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COEFFICIENTS BOUNDS IN SOME SUBCLASS OF ANALYTIC FUNCTIONS

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Abstract. In this paper we consider a class of analytic functions introduced by Mishra and Gochhayat, *Fekete-Szegö problem for a class defined by an integral operator*, Kodai Math. J., 33(2010) 310–328, which is connected with k-starlike functions through Noor operator. We find inclusion relations and coefficients bounds in this class.

1. Introduction

Let $\mathcal{H}$ denote the class of analytic functions in the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{A} \subset \mathcal{H}$ denote the class of functions of the form

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m, \tag{1.1}$$

which are analytic in the open unit disk Δ.

In [16] Noor defined an operator $I_n : \mathcal{A} \to \mathcal{A}$ for $n \in \mathbb{N} \cup \{0\}$ as follows:

$$I_n f(z) = f_n^{\dagger}(z) * f(z), \tag{1.2}$$

where $f_n^{\dagger}$ is defined by the relation

$$\frac{z}{(1-z)^{n+1}} * f_n^{\dagger}(z) = \frac{z}{(1-z)^2}. \tag{1.3}$$

It is obvious that $I_0 f(z) = zf'(z)$ and $I_1 f(z) = f(z)$. The operator $I_n f$ defined by (1.2) is called Noor operator and for $n \geqq 2$ it represent an integral operator of f. For details see [16].

It is well known that for $\alpha > 0$

$$\frac{z}{(1-z)^{\alpha}} = \sum_{m=0}^{\infty} \frac{(\alpha)_m}{m!} z^{m+1} \quad (z \in \Delta),$$

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where $(x)_n$ is the Pochhammer symbol

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = \begin{cases} 1 & \text{for } n=0, x\neq 0, \\ x(x+1)\cdots(x+n-1) & \text{for } n\in\mathbb{N}=\{1,2,3,\ldots\}. \end{cases}$$

By (1.3) we obtain

$$\sum_{m=0}^{\infty}\frac{(n+1)_m}{m!}z^{m+1} * f_n^{\dagger}(z) = \sum_{m=0}^{\infty}\frac{(2)_m}{m!}z^{m+1}. \tag{1.4}$$

Then (1.4) implies that

$$f_n^{\dagger}(z) = \sum_{m=0}^{\infty}\frac{(2)_m}{(n+1)_m}z^{m+1} \quad (z\in\Delta).$$

Therefore, if f is of the form (1.1), then

$$I_n f(z) = z+\sum_{m=2}^{\infty}\frac{(2)_{m-1}}{(n+1)_{m-1}}a_m z^m = z+\sum_{m=2}^{\infty}\frac{m!}{(n+1)_{m-1}}a_m z^m \quad (z\in\Delta). \tag{1.5}$$

A function $f(z)$ in $\mathscr{A}$ is said to be in class $\mathscr{S}^*$ of starlike functions if

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\}>0 \quad (z\in\Delta).$$

Let $\mathscr{CV}$ denote the class of all functions $f\in\mathscr{A}$ that are convex univalent. It is known that $f\in\mathscr{CV}$ if and only if $zf'\in\mathscr{S}^*$, for details see [3].

Suppose that Γ is a smooth directed curve $z=z(t)$, $t\in[t_1,t_2]$, the direction being that determines as t increases. Let $f(\Gamma)$ be the image of Γ under a function that is analytic on Γ. The arc $f(\Gamma)$ is said to be convex if the argument of the tangent to $f(\Gamma)$ is a nondecreasing function of t. In 1991 Goodman [4] investigated a class of functions mapping circular arcs contained in the unit disk, with center at an arbitrarily chosen point in Δ, onto a convex arcs. Goodman denoted the class of such functions by $\mathscr{UCV}$. Recall here his definition.

Definition 1.1 ([4]). A function $f\in\mathscr{A}$ is said to be uniformly convex in Δ, if f is convex in Δ, and has the property that for every circular arc γ, contained in Δ, with center $\zeta\in\Delta$, the arc $f(\gamma)$ is convex.

In [18] Rønning and independently in [14] Ma and Minda gave a more applicable characterization of the class $\mathscr{UCV}$, stated below.

