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A UNIQUENESS THEOREM FOR STURM-LIOUVILLE OPERATORS WITH EIGENPARAMETER DEPENDENT BOUNDARY CONDITIONS

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Abstract. In this paper, we discuss the inverse problem for Sturm-Liouville operators with boundary conditions having fractional linear function of spectral parameter on the finite interval $[0,1]$. Using Weyl m-function techniques, we establish a uniqueness theorem. i.e., If $q(x)$ is prescribed on $[0,\frac{1}{2}+\frac{\alpha}{2}]$ for some $\alpha\in[0,1)$, then the potential $q(x)$ on the interval $[0,1]$ and fractional linear function $\frac{a_2\lambda+b_2}{c_2\lambda+d_2}$ of the boundary condition are uniquely determined by a subset $S\subseteq\sigma(L)$ and fractional linear function $\frac{a_1\lambda+b_1}{c_1\lambda+d_1}$ of the boundary condition.

1. Introduction

Consider the following Sturm-Liouville operator L defined by

$$Ly=-y''+q(x)y=\lambda y(x\in[0,1]) \tag{1.1}$$

with boundary conditions

$$y'(0,\lambda)-hy(0,\lambda)=0 \tag{1.2}$$

or

$$(a_1\lambda+b_1)y(0,\lambda)-(c_1\lambda+d_1)y'(0,\lambda)=0 \tag{1.2$'$}$$

and

$$y'(1,\lambda)+Hy(1,\lambda)=0, \tag{1.3}$$

or

$$(a_2\lambda+b_2)y(1,\lambda)-(c_2\lambda+d_2)y'(1,\lambda)=0, \tag{1.3$'$}$$

respectively, where $h,H,a_k,b_k,c_k,d_k\in\mathbf{R}$, $c_1c_2\neq0$ such that

$$(-1)^k\delta_k=a_kd_k-b_kc_k>0 \qquad (k=1,2),$$

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q is real-valued function and $q \in L^2[0,1]$.

For convenience, we denote the Sturm-Liouville problem (1.1)-(1.3), the Sturm-Liouville problem (1.1), (1.2′), (1.3′) and the Sturm-Liouville problem (1.1), (1.2), (1.3′) by $B(q,h,H)$, $B(q,\delta_1,\delta_2)$ and $B(q,h,\delta_2)$, respectively.

Binding, Browne and Seddighi [1] discussed the spectral theory for the Sturm-Liouville problem $B(q,\delta_1,\delta_2)$, obtained oscillation, comparison results and asymptotic estimates of the Sturm-Liouville problem $B(q,\delta_1,\delta_2)$ and promoted Fulton's results [2]. Using nodal points as spectral data, Browne and Sleeman [3] considered the inverse nodal problem for Sturm-Liouville problem $B(q,h,\delta_2)$. Guliyev [4] found a trace formula for the Sturm-Liouville problem $B(q,\delta_1,\delta_2)$. Wang [5] considered the inverse problem for the Sturm-Liouville problem $B(q,\delta_1,\delta_2)$ and showed that the potential $q(x)$ and fractional linear functions $\frac{a_1\lambda+b_1}{c_1\lambda+d_1}$ and $\frac{a_2\lambda+b_2}{c_2\lambda+d_2}$ of the boundary conditions can be uniquely determined by a set of values of eigenfunctions at some interior point and parts of two spectra. By Weyl function techniques, Freiling and Yurko [6] discussed the inverse problems for Sturm-Liouville equations with boundary conditions polynomially dependent on the spectral parameter and provided a constructive solution of the inverse spectral problems. Sturm-Liouville problems with eigenparameter dependent boundary conditions have many applications in physics, engineering, mathematics, etc (see [1-8]).

