A UNIQUENESS THEOREM FOR STURM-LIOUVILLE OPERATORS WITH EIGENPARAMETER DEPENDENT BOUNDARY CONDITIONS

YU PING WANG

Abstract. In this paper, we discuss the inverse problem for Sturm-Liouville operators with boundary conditions having fractional linear function of spectral parameter on the finite interval [0, 1]. Using Weyl m-function techniques, we establish a uniqueness theorem. i.e., If \( q(x) \) is prescribed on \( [0, \frac{1}{2} + \frac{\alpha}{2}] \) for some \( \alpha \in [0, 1) \), then the potential \( q(x) \) on the interval \( [0, 1] \) and fractional linear function \( \frac{a_2 \lambda + b_2}{c_2 \lambda + d_2} \) of the boundary condition are uniquely determined by a subset \( S \subseteq \sigma(L) \) and fractional linear function \( \frac{a_1 \lambda + b_1}{c_1 \lambda + d_1} \) of the boundary condition.

1. Introduction

Consider the following Sturm-Liouville operator \( L \) defined by

\[
Ly = -y'' + q(x)y = \lambda y \quad (x \in [0, 1])
\]  

(1.1)

with boundary conditions

\[
y'(0, \lambda) - hy(0, \lambda) = 0 \quad (1.2)
\]

or

\[
(a_1 \lambda + b_1)y(0, \lambda) - (c_1 \lambda + d_1)y'(0, \lambda) = 0 \quad (1.2')
\]

and

\[
y'(1, \lambda) + Hy(1, \lambda) = 0, \quad (1.3)
\]

or

\[
(a_2 \lambda + b_2)y(1, \lambda) - (c_2 \lambda + d_2)y'(1, \lambda) = 0, \quad (1.3')
\]

respectively, where \( h, H, a_k, b_k, c_k, d_k \in \mathbb{R}, c_1 c_2 \neq 0 \) such that

\[
(-1)^k \delta_k = a_k d_k - b_k c_k > 0 \quad (k = 1, 2),
\]

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Let \( \sigma = \{ \lambda_n \}_{n \in \mathbb{N}_0} \) be the spectrum of the Sturm-Liouville problem \( B(q, h, H) \). If \( q(x) \) is prescribed on \([0, \frac{1}{2} + \frac{\alpha}{2}]\) for some \( \alpha \in [0, 1) \), then the potential \( q(x) \) is real-valued function and \( q \in L^2[0, 1] \).

For convenience, we denote the Sturm-Liouville problem (1.1)-(1.3), the Sturm-Liouville problem (1.1), (1.2'), (1.3') and the Sturm-Liouville problem (1.1), (1.2), (1.3') by \( B(q, h, H) \), \( B(q, \delta_1, \delta_2) \) and \( B(q, h, \delta_2) \), respectively.

Binding, Browne and Seddighi [1] discussed the spectral theory for the Sturm-Liouville problem \( B(q, \delta_1, \delta_2) \), obtained oscillation, comparison results and asymptotic estimates of the Sturm-Liouville problem \( B(q, \delta_1, \delta_2) \) and promoted Fulton’s results [2]. Using nodal points as spectral data, Browne and Sleeman [3] considered the inverse nodal problem for Sturm-Liouville problem \( B(q, h, \delta_2) \). Guliyev [4] found a trace formula for the Sturm-Liouville problem \( B(q, \delta_1, \delta_2) \). Wang [5] considered the inverse problem for the Sturm-Liouville problem \( B(q, \delta_1, \delta_2) \) and showed that the potential \( q(x) \) and fractional linear functions \( \frac{a_1 \lambda + b_1}{c_1 \lambda + d_1} \) and \( \frac{a_2 \lambda + b_2}{c_2 \lambda + d_2} \) of the boundary conditions can be uniquely determined by a set of values of eigenfunctions at some interior point and parts of two spectra. By Weyl function techniques, Freiling and Yurko [6] discussed the inverse problems for Sturm-Liouville equations with boundary conditions polynomially dependent on the spectral parameter and provided a constructive solution of the inverse spectral problems. Sturm-Liouville problems with eigenparameter dependent boundary conditions have many applications in physics, engineering, mathematics, etc (see [1-8]).

