Lambda_p L,)^{-p}\big)(\theta) \in \mathscr{M}_{e+}(S^{n-1})$.

Lemma 4.3. *Let $p \geq 1$, where p is not an even integer. Let K be an origin-symmetric convex bodies in $\mathscr{F}^n_c$ and such that $\Lambda_p K \in \mathscr{S}^n_o$. If $\Gamma(-p/2)\big(M^{1-p-n} f_p(K,\cdot)^{-\frac{p}{n+p}}\big)(\theta)$ is negative on an open subset of S^{n-1}, then there exists an origin-symmetric convex body L in $\mathbb{R}^n$, such that*

$$\gamma_n(1+p)^{-1}\big(M^{1+p} f_p(K,\cdot)\big)(\theta) \leq \gamma_n(1+p)^{-1}\big(M^{1+p} f_p(L,\cdot)\big)(\theta),$$

but

$$\Omega_p(K) > \Omega_p(L).$$

Proof. Let $\Omega = \{\theta \in S^{n-1} : \Gamma(-p/2)\big(M^{1-p-n} f_p(K,\cdot)^{-\frac{p}{n+p}}\big)(\theta) < 0\}$. From this and Remark 4.2 we know $\Lambda_p K \not\in \mathscr{I}^n_{-p}$. Then by Definition 3.2, there exists a finite Borel measure $\mu \in \mathscr{M}_e(S^{n-1})$, which is negative on some open origin-symmetric set $\Omega \subset S^{n-1}$ and such that $\Gamma(-p/2)\rho^{-p}_{\Lambda_p K} = M^{1+p}\mu$. From Definition (13), this is equivalent to $\Gamma(-p/2) f_p(K,\cdot)^{-\frac{p}{n+p}} = M^{1+p}\mu$.

We choose an even Borel measure $v \in \mathscr{M}_e(S^{n-1})$ such that the $(\gamma_n(1-p))^{-1}v$ constant is not equal to zero, $(\gamma_n(1-p))^{-1}v(\theta) \geq 0$ for $\theta \in \Omega$, and $(\gamma_n(1-p))^{-1}v(\theta) \equiv 0$, otherwise. Because $v \in \mathscr{M}_e(S^{n-1})$ and $f_p(K,\theta) = h_K^{1-p}(\theta) f_K(\theta) > 0$, one can choose a small $\varepsilon > 0$ so that, for $\theta \in S^{n-1}$ and $r > 0$,

$$f_p(L, r\theta) = f_p(K, r\theta) + \varepsilon M^{1-p-n} v(\theta) > 0.$$

By Lutwak's [14] extension of the Minkowski's existence theorem, $f_p(L,\cdot)$ defines an origin-symmetric convex body $L \in \mathscr{K}^n_c$.

Using Lemma 3.1, we have

$$\gamma_n(1+p)^{-1} M^{1+p} M^{1-p-n} v = \gamma_n(1+p)^{-1} v \geq 0,$$

then

$$\gamma_n(1+p)^{-1}\big(M^{1+p} f_p(L,\cdot)\big)(r\theta) - \gamma_n(1+p)^{-1}\big(M^{1+p} f_p(K,\cdot)\big)(r\theta)$$
$$= \varepsilon\gamma_n(1+p)^{-1} M^{1+p} M^{1-p-n} v(\theta) = \varepsilon\gamma_n(1+p)^{-1} v(\theta) \geq 0,$$

that is

$$\gamma_n(1+p)\big(M^{1+p} f_p(K,\cdot)\big)(r\theta) \leq \gamma_n(1+p)\big(M^{1+p} f_p(L,\cdot)\big)(r\theta).$$

Next, by the definition of μ, we have

$$\begin{aligned}&\Gamma(-p/2)\gamma_n(1+p)^{-1}\big(f_p(K,\cdot)^{-\frac{p}{n+p}},\ f_p(L,\cdot)-f_p(K,\cdot)\big)\\&\quad=\gamma_n(1+p)^{-1}\big(M^{1+p}\mu,\ \varepsilon M^{1-p-n}v\big)\\&\quad=\gamma_n(1+p)^{-1}\varepsilon(\mu,\ v)<0.\end{aligned}$$

From this we get

$$\Gamma(-p/2)\gamma_n(1+p)^{-1}\big(f_p(K,\theta)^{-\frac{p}{n+p}},\ f_p(L,\theta)\big)<\Gamma(-p/2)\gamma_n(1+p)^{-1}\big(f_p(K,\theta)^{-\frac{p}{n+p}},\ f_p(K,\theta)\big),$$

or

$$\Gamma(-p/2)\gamma_n(1+p)^{-1}\Omega_{-p}(L,K)<\Gamma(-p/2)\gamma_n(1+p)^{-1}\Omega_p(K).$$

Note that $p\geq 1, \Gamma(-p/2)\gamma_n(1+p)^{-1}$ is positive all along, thus

$$\Omega_{-p}(L,K)<\Omega_p(L).$$

Now we apply inequality (12), then

$$\Omega_p(L)>\Omega_{-p}(L,K)\geq\Omega_p(L)^{\frac{n+p}{n}}\Omega_p(K)^{-\frac{p}{n}},$$

this implies

$$\Omega_p(K)>\Omega_p(L).$$

Below, we begin to prove Theorem 1.1 and Theorem 1.2.