Half inverse problem for Sturm-Liouville operators consists of reconstruction of this operator by its spectrum and half of the potential. In 1978, Hochstadt and Lieberman [9] considered the half inverse problem for the Sturm-Liouville problem $B(q,h,H)$ and showed if $q(x)$ is prescribed on $[\frac{\pi}{2},\pi]$, then the potential $q(x)$ on the interval $[0,\frac{\pi}{2})$ for the Sturm-Liouville operator on the finite interval $[0,\pi]$ can be determined by one spectrum. Castillo [10] gave a counterexample which proved that the fixed boundary condition (1.3) is necessary. Suzuki [11] discussed the Sturm-Liouville problem $B(q,h,H)$. By an example, Suzuki showed that if $q_1(x)=q_2(x)$ on the interval $[0,\frac{1}{2}-\varepsilon]$ for some $0<\varepsilon<\frac{1}{2}$ and $\sigma(L_1)=\sigma(L_2)$, but $q_1(x)\neq q_2(x)$ on the interval $[0,1]$, where $\sigma(L_i)=\{\lambda_{in}\}$ is the spectrum of L_i of the corresponding Sturm-Liouville problem for the potential $q_i (i=1,2)$. One of this kind of half inverse problems for differential operators on the finite interval was considered by a number of authors (see [9-16]). Using the Weyl function techniques, Gesztesy and Simon [17] discussed the inverse problem for the Sturm-Liouville problem $B(q,h,H)$ from three spectra. Gesztesy and Simon [18] discussed the inverse problem for the Sturm-Liouville problem $B(q,h,H)$ from partial information on the potential and partial spectrum and established the following remarkable uniqueness theorem for the Sturm-Liouville problem $B(q,h,H)$, which is a generalization of Hochstadt-Lieberman theorem [9].

Theorem 1.1 ([18], Theorem 1.3)**.** *Let $\sigma(L)=\{\lambda_n\} (n\in \mathbf{N}_0)$ be the spectrum of the Sturm-Liouville problem $B(q,h,H)$. If $q(x)$ is prescribed on $[0,\frac{1}{2}+\frac{\alpha}{2}]$ for some $\alpha\in[0,1)$, then the potential $q(x)$*

a.e. on the whole interval $[0,1]$ *and coefficient* H *of the boundary condition are uniquely determined by coefficient* h *of the boundary condition and a subset* $S \subseteq \sigma(L)$ *satisfying*

$$\sharp\{\lambda \in S | \lambda \le t)\} \ge (1-\alpha)\sharp\{\lambda \in \sigma(L) | \lambda \le t)\} + \frac{\alpha}{2}, \tag{1.4}$$

for all sufficiently large $t \in \mathbf{R}^+$, *where* $\mathbf{N}_0 := \{n | n = 0,1,2,\cdots\}$.

In this paper, we consider the inverse problem for the Sturm-Liouville problem $B(q,\delta_1,\delta_2)$ from partial information on the potential and partial spectrum. We establish a Gesztesy-Simon theorem for the Sturm-Liouville problem $B(q,\delta_1,\delta_2)$. Using Freiling and Yurko's result, we show that if $q(x)$ is prescribed on $[0,\frac{1}{2}+\frac{\alpha}{2}]$ for some $\alpha \in [0,1)$, then the potential $q(x)$ a.e. on the whole interval $[0,1]$ and fractional linear function $\frac{a_2\lambda+b_2}{c_2\lambda+d_2}$ of the boundary condition are uniquely determined by fractional linear function $\frac{a_1\lambda+b_1}{c_1\lambda+d_1}$ of the boundary condition and a subset $S \subseteq \sigma(L)$ satisfying (2.4)(see below). Although the techniques used here is based on the Gesztesy-Simon method, the Sturm-Liouville problem $B(q,\delta_1,\delta_2)$ in this paper is different from the Sturm-Liouville problem $B(q,h,H)$ in [18].

From [1] and [6], we have

Lemma 1.2 ([1], [6])**.** *Let* $\{\lambda_n\}_{n=0}^{\infty}$ *be spectrum of the Sturm-Liouville problem* $B(q,\delta_1,\delta_2)$, *then* $\lambda_n (n \in \mathbf{N_0})$ *is real and simple and* λ_n *is root of* $(1.2')$ *and satisfies*

$$\lambda_0 < \lambda_1 < \lambda_2 < \cdots \to +\infty$$

and

$$\sqrt{\lambda}_n = (n-2)\pi + \frac{\omega}{n\pi} + \frac{\kappa_n}{n}, \tag{1.5}$$

where $\omega = \frac{1}{2}\int_0^{\pi} q(x)dx - \frac{a_1}{c_1} + \frac{a_2}{c_2}$ *and* $\{\kappa_n\} \in l^2$.