Definition 1.2 ([14, 18]). Let $f\in\mathscr{A}$. Then $f\in\mathscr{UCV}$ if and only if

$$\Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\left|\frac{zf''(z)}{f'(z)}\right| \quad (z\in\Delta). \tag{1.6}$$

In [10] and in the next papers of these authors generalized the notions of starlikeness and convexity. Let $0 \le k < \infty$. A function $f \in \mathscr{A}$ is said to be k-uniformly convex in Δ, if the image of every circular arc γ contained in Δ, with center ζ, is convex, where $|\zeta| \le k$. For fixed k, the class of all k-uniformly convex functions will be denoted by $k-\mathscr{UCV}$. Clearly, $0-\mathscr{UCV} = \mathscr{CV}$, and $1-\mathscr{UCV} = \mathscr{UCV}$. As with the class $\mathscr{UCV}$ it is possible to get a one-variable characterization of the class $k-\mathscr{UCV}$.

Definition 1.3 ([11])**.** Let $f \in \mathscr{A}$. Then $f \in k-\mathscr{UCV}$ iff

$$\Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} > k\left|\frac{zf''(z)}{f'(z)}\right| \quad (z\in\Delta).$$

The class $k-\mathscr{ST}$ consisting of k-starlike functions, is defined from $k-\mathscr{UCV}$ via the Alexander's transform (see [1]) i.e.

$$f \in k-\mathscr{UCV} \iff zf' \in k-\mathscr{ST}.$$

Definition 1.4 ([11])**.** Let $f \in \mathscr{A}$. Then $f \in k-\mathscr{ST}$ if and only if

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > k\left|\frac{zf'(z)}{f(z)}-1\right| \quad (z\in\Delta). \tag{1.7}$$

The class $k-\mathscr{ST}$ for $k=1$ becomes the class $\mathscr{PST}$, introduced earlier by Rønning [18]. The class $k-\mathscr{UCV}$ started earlier in papers [2, 23] with some additional conditions and without the geometric interpretation given in [11]. Recently Mishra and Gochhayat [15] defined a new class of functions using Noor operator as follows:

Definition 1.5 ([15])**.** A function $f \in \mathscr{A}$ is said to be in the class $\mathscr{M}(n,k)$, $(0 \le k < \infty;\ n \in \mathbb{N}\cup\{0\})$ if and only if $I_n f \in k-\mathscr{ST}$. Or equivalently

$$\Re\left\{\frac{z(I_nf)'(z)}{(I_nf)(z)}\right\} > k\left|\frac{z(I_nf)'(z)}{(I_nf)(z)}-1\right| \quad (z\in\Delta). \tag{1.8}$$

Note that the class $\mathscr{M}(n,k)$ unifies many subclasses of $\mathscr{A}$. In particular, $\mathscr{M}(0,0) = \mathscr{CV}$, the class of convex functions; $\mathscr{M}(0,1) = \mathscr{UCV}$, the class of uniformly convex functions; $\mathscr{M}(1,0) = \mathscr{S}^*$, the class of starlike functions; $\mathscr{M}(1,1) = \mathscr{PST}$, the class of parabolic starlike functions; $\mathscr{M}(0,k) = k-\mathscr{UCV}$ and $\mathscr{M}(1,k) = k-\mathscr{ST}$.

Let $\varphi(z) = z + a_m z^m$. It is easy to verify that $\varphi \in k-\mathscr{UCV}$ if and only if $|a_m| \le 1/[m(m+k(m-1))]$, and $\varphi \in k-\mathscr{ST}$ if and only if $|a_m| \le 1/(m+k(m-1))$. It is easy to check that for $n \in \{3,4,5,\dots\}$ we have

$$\frac{1}{m+k(m-1)}\,\frac{m!}{(n+1)_{m-1}} \le \frac{1}{m(m+k(m-1))},$$

hence, if $\varphi \in k-\mathcal{ST}$, then

$$I_n\varphi(z) = z + \frac{m!}{(n+1)_{m-1}} a_m z^m$$

is in $k-\mathcal{UCV}$ for $n \in \{3,4,5,\ldots\}$. Moreover, $I_n\varphi \notin k-\mathcal{UCV}$ for $n \in \{1,2\}$. It would be interesting to check this property of the Noor operator for other functions in $k-\mathcal{ST}$.

