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LEFT PRIME WEAKLY REGULAR NEAR-RINGS

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Abstract. In this paper we introduce the notion of left prime weakly regular, left prime weakly π-regular and left prime pseudo π-regular near-rings. We also introduce the concept of strong left prime weakly regular near-rings. We have obtained conditions for a near-ring N to be left prime pseudo π-regular. We have also obtained conditions for a near-ring N to be strong left prime pseudo π-regular. Finally we have answered the open question given in [1].

Introduction

Throughout this paper N stands for a zero-symmetric near-ring. Let $P_0(N)$ denote the prime radical and $N(N)$ denote the set of nilpotent elements of N. From [2] an ideal I of N is a 2-primal ideal of N if $P_0(N/I) = N(N/I)$. If I is the zero ideal of N, then N is a 2-primal near-ring (i.e. $P_0(N) = N(N)$). A near-ring N is said to be reduced if $N(N) = 0$.

Recall from [5] that an ideal P is called a minimal prime ideal of an ideal I if P is minimal in the set of all prime ideals containing I. An ideal I of N is a completely prime ideal (completely semiprime ideal) if for a, $b \in N$, $ab \in I$ implies $a \in I$ or $b \in I$ ($a^2 \in I$ implies $a \in I$).

N is said to fulfill the insertion of factors property (IFP) provided that for all a, b, $x \in N$, $ab = 0$ implies $axb = 0$. Also for $X \subseteq N$, $(0 : X)$ and $\langle x \rangle$ denote the left annihilator of X and the ideal of N generated by x respectively.

Birkenmeier and Groenewald [1] introduced left weakly regularity in near-rings. In this paper we introduce the concept of left prime weakly regularity in near-rings.

Definition 1. (i) A near-ring N is said to be left (right) prime weakly regular if for a given $x \in N$ there exists a minimal prime ideal P of $\langle x \rangle$ such that $x \in Px$ ($x \in xP$).

(ii) N is said to be left (right) prime weakly π-regular if for a given $x \in N$ there exists a natural number $n = n(x)$ and a minimal prime ideal P of $\langle x^n \rangle$ such that $x^n \in Px^n$ ($x^n \in x^nP$).

(iii) N is said to be left (right) prime pseudo π-regular if for a given $x \in N$ there exists a natural number $n = n(x)$ and a minimal prime ideal P of $\langle x \rangle$ such that $x^n \in Px^n$ ($x^n \in x^nP$).

Following G. F. Birkenmeier and N. J. Groenewald [1], N is said to be left weakly regular if $x \in \langle x \rangle x$ for all $x \in N$, N is said to be left weakly π-regular if $x^n \in \langle x^n \rangle x^n$ for

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all $x \in N$ and for a natural number $n = n(x)$ and N is said to be left pseudo π-regular if $x^n \in \langle x \rangle x^n$ for all $x \in N$ and for a natural number $n = n(x)$.

If N is left weakly regular then clearly N is left prime weakly regular. But the converse is not true as the following example shows.

Example 1. Let $N = \begin{bmatrix} F & F \\ O & F \end{bmatrix}$ where $F = \{0, 1\}$ is the field under addition modulo 2 and multiplication modulo 2. Then N is not left weakly regular, since if $x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then $x \notin \langle x \rangle x = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. But N is left prime weakly regular.

Lemma 2. *Let N be reduced. Then N is left prime weakly regular if and only if N is left prime pseudo π-regular.*

Proof. Assume that N is left prime pseudo π-regular. Let $x \in N$. Then there exists a minimal prime ideal P of $\langle x \rangle$ and a natural number n such that $x^n \in Px^n$. So there exists an element $a \in P$ such that $x^n = ax^n$. If $n = 1$, then $x = ax \in Px$ and therefore N is left prime weakly regular. For $n > 1$, $(x - ax)x^{n-1} = x^n - ax^n = x^n - x^n = 0$. Since N is reduced, N has the IFP. So $((x - ax)x)^{n-1} = 0$ and hence $(x - ax)x = 0$. Thus $x(x - ax) = 0$. Since $(x - ax)^2 = (x - ax)(x - ax) = x(x - ax) - ax(x - ax) = 0$, $x - ax = 0$. Thus $x = ax \in Px$. Hence N is left prime weakly regular. Clearly if N is left prime weakly regular then N is left prime pseudo π-regular.