Half inverse problem for Sturm-Liouville operators consists of reconstruction of this operator by its spectrum and half of the potential. In 1978, Hochstadt and Lieberman [9] considered the half inverse problem for the Sturm-Liouville problem \( B(q, h, H) \) and showed if \( q(x) \) is prescribed on \([\frac{2}{7}, \pi]\), then the potential \( q(x) \) on the interval \([0, \frac{2}{7}]\) for the Sturm-Liouville operator on the finite interval \([0, \pi]\) can be determined by one spectrum. Castillo [11] discussed the Sturm-Liouville problem \( B(q, h, H) \). By an example, Suzuki showed that if \( q_1(x) = q_2(x) \) on the interval \([0, \frac{1}{2} - \varepsilon]\) for some \( 0 < \varepsilon < \frac{1}{2} \) and \( \sigma(L_1) = \sigma(L_2) \), but \( q_1(x) \neq q_2(x) \) on the interval \([0, 1]\), where \( \sigma(L_i) = \{ \lambda_{i,n} \} \) is the spectrum of \( L_i \) of the corresponding Sturm-Liouville problem for the potential \( q_i(i = 1, 2) \). One of this kind of half inverse problems for differential operators on the finite interval was considered by a number of authors (see [9-16]). Using the Weyl function techniques, Gesztesy and Simon [17] discussed the inverse problem for the Sturm-Liouville problem \( B(q, h, H) \) from three spectra. Gesztesy and Simon [18] discussed the inverse problem for the Sturm-Liouville problem \( B(q, h, H) \) from partial information on the potential and partial spectrum and established the following remarkable uniqueness theorem for the Sturm-Liouville problem \( B(q, h, H) \), which is a generalization of Hochstadt-Lieberman theorem [9].

Theorem 1.1 ([18], Theorem 1.3). Let \( \sigma(L) = \{ \lambda_{i,n} \}_{n \in \mathbb{N}_0} \) be the spectrum of the Sturm-Liouville problem \( B(q, h, H) \). If \( q(x) \) is prescribed on \([0, \frac{1}{2} + \frac{\alpha}{2}]\) for some \( \alpha \in [0, 1) \), then the potential \( q(x) \)
a.e. on the whole interval $[0, 1]$ and coefficient $H$ of the boundary condition are uniquely determined by coefficient $h$ of the boundary condition and a subset $S \subseteq \sigma(L)$ satisfying

$$
\# \{ \lambda \in S | \lambda \leq t \} \geq (1 - a)\# \{ \lambda \in \sigma(L) | \lambda \leq t \} + \frac{a}{2},
$$

(1.4)

for all sufficiently large $t \in \mathbb{R}^+$, where $N_0 := \{ n | n = 0, 1, 2, \cdots \}$.

In this paper, we consider the inverse problem for the Sturm-Liouville problem $B(q, \delta_1, \delta_2)$ from partial information on the potential and partial spectrum. We establish a Gesztesy-Simon theorem for the Sturm-Liouville problem $B(q, \delta_1, \delta_2)$. Using Freiling and Yurko’s result, we show that if $q(x)$ is prescribed on $[0, \frac{1}{2} + \frac{\alpha}{2}]$ for some $\alpha \in (0, 1)$, then the potential $q(x)$ a.e. on the whole interval $[0, 1]$ and fractional linear function $\frac{a_2 \lambda + b_2}{c_2 \lambda + d_2}$ of the boundary condition are uniquely determined by fractional linear function $\frac{a_1 \lambda + b_1}{c_1 \lambda + d_1}$ of the boundary condition and a subset $S \subseteq \sigma(L)$ satisfying (2.4) (see below). Although the techniques used here is based on the Gesztesy-Simon method, the Sturm-Liouville problem $B(q, \delta_1, \delta_2)$ in this paper is different from the Sturm-Liouville problem $B(q, h, H)$ in [18].

From [1] and [6], we have

**Lemma 1.2** ([1], [6]). Let $(\lambda_n)_{n=0}^{\infty}$ be spectrum of the Sturm-Liouville problem $B(q, \delta_1, \delta_2)$, then $\lambda_n (n \in N_0)$ is real and simple and $\lambda_n$ is root of (1.2') and satisfies

$$
\lambda_0 < \lambda_1 < \lambda_2 < \cdots \rightarrow +\infty
$$

and

$$
\sqrt{\lambda_n} = (n - 2)\pi + \frac{\omega}{n\pi} + \frac{\kappa_n}{n},
$$

(1.5)

where $\omega = \frac{1}{2} \int_0^\pi q(x) dx - \frac{a_1}{c_1} + \frac{a_2}{c_2}$ and $\{ \kappa_n \} \in l^2$.