Conjecture. If $f \in k-\mathcal{ST}$ and $n \in \{3,4,5,\ldots\}$, then

$$I_n f \in k-\mathcal{UCV}.$$

Our aim in this paper is to find coefficient bounds and coefficient inequalities for the class $\mathcal{M}(n,k)$.

In the present investigation we also need the following definitions and notations, for the presentation of our results.

For arbitrary chosen $k \in [0,\infty)$ let Ω_k denote the domain

$$\Omega_k = \{u+iv : u^2 > k^2(u-1)^2 + k^2v^2, u > 0\}. \tag{1.9}$$

Note that $1 \in \Omega_k$ for all k and each Ω_k is convex and symmetric in the real axis. Ω_0 is nothing but the right half-plane and when $0 < k < 1$, Ω_k is an unbounded domain contained in the right branch of a hyperbola. When $k = 1$, the domain Ω_1 is still unbounded domain enclosed by the parabola $v^2 = 2u-1$. When $k > 1$, the domain Ω_k becomes bounded domain being the interior of a ellipse. Note also that for no choice of parameter k, Ω_k reduces to a disk.

Under the above notations we may rewrite the Definition 3, as follows

$$f \in k-\mathcal{UCV} \Leftrightarrow f \in \mathcal{A} \text{ and } 1 + \frac{zf''(z)}{f'(z)} \in \Omega_k \ \ (z \in \Delta). \tag{1.10}$$

Let $\mathcal{P}$ denote the class of Caratheodory functions, e.g.

$$\mathcal{P} = \{p : p \text{ analytic in } \Delta, p(0) = 1, \Re\{p(z)\} > 0\}, \tag{1.11}$$

and let p_k denote a conformal mapping of Δ onto Ω_k determined by conditions $p_k(0) = 1$, $\Re\{p_k'(0)\} > 0$. Then we have

$$p_1(z) = 1 + \frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, \ \ z \in \Delta, \tag{1.12}$$

and if $0 \le k < 1$, then

$$p_k(z) = \frac{1}{1-k^2}\cosh\left\{\left(\frac{2}{\pi}\arccos k\right)\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right\} - \frac{k^2}{1-k^2}, \ \ z \in \Delta, \tag{1.13}$$

moreover, if $k > 1$, then

$$p_k(z) = \frac{1}{k^2-1}\sin\left(\frac{\pi}{2K(\kappa)}\int_0^{\frac{u(z)}{\sqrt{\kappa}}}\frac{\mathrm{d}t}{\sqrt{1-t^2}\sqrt{1-\kappa^2t^2}}\right)+\frac{k^2}{k^2-1},\quad z\in\Delta, \tag{1.14}$$

where

$$u(z) = \frac{z-\sqrt{\kappa}}{1-\sqrt{\kappa}z},\quad z\in\Delta,$$

and $\kappa \in (0,1)$ is chosen such that $k = \cosh(\pi K'(\kappa)/(4K(\kappa)))$. Here $K(\kappa)$ is Legendre's complete elliptic integral of first kind and $K'(\kappa) = K(\sqrt{1-\kappa^2})$. For more details about p_k see [4-8].

If $f, g \in \mathcal{H}$, then the function f is said to be subordinate to g, written as $f(z) \prec g(z)$ $(z \in \Delta)$, if there exists a Schwarz function $w \in \mathcal{H}$ with $w(0) = 0$ and $|w(z)| < 1$, $z \in \Delta$ such that $f(z) = g(w(z))$. In particular, if g is univalent in Δ, then we have the following equivalence:

$$f(z) \prec g(z) \iff f(0) = g(0) \text{ and } f(\Delta) \subset g(\Delta). \tag{1.15}$$

In terms of subordination we can write

$$f \in \mathcal{M}(n,k) \Leftrightarrow \left[f \in \mathcal{A} \text{ and } \frac{z[I_nf(z)]'}{I_nf(z)} \prec p_k(z) \quad (z\in\Delta)\right]. \tag{1.16}$$

2. Preliminary lemmas

We need the following results in our investigation:

Lemma A.[7] *Let $k \in [0,\infty)$, be fixed and p_k be the Riemann map of Δ on to Ω_k, satisfying $p_k(0) = 1$, $\Re\{p_k'(0)\} > 0$. If $p_k(z) = 1 + Q_1(k)z + Q_2(k)z^2 + \ldots$, $(z \in \Delta)$, then*

$$Q_1(k) = \begin{cases} 2 & for\ k = 0, \\ \frac{2A^2}{1-k^2} & for\ k\in(0,1), \\ \frac{8}{\pi^2} & for\ k = 1, \\ \frac{\pi^2}{4(k^2-1)K^2(\kappa)(1+\kappa)\sqrt{\kappa}} & for\ k > 1, \end{cases} \tag{2.1}$$

where $A = (2/\pi)\arccos k$ while κ and $K(\kappa)$ are the same as in (1.14).

Lemma B.[17] *Let*

$$h(z) = 1+\sum_{n=1}^{\infty}c_nz^n \prec 1+\sum_{n=1}^{\infty}C_nz^n = H(z)\ (z\in\Delta). \tag{2.2}$$

If the function H is univalent in Δ and $H(\Delta)$ is a convex set, then

$$|c_n| \le |C_1|. \tag{2.3}$$

Lemma C.[21] *If $f \in \mathcal{CV}$, $g \in \mathcal{S}^*$, then for each analytic function h in Δ,*

$$\frac{(f * hg)(\Delta)}{(f * g)(\Delta)} \subset \overline{co} h(\Delta), \tag{2.4}$$

where $\overline{co} h(\Delta)$ denotes the closed convex hull of $h(\Delta)$.

Lemma D. *Let $0 < \alpha \leq \beta$. If $\beta \geq 2$ or if $\alpha + \beta \geq 3$, then the function*

$$h(z) = \sum_{m=0}^{\infty} \frac{(\alpha)_m}{(\beta)_m} z^{m+1} \; (z \in \Delta) \tag{2.5}$$

belongs to the class $\mathcal{CV}$ of convex functions.

Lemma D is a special case of Theorem 2.12 or Theorem 2.13 contained in [19].

3. Main results

Theorem 1. *Let f be in the class $\mathcal{M}(n,k)$. If f is of the form (1.1), then*

$$|a_2| \leq \frac{Q_1(k)(n+1)}{2} \tag{3.1}$$

and

$$|a_m| \leq \frac{(n+1)_{m-1}\, Q_1(k)}{(m-1)(2)_{m-1}} \prod_{s=3}^{m} \left(1 + \frac{Q_1(k)}{s-2}\right) (m \geqq 3), \tag{3.2}$$

where $Q_1(k)$ is described in (2.1).

Proof. Let f given by (1.1), belong to $\mathcal{M}(n,k)$, also let $I_n f(z) = z + \sum_{m=2}^{\infty} b_m z^m = F(z)$, where

$$b_m = \frac{(2)_{m-1}}{(n+1)_{m-1}} a_m \tag{3.3}$$

and define

$$\phi(z) = \frac{zF'(z)}{F(z)} = 1 + \sum_{m=1}^{\infty} c_m z^m.$$

Then $\phi \prec p_k$, where p_k is the function given by (1.12), (1.13) and (1.14) depending on k. The function p_k is univalent in Δ and $p_k(\Delta) = \Omega_k$ which is convex region (see (1.9)). Using Rogosinski's Lemma B and (2.1) of Lemma A, we have $|c_m| \leq Q_1$. Now, writing $zF'(z) = \phi(z)F(z)$ and comparing the coefficients of z^n on both sides, we get

$$(m-1)b_m = \sum_{k=1}^{m-1} c_{m-k} b_k.$$

From this we get $|b_2| = |c_1| \leq Q_1$, which in view of (3.3) gives (3.1). If we choose f to be that function for which $\frac{zF'(z)}{F(z)} = p_k(z)$, then f is a function in $\mathcal{M}(n,k)$ with $a_2 = Q_1(n+1)/2$, which shows that this result is sharp. Further

$$|b_3| \leq \frac{1}{2}|c_2 + c_1 b_2| \leq \frac{1}{2}(|c_2| + |c_1||b_2|) \leq \frac{1}{2} Q_1(1 + Q_1).$$