Proof of Theorem 1.1. From Lemma 4.1 and (20) we know that $\Pi K\subset\Pi L$ is equivalent to

$$\gamma_n(2)^{-1}\big(M^2 f(K,\cdot)\big)(\theta)\leq\gamma_n(2)^{-1}\big(M^2 f(L,\cdot)\big)(\theta),\theta\in S^{n-1}.$$

Taking $p=1$ in Lemma 4.1, if the condition $\Gamma(-1/2)\big(M^{-n}f(L,\cdot)^{-\frac{1}{n+1}}\big)(\theta)\in\mathcal{M}_{e+}(S^{n-1})$ is true for all $\theta\in S^{n-1}$, then Winterniz problem for projection bodies has an affirmative answer for this L and any K.

Similarly, taking $p=1$ in Lemma 4.3, if the curvature function $f(K,\cdot)$ is positive on S^{n-1} and $\Gamma(-1/2)\times\big(M^{-n}f(K,\cdot)^{-\frac{1}{n+1}}\big)(\theta)$ is negative on an open subset of S^{n-1}, then there exists an origin-symmetric convex body L such that Winterniz problem for projection bodies has an negative answer.

Therefore, using the equivalence of (1) and (3) in Remark 4.2, we can seen that for a given dimension n the answer of Winterniz problem for projection bodies is affirmative if and only if all convex bodies $Q\in\mathcal{F}_o^n$ with $\Lambda_1 Q\in\mathcal{S}_o^n$, such that $\Lambda_1 Q\in\mathcal{I}_{-1}^n$. According to the equivalence of (1) and (2) in Lemma 3.7, then this is equivalent to saying that any $n-$dimension normed space $(\mathbb{R}^n, ||\cdot||_{\Lambda_1 Q})$ can be isometrically embedded into L_1 , which is true if and only if for any $n \le 2$ (see [2, 6]).

Proof of Theorem 1.2. Let $p > 1$ and p is not an even integer. We will prove that for a given dimension n the answer of L_p−Winterniz problem for Firey projections is affirmative if and only if all convex bodies $Q \in \mathscr{F}_o^n$ with $\Lambda_p Q \in \mathscr{S}_o^n$, such that $\Lambda_p Q \in \mathscr{I}_{-p}^n$. Using the same argument as in Theorem 1.1 of the proof, we according to Lemma 3.7, this is equivalent to saying that any n−dimensional normed space $(\mathbb{R}^n, ||\cdot||_{\Lambda_p Q})$ can be isometrically embedded into L_p, which is not true for $n \ge 2$ (see [6]). Thus, for $p > 1$ and p is not an even integer, L_p−Winterniz monotonicity problem for L_p−Firey projections has a negative answer if and only if for $p > 1$ and $n \ge 2$.

Finally, we prove that the answer is always negative if p is an even integer. It turns out that for any body $K \subset \mathbb{R}^n$ there exists a body $L \subset \mathbb{R}^n$ such that the Firey projections of bodies K and L are equal but their L_p−affine surface area are different.

Let p be an even integer. Then $|x\cdot\xi|^p = (x\cdot\xi)^p$, and there exists a nonzero continuous even function g on S^{n-1} such that (see [19])

$$\int_{S^{n-1}} |x\cdot\xi|^p g(x)dx = 0, \quad \forall \xi \in S^{n-1}. \tag{27}$$

Indeed, if $p = 2k$, then $(x\cdot\xi)^{2k}$ is a polynomial of degree $2k$ with coefficients depending on ξ. So, it is enough to construct a nontrivial even function g, satisfying

$$\int_{S^{n-1}} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} g(x)dx = 0,$$

for all integer powers $0 \le i_j \le 2k$ such that $i_1 + i_2 + \cdots + i_n = 2k$. Taking $g(x) = \sum_{l=1}^m c_l x_1^{2l}$ and solving the system of linear equations, one can find a nontrivial solution $c_1, c_2, \dots, c_m$ provided m is big enough.

Consider an origin-symmetric convex body K in $\mathbb{R}^n$ with a strictly positive L_p-curvature function (i.e., $f_p(K,\xi) > 0, \forall\, \xi \in S^{n-1}$). Without loss of generality, we may assume that

$$\int_{S^{n-1}} f_p(K,\xi)^{-\frac{p}{n+p}} g(\xi)d\xi \ge 0, \tag{28}$$

(otherwise consider$-g(\xi)$ instead of $g(\xi)$). Choose $\varepsilon > 0$ such that

$$f_p(K,\xi) - \varepsilon g(\xi) > 0, \quad \forall \xi \in S^{n-1}.$$

Since $f_p(K,\theta) = h_K^{1-p}(\theta) f(K,\theta) > 0$, using the existence theorem for L_p-curvature functions (see [14]), we conclude that there exists an origin-symmetric convex body L in $\mathbb{R}^n$ such that

$$f_p(L,\xi) = f_p(K,\xi) - \varepsilon g(\xi). \tag{29}$$

Now multiply both sides by $|x\cdot\xi|^p$ and integrating, then

$$\int_{S^{n-1}} |x\cdot\xi|^p f_p(L,\xi)d\xi = \int_{S^{n-1}} |x\cdot\xi|^p f_p(K,\xi)d\xi - \varepsilon\int_{S^{n-1}} |x\cdot\xi|^p g(\xi)d\xi.$$

Applying (27) and (1), we get that $h(\Pi_p L, x) = h(\Pi_p K, x)$, i.e., $\Pi_p L = \Pi_p K$.