Suppose $\varphi(x, \lambda), \theta(x, \lambda)$ are the two fundamental solutions of the equation (1.1) and satisfy

$$
\varphi(1, \lambda) = 1, \varphi'(1, \lambda) = 0, \theta(1, \lambda) = 0 \text{ and } \theta'(1, \lambda) = 1,
$$

then the solution of the equation (1.1) satisfying $y(1, \Lambda) = c_2 \lambda + d_2$ and $y'(1, \Lambda) = a_2 \lambda + b_2$ is

$$
y(x, \Lambda) = (c_2 \lambda + d_2) \varphi(x, \Lambda) + (a_2 \lambda + b_2) \theta(x, \Lambda).
$$

By virtue of [26-28], for sufficiently large $|\lambda|$, this yields

$$
\varphi(x, \Lambda) = \cos \sqrt{\lambda}(1 - x) + O(e^{lim \sqrt{\lambda}(1-x)}),
$$

(1.6)

$$
\varphi'(x, \Lambda) = \sqrt{\lambda} \sin \sqrt{\lambda}(1 - x) + O(e^{lim \sqrt{\lambda}(1-x)}),
$$

(1.7)
\[ \theta(x, \lambda) = \frac{\sin \sqrt{\lambda}(1-x)}{\sqrt{\lambda}} + O\left(\frac{e^{\sqrt{\lambda}(1-x)}}{\sqrt{\lambda}}\right) \] 

and

\[ \theta'(x, \lambda) = -\cos \sqrt{\lambda}(1-x) + O\left(\frac{e^{\sqrt{\lambda}(1-x)}}{\sqrt{\lambda}}\right). \] 

Hence, we obtain the following asymptotic formulae

\[ y(x, \lambda) = c_2 \lambda \cos \sqrt{\lambda}(1-x) + O\left(\frac{e^{\sqrt{\lambda}(1-x)}}{\lambda}\right) \] 

and

\[ y'(x, \lambda) = c_2 \lambda^2 \sin \sqrt{\lambda}(1-x) + O\left(\frac{e^{\sqrt{\lambda}(1-x)}}{\lambda}\right). \] 

Define the Weyl \( m \)-function \( m(x, \lambda) \) by

\[ m(x, \lambda) = \frac{y'(x, \lambda)}{y(x, \lambda)}. \] 

Let

\[ M(\lambda) := \frac{y'(0, \lambda)}{(c_1 \lambda + d_1) y'(0, \lambda) - (a_1 \lambda + b_1) y(0, \lambda)}, \]

which is called the Weyl function of the Sturm-Liouville problem \( B(q, \delta_1, \delta_2) \).

In virtue of [6], we present the following lemma, which is important for us to show the main theorem in this paper.

**Lemma 1.3 ([6]).** Let \( M(\lambda) \) be the Weyl function of the Sturm-Liouville problem \( B(q, \delta_1, \delta_2) \). Then \( M(\lambda) \) uniquely determines fractional linear function \( \frac{a_j \lambda + b_j}{c_j \lambda + d_j} \) of the boundary condition as well as \( q(x) \) (a.e.) on \([0, 1]\).

**2. Main results and Proofs**

Consider the following Sturm-Liouville operator \( L_j(j = 1, 2) \) satisfying

\[ L_j u = -u'' + q_j(x) u = \lambda u(x \in [0, 1]) \] 

with boundary conditions

\[ (a_1 \lambda + b_1) u(0, \lambda) - (c_1 \lambda + d_1) u'(0, \lambda) = 0 \]

and

\[ (a_j \lambda + b_j) u(1, \lambda) - (c_j \lambda + d_j) u'(1, \lambda) = 0, \]

respectively, where \( a_1, b_1, c_1, d_1, a_j, b_j, c_j, d_j \in \mathbb{R}, c_1 c_{12} c_{22} \neq 0 \) such that

\[ \delta_1 = a_1 d_1 - b_1 c_1 < 0, \quad \delta_{j2} = a_j d_{j2} - b_j c_{j2} > 0, \]

\( q_1, q_2 \) are real-valued functions and \( q_1, q_2 \in L^2[0, 1] \).