Suppose $\varphi(x,\lambda), \theta(x,\lambda)$ are the two fundamental solutions of the equation (1.1) and satisfy

$$\varphi(1,\lambda) = 1, \varphi'(1,\lambda) = 0, \theta(1,\lambda) = 0 \ and\ \theta'(1,\lambda) = 1,$$

then the solution of the equation (1.1) satisfying $y(1,\lambda) = c_2\lambda + d_2$ and $y'(1,\lambda) = a_2\lambda + b_2$ is

$$y(x,\lambda) = (c_2\lambda + d_2)\varphi(x,\lambda) + (a_2\lambda + b_2)\theta(x,\lambda).$$

By virtue of [26-28], for sufficiently large $|\lambda|$, this yields

$$\varphi(x,\lambda) = \cos\sqrt{\lambda}(1-x) + O\left(\frac{e^{|Im\sqrt{\lambda}|(1-x)}}{\sqrt{\lambda}}\right), \tag{1.6}$$

$$\varphi'(x,\lambda) = \sqrt{\lambda}\sin\sqrt{\lambda}(1-x) + O(e^{|Im\sqrt{\lambda}|(1-x)}), \tag{1.7}$$

$$\theta(x,\lambda) = \frac{\sin\sqrt{\lambda}(1-x)}{\sqrt{\lambda}} + O(\frac{e^{|Im\sqrt{\lambda}|(1-x)}}{\lambda}) \tag{1.8}$$

and

$$\theta'(x,\lambda) = -\cos\sqrt{\lambda}(1-x) + O(\frac{e^{|Im\sqrt{\lambda}|(1-x)}}{\sqrt{\lambda}}). \tag{1.9}$$

Hence, we obtain the following asymptotic formulae

$$y(x,\lambda) = c_2\lambda\cos\sqrt{\lambda}(1-x) + O(\sqrt{\lambda}e^{|Im\sqrt{\lambda}|(1-x)}) \tag{1.10}$$

and

$$y'(x,\lambda) = c_2\lambda^{\frac{3}{2}}\sin\sqrt{\lambda}(1-x) + O(\lambda e^{|Im\sqrt{\lambda}|(1-x)}). \tag{1.11}$$

Define the Weyl m-function $m(x,\lambda)$ by

$$m(x,\lambda) = \frac{y'(x,\lambda)}{y(x,\lambda)}. \tag{1.12}$$

Let

$$M(\lambda) := \frac{y'(0,\lambda)}{(c_1\lambda+d_1)y'(0,\lambda)-(a_1\lambda+b_1)y(0,\lambda)}, \tag{1.13}$$

which is called the Weyl function of the Sturm-Liouville problem $B(q,\delta_1,\delta_2)$.

In virtue of [6], we present the following lemma, which is important for us to show the main theorem in this paper.

Lemma 1.3 ([6])**.** *Let $M(\lambda)$ be the Weyl function of the Sturm-Liouville problem $B(q,\delta_1,\delta_2)$. Then $M(\lambda)$ uniquely determines fractional linear function $\frac{a_2\lambda+b_2}{c_2\lambda+d_2}$ of the boundary condition as well as $q(x)$ (a.e.) on* $[0,1]$.

2. Main results and Proofs

Consider the following Sturm-Liouville operator $L_j(j=1,2)$ satisfying

$$L_j u = -u'' + q_j(x)u = \lambda u(x\in[0,1]) \tag{2.1}$$

with boundary conditions

$$(a_1\lambda+b_1)u(0,\lambda)-(c_1\lambda+d_1)u'(0,\lambda) = 0 \tag{2.2}$$

and

$$(a_{j2}\lambda+b_{j2})u(1,\lambda)-(c_{j2}\lambda+d_{j2})u'(1,\lambda) = 0, \tag{2.3}$$

respectively, where $a_1,b_1,c_1,d_1,a_{j2},b_{j2},c_{j2},d_{j2}\in\mathbf{R}$, $c_1c_{12}c_{22}\neq 0$ such that

$$\delta_1 = a_1d_1-b_1c_1<0, \qquad \delta_{j2} = a_{j2}d_{j2}-b_{j2}c_{j2}>0,$$

q_1,q_2 are real-valued functions and $q_1,q_2\in L^2[0,1]$.

We have the following uniqueness theorem for the boundary-value problem $B(q,\delta_1,\delta_{j2})$.