Theorem 3. *Let N be an IFP near-ring with left unity e. Then the following are equivalent:*

(i) *N is left prime pseudo π-regular.*

(ii) *$N = (0 : a^k) + P$, where P is a minimal prime ideal of $\langle a \rangle$ and for some positive integer k.*

Proof. (i)$\Rightarrow$(ii). Assume that N is left prime pseudo π-regular. Let $a \in N$. Then there exists a minimal prime ideal P of $\langle a \rangle$ and a positive integer k such that $a^k \in Pa^k$. So there exists $s \in P$ such that $a^k = sa^k$. Thus $(e - s) \in (0 : a^k)$. Then for any $n \in N$, we have $n = (e - s + s)n = (e - s)n + sn$. Since $(0 : a^k)$ is an ideal of N, $(e - s)n \in (0 : a^k)$. Hence $N = (0 : a^k) + P$.

(ii)$\Rightarrow$(i) Let $N = (0 : a^k) + P$ for some positive integer k. Then there exists $r \in (0 : a^k)$ and $s \in P$ such that $e = r + s$. Thus $a^k = ra^k + sa^k = sa^k \in Pa^k$. Hence N is left prime pseudo π-regular.

Lemma 4. *If $I = (0 : a)$ is a 2-primal ideal and if $\bar{k}\bar{a} \in N(\bar{N})$ where $\bar{N} = N/I$, then $ka^2 \in N(N)$.*

Proof. Let $\bar{k}\bar{a} \in N(\bar{N})$. Since $(0 : a)$ is a 2-primal ideal, $N(\bar{N})$ is an ideal. Thus $\bar{k}\bar{a}^2 \in N(\bar{N})$. So there exists a positive integer j, such that $(\bar{k}\bar{a}^2)^j = \bar{0}$. Thus $(\bar{k}\bar{a}^2)^j(\bar{k}\bar{a}) = \bar{0}$ and hence $(ka^2)^{j+1} = 0$. Therefore $ka^2 \in N(N)$.

Theorem 5. *Let N be a near-ring with left unity e such that every completely prime ideal is maximal. If $a \in N$ is such that $(0 : a)$ is a 2-primal ideal of N, then there exists $s \in \langle a\rangle$ such that $a^2 = x + sa^2$ where $s \in \langle a\rangle$ and $x \in N(N)$.*

Proof. Let $0 \neq a \in N$ be such that $(0 : a)$ is 2-primal. Let $\bar{N} = N/(0 : a)$. Since every completely prime ideal of N is maximal, every completely prime ideal of $\bar{N}$ is also maximal. Let M be the multiplicative semigroup generated by all elements of the form $\bar{a} - \bar{x}\bar{a}$, where $x \in \langle a\rangle$. Now we claim $P_0(\bar{N}) \cap M \neq \varnothing$.