We now proceed by induction. Assume that

$$|b_k| \leq \frac{Q_1}{k-1}(1+Q_1)(1+Q_1/2)\ldots(1+Q_1/(m-2)), \text{ for } k = 3, 4, \ldots, m-1.$$

Then

$$\begin{aligned}(m-1)|b_m| &\leq \sum_{k=1}^{m-1} |c_{m-k}||b_k| \leq Q_1 \sum_{k=1}^{m-1} |b_k| \\ &\leq Q_1 \left(1+Q_1+\frac{Q_1}{2}(1+Q_1)+\frac{Q_1}{3}(1+Q_1)(1+\frac{Q_1}{2})+\ldots \right.\\ &\left. + \frac{Q_1}{m-2}(1+Q_1)(1+Q_1/2)\ldots\left(1+\frac{Q_1}{m-3}\right)\right) \\ &= Q_1(1+Q_1)(1+Q_1/2)\ldots\left(1+\frac{Q_1}{m-2}\right),\end{aligned}$$

and hence

$$|b_m| \leq \frac{Q_1}{(m-1)} \prod_{s=3}^{m} \left(1+\frac{Q_1}{s-2}\right) \quad (m \geqq 3).$$

Putting the value of b_m from (3.3) we get the desired result. □

Theorem 2. *The function $k(z) = z/(1-Az)^2$ is in $\mathcal{M}(1,k) = k-\mathcal{ST}$ if and only if*

$$|A| \leq \frac{1}{2k+1}. \tag{3.4}$$

Proof. Using Definition 4, $k(z) \in k-\mathcal{ST}$ if and only if

$$k\left|\frac{2Az}{1-Az}\right| < \Re\left(\frac{1+Az}{1-Az}\right) \ (z \in \Delta).$$

It is suffices to study above for $|z| = 1$. Setting $|A| = r$ and $Az = re^{i\phi}$ in above, we have

$$k\left|\frac{2re^{i\phi}}{1-re^{i\phi}}\right| \leq \Re\left(\frac{1+re^{i\phi}}{1-re^{i\phi}}\right). \tag{3.5}$$

On simplification, we see that

$$\Re\left(\frac{1+re^{i\phi}}{1-re^{i\phi}}\right) = \frac{1-r^2}{|1-re^{i\phi}|^2}.$$

So (3.5) is equivalent to

$$2kr \leq \frac{1-r^2}{[1-2r\cos\phi+r^2]^{1/2}}. \tag{3.6}$$

The right-hand side of (3.6) is seen to have a minimum for $\phi = \pi$, and this minimal value is $1-r$. Hence, a necessary and sufficient condition for (3.6) is $2rk \leq 1-r$ or $|A| = r \leq 1/2k+1$. □

Remark 1. If $A = 1$, then $k(z)$ is a Koebe function and (3.4) forces $k = 0$, i.e. Koebe function belongs to class $k-\mathcal{ST}$ if and only if $k = 0$.

Theorem 3. *The function $f(z) = z + a_m z^m$ is in $\mathcal{M}(n,k)$ if and only if*

$$|a_m| \le \frac{(n+1)_{m-1}}{(2)_{m-1}\,(mk+m-k)}\ (m \geqq 2).$$

Proof. Let $I_n f(z) = z + b_m z^m = F(z)$, where b_m is given by (3.3). It is sufficient to study (1.8) for $|z| = 1$. Setting $|b_m| = r$ and $b_m z^{m-1} = re^{i\phi}$. Then (1.8) for this f will be

$$k\left|\frac{(m-1)re^{i\phi}}{1-re^{i\phi}}\right| \le \Re\left(\frac{1+mre^{i\phi}}{1-re^{i\phi}}\right).$$

Following the same steps as in Theorem 2, we get desired result. □

Remark 2. For particular values of $m,\ n,\ k$, Theorem 3, provides functions belonging to the class $\mathcal{M}(n,k)$. For example, if $m = 2,\ n = 1,\ k = 1$ then $|a_2| \le 1/3$. So, if we take $f(z) = z + z^2/3$, then $f \in \mathcal{PST}$.

Remark 3. Putting $n = 1$ and $k = 1$ in Theorem 1, 2 and 3 we get the Theorem 5, 3 and 2 of Rønning [18] respectively.