Theorem 2.4. *Let $\sigma(L_j) = \{\lambda_{jn}\}(j = 1,2, n \in \mathbf{N_0})$ be the spectrum of the boundary-value problem $B(q,\delta_1,\delta_{j2})$. For some $\alpha \in [0,1)$ and sufficiently small $\varepsilon(\varepsilon > 0)$, If*

$$q_1(x) = q_2(x), x \in [0, \frac{1}{2} + \frac{\alpha}{2}]$$

and $S \subseteq \sigma(L_1) \bigcap \sigma(L_2)$ satisfying

$$\sharp\{\lambda \in S | \lambda \le t)\} \ge (1-\alpha)\sharp\{\lambda \in \sigma(L_1) | \lambda \le t)\} + \frac{5\alpha}{2} - \frac{1}{2} + \varepsilon, \tag{2.4}$$

for all sufficiently large $t \in \mathbf{R}^+$, then

$$q_1(x) = q_2(x) \ a.e. \ on \ [0,1]$$

and

$$\frac{a_{12}\lambda + b_{12}}{c_{12}\lambda + d_{12}} = \frac{a_{22}\lambda + b_{22}}{c_{22}\lambda + d_{22}}.$$

Remark. When $\alpha = 0$, we prove the half inverse problem for the Sturm-Liouville operator with boundary conditions having fractional linear function of spectral parameter on the finite interval $[0,1]$.

Proof. Let $u_j(x,t)(j = 1,2)$ be the solution of the equation (2.1) satisfying $u_j(1,t) = c_{j2}\lambda + d_{j2}$ and $u_j'(1,t) = a_{j2}\lambda + b_{j2}$.

Denote

$$\begin{aligned} F(\lambda) &= u_1(\frac{1}{2}+\frac{\alpha}{2},\lambda)u_2'(\frac{1}{2}+\frac{\alpha}{2},\lambda) - u_1'(\frac{1}{2}+\frac{\alpha}{2},\lambda)u_2(\frac{1}{2}+\frac{\alpha}{2},\lambda) \\ &= u_1'(\frac{1}{2}+\frac{\alpha}{2},\lambda)u_2'(\frac{1}{2}+\frac{\alpha}{2},\lambda)(m_1^{-1}(\frac{1}{2}+\frac{\alpha}{2},\lambda) - m_2^{-1}(\frac{1}{2}+\frac{\alpha}{2},\lambda)) \end{aligned} \tag{2.5}$$

and

$$\omega(\lambda) = (a_1\lambda + b_1)u(0,\lambda) - (c_1\lambda + d_1)u'(0,\lambda). \tag{2.6}$$

Then

$$F(\lambda_n) = 0, \forall \lambda_n \in S \subseteq \sigma(L_1) \bigcap \sigma(L_2)$$

From (1.10) and (1.11), we obtain

$$|F(\lambda)| = O(\lambda^2 e^{|Im\sqrt{\lambda}|(1-\alpha)}) \tag{2.7}$$

and

$$|\omega(\lambda)| = O(\lambda^{\frac{5}{2}} e^{|Im\sqrt{\lambda}|}). \tag{2.8}$$

Define the function $G(\lambda)$ by

$$G(\lambda) = \prod_{\lambda_n \in S} (1 - \frac{\lambda}{\lambda_n}). \tag{2.9}$$

and

$$\psi(\lambda) = \frac{F(\lambda)}{G(\lambda)}. \tag{2.10}$$

Then, $\psi(\lambda)$ is an entire function. For convenience, we denote

$$N_G(t) = \sharp\{\lambda_n \in S | \lambda_n \le t\}, \quad N_\omega(t) = \sharp\{\lambda_n \in \sigma(L) | \lambda_n \le t\}.$$

By virtue of (2.4), this yields

$$N_G(t) \ge (1-\alpha) N_\omega(t) + \frac{5\alpha - 1}{2} + \varepsilon. \tag{2.11}$$