Suppose $P_0(\bar{N}) \cap M = \varnothing$. Let $S = \{I/I$ is completely semiprime ideal with $I \cap M = \varnothing\}$. Then S is nonempty. Using Zorn's Lemma, S has a maximal element, say P. Then P is completely prime ideal of N with $P \cap M = \varnothing$. Now $\langle\bar{a}\rangle \subseteq \bar{P}$ or there exists $\bar{\alpha} \in \langle\bar{a}\rangle$ such that $\bar{\alpha} \notin \bar{P}$. If $\langle\bar{a}\rangle \subseteq \bar{P}$, then $\bar{a} - \bar{x}\bar{a} \in \bar{P} \cap M \neq \varnothing$ for any $\bar{x} \in \bar{N}$, which is a contradiciton. So assume that there exists $\bar{\alpha} \in \langle\bar{a}\rangle$ such that $\bar{\alpha} \notin \bar{P}$. Since $\bar{P}$ is maximal, we have $\bar{P} + \langle\bar{\alpha}\rangle = \bar{N}$. Thus $\bar{e} = \bar{p} + \bar{t}$ for some $\bar{p} \in \bar{P}$ and $\bar{t} \in \langle\bar{\alpha}\rangle \subseteq \langle\bar{a}\rangle$. Then $\bar{a} - \bar{t}\bar{a} = (\bar{e} - \bar{t})\bar{a} = \bar{p}\bar{a} \in \bar{P} \cap M \neq \varnothing$, which is again a contradiction. Therefore $P_0(\bar{N}) \cap M \neq \varnothing$. So we have $(\bar{a} - \bar{t}_1\bar{a})(\bar{a} - \bar{t}_2\bar{a}) \cdots (\bar{a} - \bar{t}_n\bar{a}) \in P_0(\bar{N})$ for some $\bar{t}_i \in \langle\bar{a}\rangle$. Since $(0 : a)$ is 2-primal, $P_0(\bar{N}) = N(\bar{N})$. Thus $N(\bar{N})$ is completely semiprime ideal. So there exists $\bar{s} \in \langle\bar{a}\rangle$ such that $(\bar{e} - \bar{s})\bar{a}^n \in N(\bar{N})$. Then $((\bar{e} - \bar{s})\bar{a})^n \in N(\bar{N})$ and hence $(\bar{e} - \bar{s})\bar{a} = \bar{k}\bar{a} \in N(\bar{N})$ where $\bar{k} = \bar{e} - \bar{s}$. Thus $(\bar{e} - \bar{s} - \bar{k})\bar{a} = \bar{0}$ and hence $(e - s - k)a^2 = 0$. Therefore $a^2 = ka^2 + sa^2$. Since $\bar{k}\bar{a} \in N(\bar{N})$, $ka^2 \in N(N)$ by Lemma 4. Hence $a^2 = x + sa^2$ where $s \in \langle a\rangle$ and $x = ka^2 \in N(N)$.

Corollary 6. ([1], Proposition 3.2 [i]) *Let N be an IFP right near-ring with left unity e such that every completely prime ideal is maximal. If $a \in N$ such that $(0 : a)$ is a 2-primal ideal of N, then there exists $s \in \langle a\rangle$ such that $a^3 = sa^3 + x$, where $x \in N(N)$.*

Definition 7. (i) A near-ring N is said to be strong left prime weakly regular if for a given $x \in N$, $x \in Px$ for every prime ideal P of $\langle x\rangle$.

(ii) N is said to be strong left prime weakly π-regular if for a given $x \in N$, there exists a positive integer n such that $x^n \in Px^n$ for every prime ideal P of $\langle x^n\rangle$.

(iii) N is said to be strong left prime pseudo π-regular if for a given $x \in N$ there exists a positive integer n such that $x^n \in Px^n$ for every prime ideal P of $\langle x\rangle$.

If N is left weakly regular then clearly N is strong left prime weakly regular. For example, $(Z_6, \oplus, \odot)$ is strong left prime weakly regular, where $\oplus$ and $\odot$ are the addition modulo 6 and multiplication modulo 6 respectively.

If N is strong left prime weakly regular, then N is left prime weakly regular. But the converse is not true as the following example shows.

Example 2. The near-ring N in Example (1) is left prime weakly regular but not strong left prime weakly regular, since for $P = \left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$ and for $x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $x \notin Px = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Theorem 8. *If N is 2-primal and N is strong left prime pseudo π-regular, then every prime ideal P of N is completely prime.*

Proof. Let P be any prime ideal of N. Since N is 2-primal, $P_0(N) = N(N)$ and hence $P_0(N)$ is completely semiprime. So by Proposition 2.1 of [1] there exists a prime ideal X of N which is minimal among prime ideals and $X \subset P$ is completely prime.