We have the following uniqueness theorem for the boundary-value problem \( B(q, \delta_1, \delta_{j2}) \).
Theorem 2.4. Let $\sigma(L_j) = \{\lambda_{jn}\}(j = 1, 2, n \in \mathbb{N}_0)$ be the spectrum of the boundary-value problem $B(q, \delta_1, \delta_2)$. For some $\alpha \in [0, 1)$ and sufficiently small $\epsilon (\epsilon > 0)$, if

$$q_1(x) = q_2(x), x \in [0, \frac{1}{2} + \frac{\alpha}{2}]$$

and $S \subseteq \sigma(L_1) \cap \sigma(L_2)$ satisfying

$$\#\{\lambda \in S| \lambda \leq t\} \geq (1 - \alpha)\#\{\lambda \in \sigma(L_1)| \lambda \leq t\} + \frac{5\alpha}{2} - \frac{1}{2} + \epsilon,$$  \hspace{1cm} (2.4)

for all sufficiently large $t \in \mathbb{R}^+$, then

$$q_1(x) = q_2(x) \text{ a.e. on } [0, 1]$$

and

$$\frac{a_{12}\lambda + b_{12}}{c_{12}\lambda + d_{12}} = \frac{a_{22}\lambda + b_{22}}{c_{22}\lambda + d_{22}}.$$  

Remark. When $\alpha = 0$, we prove the half inverse problem for the Sturm-Liouville operator with boundary conditions having fractional linear function of spectral parameter on the finite interval $[0, 1]$.

Proof. Let $u_j(x, t)(j = 1, 2)$ be the solution of the equation (2.1) satisfying $u_j(1, t) = c_j \lambda + d_j$ and $u'_j(1, t) = a_j \lambda + b_j$.

Denote

$$F(\lambda) = u_1\left(\frac{1}{2} + \frac{\alpha}{2}, \lambda\right)u'_2\left(\frac{1}{2} + \frac{\alpha}{2}, \lambda\right) - u'_1\left(\frac{1}{2} + \frac{\alpha}{2}, \lambda\right)u_2\left(\frac{1}{2} + \frac{\alpha}{2}, \lambda\right)$$

$$= u'_1\left(\frac{1}{2} + \frac{\alpha}{2}, \lambda\right)u'_2\left(\frac{1}{2} + \frac{\alpha}{2}, \lambda\right)m^{-1}\left(\frac{1}{2} + \frac{\alpha}{2}, \lambda\right) - m^{-1}\left(\frac{1}{2} + \frac{\alpha}{2}, \lambda\right)$$

and

$$\omega(\lambda) = (a_1 \lambda + b_1)u(0, \lambda) - (c_1 \lambda + d_1)u'(0, \lambda).$$

Then

$$F(\lambda_n) = 0, \forall \lambda_n \in S \subseteq \sigma(L_1) \cap \sigma(L_2)$$

From (1.10) and (1.11), we obtain

$$|F(\lambda)| = O(\lambda^2 e^{l\sqrt{\lambda}(1-\alpha)})$$

and

$$|\omega(\lambda)| = O(\lambda^\frac{5}{2} e^{l\sqrt{\lambda}}).$$

Define the function $G(\lambda)$ by

$$G(\lambda) = \prod_{\lambda_n \in S} \left(1 - \frac{\lambda}{\lambda_n}\right).$$

(2.9)
and
\[ \psi(\lambda) = \frac{F(\lambda)}{G(\lambda)}. \]  

(2.10)

Then, \( \psi(\lambda) \) is an entire function. For convenience, we denote
\[ N_G(t) = \sharp \{ \lambda_n \in S | \lambda_n \leq t \}, \quad N_\omega(t) = \sharp \{ \lambda_n \in \sigma(L) | \lambda_n \leq t \}. \]

By virtue of (2.4), this yields
\[ N_G(t) \geq (1 - \alpha) N_\omega(t) + \frac{5\alpha - 1}{2} + \varepsilon. \]  

(2.11)

Since \( \omega(\lambda) \) is an entire function in \( \lambda \) of order \( \frac{1}{2} \), there exists a positive constant \( c \) such that
\[ N_G(t) \leq N_\omega(t) \leq c t^{\frac{1}{2}}, \]  

(2.12)