Since $\omega(\lambda)$ is an entire function in λ of order $\frac{1}{2}$, there exists a positive constant c such that

$$N_G(t) \le N_\omega(t) \le c t^{\frac{1}{2}}, \tag{2.12}$$

Without loss of generality, let us to assume $\lambda_n > 1 (n \in \mathbf{N_0})$, then $N_G(1) = N_\omega(1) = 0$. For a fixed x ($x \in \mathbf{R}$) and $|x|$ sufficiently large, we have

$$\begin{aligned}
\ln|G(\mathrm{i}x)| &= \frac{1}{2} \ln G(\mathrm{i}x)\overline{G(\mathrm{i}x)} = \frac{1}{2} \sum_{\lambda_n \in S} \ln(1 - \frac{ix}{\lambda_n})(1 + \frac{\mathrm{i}x}{\lambda_n}) \\
&= \frac{1}{2} \sum_{\lambda_n \in S} \ln(1 + \frac{x^2}{\lambda_n^2}) = \frac{1}{2} \int_1^\infty \ln(1 + \frac{x^2}{t^2}) \mathrm{d} N_G(t) \\
&= \frac{1}{2} \ln(1 + \frac{x^2}{t^2}) N_G(t)|_1^\infty - \frac{1}{2} \int_1^\infty N_G(t) \mathrm{d}[\ln(1 + \frac{x^2}{t^2})].
\end{aligned}$$

For sufficiently large t, since

$$\ln(1 + \frac{x^2}{t^2}) = O(\frac{1}{t^2}),$$

then

$$\lim_{n \to \infty} \ln(1 + \frac{x^2}{t^2}) N_G(t) = 0$$

and

$$\lim_{n \to \infty} \ln(1 + \frac{x^2}{t^2}) N_\omega(t) = 0.$$

Therefore

$$\begin{aligned}
\ln|G(\mathrm{i}x)| &= \int_1^\infty \frac{x^2}{t^3 + t x^2} N_G(t) \mathrm{d}t \\
&\ge (1-\alpha) \int_1^\infty \frac{x^2}{t^3 + t x^2} N_\omega(t) \mathrm{d}t + (\frac{5\alpha - 1}{2} + \varepsilon) \int_1^\infty \frac{x^2}{t^3 + t x^2} \mathrm{d}t \\
&= (1-\alpha) \ln|\omega(\mathrm{i}x)| + \frac{1}{2}(\frac{5\alpha - 1}{2} + \varepsilon) \ln(1 + x^2).
\end{aligned} \tag{2.13}$$

By virtue of (2.13), this yields

$$|G(\mathrm{i}x)| \ge |\omega(ix)|^{1-\alpha} (1 + x^2)^{\frac{5\alpha - 1}{4} + \frac{\varepsilon}{2}}. \tag{2.14}$$

From (2.10) and (2.14), we have

$$|\psi(ix)| = |\frac{F(ix)}{G(ix)}| = O(\frac{1}{|x|^{\varepsilon}})(\varepsilon > 0) \tag{2.15}$$

for $|x|$ sufficiently large.

By the Phragmén-Lindelöf theorem, we get

$$\psi(\lambda) = 0,\ \forall\lambda \in \mathbf{C}. \tag{2.16}$$

i.e,

$$F(\lambda) = 0,\ \forall\lambda \in \mathbf{C}. \tag{2.17}$$

Therefore,

$$u_1(\frac{1}{2}+\frac{\alpha}{2},\lambda)u_2'(\frac{1}{2}+\frac{\alpha}{2},\lambda) - u_1'(\frac{1}{2}+\frac{\alpha}{2},\lambda)u_2(\frac{1}{2}+\frac{\alpha}{2},\lambda) = 0,\ \forall\lambda \in \mathbf{C}. \tag{2.18}$$

In virtue of $q_1(x) = q_2(x)$ on $[0, \frac{1}{2}+\frac{\alpha}{2}]$ together with (2.18), we obtain

$$u_1(0,\lambda)u_2'(0,\lambda) - u_1'(0,\lambda)u_2(0,\lambda) = 0,\ \forall\lambda \in \mathbf{C}. \tag{2.19}$$

From (2.19), we have

$$M_1(\lambda) = M_2(\lambda),\ \forall\lambda \in \mathbf{C}. \tag{2.20}$$

By virtue of Lemma 1.3 together with (2.20), we obtain

$$q_1(x) = q_2(x)\ a.e.\ on\ [0,1]\ and\ \frac{a_{12}\lambda + b_{12}}{c_{12}\lambda + d_{12}} = \frac{a_{22}\lambda + b_{22}}{c_{22}\lambda + d_{22}}.$$

This completes the proof of Theorem 2.1.

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