Suppose Q is any prime ideal such that $X \subset Q$. Let $a \in Q$ be such that $a \notin X$. Since N is strong left prime weakly regular, $a^k \in Qa^k$ for some positive integer k. Thus $a^k = qa^k$ for some $q \in Q$. So for $n \in N$, $na^k = nqa^k$. Hence $(n - nq)a^k = 0 \in X$. Since X is completely prime and $a \notin X$, we have $n - nq \in X \subset Q$. Thus $n \in Q$, which implies $Q = N$. Hence $X = P$ and P is completely prime.

Definition 9. A near-ring N is said to satisfy PI condiiton if for any ideal I of N and for any $x \in I$, there exists a prime ideal P of N such that $\langle x \rangle \subseteq P \subseteq I$.

Theorem 10. *If N is 2-primal with PI condition and strong left prime pseudo π-regular, then every prime ideal of N is maximal.*

Proof. Let P be any prime ideal of N. Since N is 2-primal, $P_0(N)$ is completely semiprime. So by Propositon 2.1 of (i), there exists a completely prime ideal X of N such that $X \subset P$. Let I be any ideal such that $X \subset I$ and let $a \in I$. Since N satisfies PI condition, there exists a prime ideal Q of N such that $\langle a \rangle \subseteq Q \subseteq I$. Since N is strong left prime psuedo π-regular, we have $a^k \in Qa^k$ for some positive integer k. As in the case of Theorem 8, we get $Q = N$. Therefore $I = N$. Thus $X = P$ and hence P is maximal.

Following [1], N satisfies the CZ1 condition if for any x, $y \in N$ and positive integer k such that $(xy)^k = 0$, then there exists a positive integer m such that $x^m y^m = 0$.

Theorem 11. *Let N be a near-ring with left unity e which satisfies the $CZ1$ and PI condition. Suppose that $(0 : a)$ is a 2-primal ideal for all $a \in N$. Then the following are equivalent:*

(i) *N is left pseudo π-regular.*
(ii) *N is strong left prime pseudo π-regular.*
(iii) *Every prime ideal is maximal.*
(iv) *Every completely prime ideal is maximal.*

Proof. (i)⇒(ii) Since N is left pseudo π-regular, for every $x \in N$ there exists a natural number $n = n(x)$ such that $x^n \in \langle x \rangle x^n$. Therefore $x^n \in Px^n$ for every prime ideal P of $\langle x \rangle$. Hence N is strong left prime pseudo π-regular.

(ii)⇒(iii) Since N is strong left prime pseudo π-regular, by Theorem 10 every prime ideal of N is maximal.

(iii)⇒(iv) Proof is immediate.

(iv)⇒(i) Let $a \in N$. Since every completely prime ideal is maximal, by Theorem 5 there exists $s \in \langle a \rangle$ such that $a^2 = x + sa^2$ for some $x \in N(N)$. Thus $((e - s)a^2)^k = 0$ for some positive integer k. Since N satsifies CZ1 condition, there exists a positive integer m such that $(e-s)^m a^{2m} = 0$. Thus $(e-\bar{s})a^{2m} = 0$ for some $\bar{s} \in \langle a \rangle$. Hence $a^{2m} = \bar{s}a^{2m} \in \langle a \rangle a^{2m}$.

Birkenmeier and Groenewald [1] have raised the following question: Is a reduced left weakly regular near-ring (with unity) also a right weakly regular? We have answered affirmatively.

Theorem 12. *A reduced left weakly regular near-ring is also a right weakly regular.*

Proof. Let N be a left weakly regular near-ring. Let $x \in N$. Then $x \in \langle x \rangle x$. So there exists $a \in \langle x \rangle$ such that $x = ax$. Thus $(x - xa)x = x^2 - xax = 0$. Since N is reduced, we have $x(x - xa) = 0$ and hence $xa(x - xa) = 0$. Then $(x - xa)^2 = (x - xa)(x - xa) = x(x - xa) - xa(x - xa) = 0$ and hence $x - xa = 0$. Thus $x = xa \in x\langle x \rangle$. Therefore N is right weakly regular.

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