Theorem 4. *Assume that $n_1 \le n_2,\ n_1, n_2 \in \mathbb{N} \cup \{0\}$. Then*

$$\mathcal{M}(n_1,k) \subset \mathcal{M}(n_2,k) \tag{3.7}$$

for all $k \in [0,\infty)$.

Proof. Let $f \in \mathcal{M}(n_1,k)$. By the definition of the class $\mathcal{M}(n_1,k)$ we have

$$\frac{z[I_{n_1} f(z)]'}{I_{n_1} f(z)} = p_k\{\omega(z)\} \quad (z \in \Delta), \tag{3.8}$$

where p_k is convex univalent with $p_k(\Delta) = \Omega_k$ and $|\omega(z)| < 1$ in Δ with $\omega(0) = 0 = p_k(0) - 1$. Let us denote

$$f_{n_1,n_2}(z) = \sum_{m=0}^{\infty} \frac{(n_1+1)_m}{(n_2+1)_m} z^{m+1} \quad (z \in \Delta). \tag{3.9}$$

Then we have

$$f^{\dagger}_{n_2}(z) = f^{\dagger}_{n_1}(z) * f_{n_1,n_2}(z). \tag{3.10}$$

Applying (1.2), (3.8), (3.10) and the properties of convolution we get

$$\frac{z\left[I_{n_2} f(z)\right]'}{I_{n_2} f(z)} = \frac{z(f^{\dagger}_{n_2} * f)'(z)}{(f^{\dagger}_{n_2} * f)(z)} = \frac{z\left(f^{\dagger}_{n_1} * f_{n_1,n_2} * f\right)'(z)}{(f^{\dagger}_{n_1} * f_{n_1,n_2} * f)(z)}$$

$$= \frac{f_{n_1,n_2}(z) * z\left[I_{n_1} f(z)\right]'}{f_{n_1,n_2}(z) * I_{n_1} f(z)}$$

$$= \frac{f_{n_1,n_2}(z) * p_k[\omega(z)] I_{n_1} f(z)}{f_{n_1,n_2}(z) * I_{n_1} f(z)}. \tag{3.11}$$

Moreover, it follows from (3.8) that $I_{n_1} f \in k - \mathcal{ST} \subset \mathcal{S}^*$ and it follows from Lemma D that $f_{n_1,n_2} \in \mathcal{CV}$. Then using Lemma C to (3.11), we obtain

$$\frac{f_{n_1,n_2} * p_k(\omega) I_{n_1} f}{f_{n_1,n_2} * I_{n_1} f}(\Delta) \subset \overline{co}\, p_k[\omega(\Delta)] \subset p_k(\Delta), \tag{3.12}$$

because p_k is convex univalent. By (1.15) the function (3.11) is subordinated to p_k, and so $f \in \mathcal{M}(n_2, k)$. □

Corollary 1. *The following relations are satisfied*

$$k - \mathcal{ST} = \mathcal{M}(1,k) \subset \mathcal{M}(n,k),$$

for all $k \in [0,\infty)$ *and for all* $n \in \mathbb{N}$.

Theorem 5. *Assume that* $0 \le k_1 \le k_2 < \infty$. *Then*

$$\mathcal{M}(n,k_2) \subset \mathcal{M}(n,k_1) \tag{3.13}$$

for all $n \in \mathbb{N} \cup \{0\}$.

Proof. Let $f \in \mathcal{M}(n,k_2)$. By the definition of the class $\mathcal{M}(n,k_2)$ we have

$$\frac{z[I_n f(z)]'}{I_n f(z)} = p_{k_2}\{\omega(z)\} \subset p_{k_1}\{\omega(z)\} \quad (z \in \Delta), \tag{3.14}$$

because p_{k_i}, $i = 1,2$, are convex univalent with $p_{k_2} \prec p_{k_1}$. Therefore, $f \in \mathcal{M}(n,k_1)$. □

Corollary 2. *The following inclusion relations are satisfied*

$$\mathcal{M}(n,k) \subset \mathcal{M}(n,0) \supset \mathcal{M}(1,0) = \mathcal{S}^*,$$

for all $k \in [0,\infty)$ *and for all* $n \in \mathbb{N} \cup \{0\}$.

Proof. The first relation is a simple consequence of Theorem 5 while the second one of Theorem 4. □

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