Without loss of generality, let us assume \( \lambda_n > 1(n \in \mathbb{N}_0) \), then \( N_G(1) = N_\omega(1) = 0 \). For a fixed \( x \, (x \in \mathbb{R}) \) and \( |x| \) sufficiently large, we have
\[
\ln |G(ix)| = \frac{1}{2} \ln G(ix) G(ix) = \frac{1}{2} \sum_{\lambda_n \in S} \ln(1 - \frac{ix}{\lambda_n})(1 + \frac{ix}{\lambda_n})
\]
\[
= \frac{1}{2} \sum_{\lambda_n \in S} \ln(1 + \frac{x^2}{\lambda_n^2}) = \frac{1}{2} \int_1^\infty \ln(1 + \frac{x^2}{t^2}) dN_G(t)
\]
\[
= \frac{1}{2} \ln(1 + \frac{x^2}{t^2}) N_G(t) \big|_1^\infty - \frac{1}{2} \int_1^\infty N_G(t) d[\ln(1 + \frac{x^2}{t^2})].
\]

For sufficiently large \( t \), since
\[ \ln(1 + \frac{x^2}{t^2}) = O\left(\frac{1}{t^2}\right), \]
then
\[ \lim_{n \to \infty} \ln(1 + \frac{x^2}{t^2}) N_G(t) = 0 \]
and
\[ \lim_{n \to \infty} \ln(1 + \frac{x^2}{t^2}) N_\omega(t) = 0. \]

Therefore
\[
\ln |G(ix)| = \int_1^\infty \frac{x^2}{t^2 + tx^2} N_G(t) dt
\]
\[
\geq (1 - \alpha) \int_1^\infty \frac{x^2}{t^2 + tx^2} N_\omega(t) dt + \frac{5\alpha - 1}{2} + \varepsilon \int_1^\infty \frac{x^2}{t^2 + tx^2} dt
\]
\[
= (1 - \alpha) \ln |\omega(ix)| + \frac{1}{2} \left(\frac{5\alpha - 1}{2} + \varepsilon\right) \ln(1 + x^2). \]  

(2.13)

By virtue of (2.13), this yields
\[ |G(ix)| \geq |\omega(ix)|^{1 - \alpha} (1 + x^2)^{\frac{5\alpha - 1}{4} + \frac{\varepsilon}{2}}. \]  

(2.14)
From (2.10) and (2.14), we have

\[ |\psi(ix)| = \left| \frac{F(ix)}{G(ix)} \right| = O\left( \frac{1}{|x|^\varepsilon} \right) (\varepsilon > 0) \] (2.15)

for \(|x|\) sufficiently large.

By the Phragmén-Lindelöf theorem, we get

\[ \psi(\lambda) = 0, \ \forall \lambda \in \mathbb{C}. \] (2.16)

i.e,

\[ F(\lambda) = 0, \ \forall \lambda \in \mathbb{C}. \] (2.17)

Therefore,

\[ u_1'(\frac{1}{2} + \frac{\alpha}{2}, \lambda) u_2'(\frac{1}{2} + \frac{\alpha}{2}, \lambda) - u_2'(\frac{1}{2} + \frac{\alpha}{2}, \lambda) u_1'(\frac{1}{2} + \frac{\alpha}{2}, \lambda) = 0, \ \forall \lambda \in \mathbb{C}. \] (2.18)

In virtue of \(q_1(x) = q_2(x)\) on \([0, \frac{1}{2} + \frac{\alpha}{2}]\) together with (2.18), we obtain

\[ u_1(0, \lambda) u_2'(0, \lambda) - u_2'(0, \lambda) u_1(0, \lambda) = 0, \ \forall \lambda \in \mathbb{C}. \] (2.19)

From (2.19), we have

\[ M_1(\lambda) = M_2(\lambda), \ \forall \lambda \in \mathbb{C}. \] (2.20)

By virtue of Lemma 1.3 together with (2.20), we obtain

\[ q_1(x) = q_2(x) \text{ a.e. on } [0, 1] \text{ and } \frac{a_{12}\lambda + b_{12}}{c_{12}\lambda + d_{12}} = \frac{a_{22}\lambda + b_{22}}{c_{22}\lambda + d_{22}}. \]

This completes the proof of Theorem 2.1.

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\textbf{References}


Department of Applied Mathematics, Nanjing Forestry University, Nanjing, 210037, Jiangsu, People’s Republic of China.

E-mail: ypwang@njfu.com.cn