On the other hand, using (28), (29) and inequality (12), we have

$$\begin{aligned}\Omega_p(K) &= \int_{S^{n-1}} f_p(K,\xi)^{\frac{n}{n+p}} d\xi\\ &= \int_{S^{n-1}} f_p(K,\xi)^{-\frac{p}{n+p}} f_p(K,\xi)d\xi\\ &= \int_{S^{n-1}} f_p(K,\xi)^{-\frac{p}{n+p}} (f_p(L,\xi)+\varepsilon g(\xi))d\xi\\ &\geq \int_{S^{n-1}} f_p(K,\xi)^{-\frac{p}{n+p}} f_p(L,\xi)d\xi\\ &= \Omega_{-p}(L,K)\\ &\geq \Omega_p(L)^{\frac{n+p}{n}}\Omega_p(K)^{-\frac{p}{n}}.\end{aligned}$$

The last inequality in the above formula is the equality holds if and only if K and L are dilates. Therefore, $\Omega_p(K) = \Omega_p(L)$ must implies that $K = L$, but by (29) this contradicts with the uniqueness of L_p-curvature function. Then there must be $\Omega_p(K) > \Omega_p(L)$. The proof of Theorem 1.2 is completed.

References

[1] W. Blaschke, Vorlesungen über Differentialgeometric, II:Affine Differentialgeometrie, Springer, Berlin, 1923.

[2] L. Dor, *Potentials and isometric embeddings in L_1*, Israel J. Math., **24**(1976), 260–268.

[3] R. J. Gardner, Geometric tomography, Encyclopedia of Mathematics and its Application 58, Cambridge University Press, New York, 1995.

[4] A. Koldobsky, Fourier analysis in convex geometry, Mathematical Surveys and Monographs, American Mathematical Society, Providence RI, 2005.

[5] A. Koldobsky, *Positive definite distributions and subspaces of L_{-p} with applications to stable processes*, Canad. Math. Bull., **42**(1999), 344–353.

[6] A. Koldobsky, *Generalized Lévy representation of norms and isometric embeddings into L_p spaces*, Ann. Inst. H. Poincaré Sér. B., **28**(1992), 335–353.

[7] E. Lutwak, *Mixed affine surface area*, J. Math. Anal. Appl., **125**(1987), 351–360.

[8] E. Lutwak, *Centroid bodies and dual mixed volumes*, Proc. London Math.Soc., **60**(1990), 365–391.

[9] E. Lutwak, *The Brunn-Minkowski-Firey theory II: Affine and geominimal surface area*, Adv. Math., **118**(1996), 244–294.

[10] E. Lutwak, D. Yang and G. Zhang, *L_p affine isoperimetric inequalities*, J. Differential Geom., **56**(2000), 111–132.

[11] E. Lutwak, *The Brunn-Minkowski-Firey Theory i : Mixed volumes and the Minkowski Problem*, J. Differential Geom., **38**(1993), 131–150.

[12] E. Lutwak, *Extended affine surface area*, Adv. Math., **85**(1991), 39–68.

[13] E. Lutwak, *On some affine isoperimetric inequalities*, J. Differential Geom., **23**(1986), 1-13.

[14] E. Lutwak, D. Yang and G. Zhang, *On the L_p−Minkowski problem*, Trans. Amer. Math. Soc., **356**(2004), 4359–4370.

[15] K. Leichtweiß, *Bemerkungen zur Definition einer erweiterten Affinoberfläche von E.Lutwak*, Manuscripta Math., **65**(1989), 181–197.

[16] C. M. Petty, *Affine isoperimetric problems*, Ann. New York Acad. Sci., **440**(1985), 113–127.

[17] B. Rubin, *Intersection bodies and generalized cosine transforms*, Advances in Mathematics, **218**(2008), 696–727. In: http://arxiv.org/PS_cache/arxiv/pdf/0704/0704.0061v2.pdf.

[18] B. Rubin, *Inversion of fractional integrals related to the spherical Radon transform*, Journal of Functional Analysis, **157**(1998), 470–487.

[19] D. Ryabogin and A. Zvavitch, *The Fourier transform and Firey projections of convex bodies*, Indiana Univ. Math. J., **53**(2004), 667–682.

[20] R. Schneider, Convex Bodies: The Brunn-Minkowski theory, Cambridge Univ.Press, Cambridge, 1993.

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