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OTHER CHARACTERIZATIONS OF β-θ-R_0 TOPOLOGICAL SPACES

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Abstract. In this paper we give other characterizations of β-θ-R_0 and also introduce a new separation axiom called β-θ-R_1. It turns out that β-θ-R_1 is stronger that β-θ-R_0.

1. Introduction

The notion of R_0 topological spaces was introduced by Shanin [14] in 1943. By definition, a topological space is R_0 if every open set contains the closure of each of its singletons. Later, Davis [7] rediscovered it and studied some properties of this weak separation axiom. Many researchers investigated further properties of R_0 topological spaces and many interesting results have been obtained in various contexts (see: [8], [9], [10], [13]). In 2003, Noiri [12] introduce the notion of β-θ-closed sets. In 2011, Caldas [3, 4, 5, 6] introduced and investigated properties of β-θ-closed sets and of the separation axiom β-θ-R_0. In this paper, we give some properties and other characterizations of β-θ-R_0. We also introduce a new separation axiom called β-θ-R_1. It turns out that β-θ-R_1 is stronger that β-θ-R_0.

2. Preliminaries

Since we shall require the following known definitions, notations and some properties, we recall them in this section.

Let (X,τ) be a topological space and S a subset of X. We denote the closure and the interior of S by $Cl(S)$ and $Int(S)$, respectively. A subset S is said to be β-open [1, 2] if $S \subset Cl(Int(Cl(S)))$. The complement of a β-open set is said to be β-closed [1]. The intersection of all β-closed sets containing S is called the β-closure [2] of S and is denoted by $\beta Cl(S)$. A subset S is said to be β-regular [12] if it is both β-open and β-closed. The family of all β-open sets (resp. β-regular sets) of (X,τ) is denoted by $\beta O(X,\tau)$ (resp. $\beta R(X,\tau)$). The β-θ-closure of S [12], denoted by $\beta Cl_\theta(S)$, is defined to be the set of all $x \in X$ such that $\beta Cl(O) \cap S \neq \emptyset$ for

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every $O \in \beta O(X,\tau)$ with $x \in O$. The set $\{x \in X : \beta Cl_\theta(O) \subset S\ for\ some\ O \in \beta(X,x)\}$ is called the β-θ- interior of S and is denoted by $\beta Int_\theta(S)$. A subset S is said to be β-θ-closed [12] if $S = \beta Cl_\theta(S)$. The complement of a β-θ-closed set is said to be β-θ-open. The family of all β-θ-open (resp. β-θ-closed) subsets of X is denoted by $\beta\theta O(X,\tau)$ or $\beta\theta O(X)$ (resp. $\beta\theta C(X,\tau)$). We set $\beta\theta O(X,x) = \{U : x \in U \in \beta\theta O(X,\tau)\}$ and $\beta\theta C(X,x) = \{U : x \in U \in \beta\theta C(X,\tau)\}$.

We recall the following results which were obtained by Noiri [12].

Lemma 2.1 ([12])**.** *Let A be a subset of a topological space (X,τ).*
(i) If $A \in \beta O(X,\tau)$, then $\beta Cl(A) \in \beta R(X)$.
(ii) $A \in \beta R(X)$ if and only if $A \in \beta\theta O(X) \cap \beta\theta C(X)$.

Lemma 2.2 ([12])**.** *For the β-θ-closure of a subset A of a topological space (X,τ), the following properties are held:*

(i) *$A \subset \beta Cl(A) \subset \beta Cl_\theta(A)$ and $\beta Cl(A) = \beta Cl_\theta(A)$ if $A \in \beta O(X)$.*
(ii) *If $A \subset B$, then $\beta Cl_\theta(A) \subset \beta Cl_\theta(B)$.*
(iii) *If $A_\alpha \in \beta\theta C(X)$ for each $\alpha \in A$, then $\bigcap\{A_\alpha \mid \alpha \in A\} \in \beta\theta C(X)$.*
(iv) *If $A_\alpha \in \beta\theta O(X)$ for each $\alpha \in A$, then $\bigcup\{A_\alpha \mid \alpha \in A\} \in \beta\theta O(X)$.*
(v) *$\beta Cl_\theta(\beta Cl_\theta(A)) = \beta Cl_\theta(A)$ and $\beta Cl_\theta(A) \in \beta\theta C(X)$.*

The union of two β-θ-closed sets is not necessarily β-θ-closed as showed in the following example.

Example 2.3 ([12])**.** Let $X = \{a,b,c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}\}$. The subsets $\{a\}$ and $\{b\}$ are β-θ-closed in (X,τ) but $\{a,b\}$ is not β-θ-closed.

3. Other characterizations of β-θ-R_0 spaces

Definition 1. Let (X,τ) be a topological space and $A \subset X$. Then the β-θ-kernel of A [3], denoted by $\beta\mathrm{Ker}_\theta(A)$ is defined to be the set $\beta\mathrm{Ker}_\theta(A) = \cap\{G \in \beta\theta O(X,\tau) \mid A \subset G\}$.

Lemma 3.1 ([3])**.** (1) *Let (X,τ) be a topological space and $x \in X$. Then:*
$y \in \beta Ker_\theta(\{x\})$ if and only if $x \in \beta Cl_\theta(\{y\})$.

(2) *Let (X,τ) be a topological space and A a subset of X. Then:*
$\beta Ker_\theta(A) = \{x \in X \mid \beta Cl_\theta(\{x\}) \cap A \neq \emptyset\}$.

(3) *The following statements are equivalent for any points x and y in a topological space (X,τ):*

(i) *$\beta Ker_\theta(\{x\}) \neq \beta Ker_\theta(\{y\})$;*
(ii) *$\beta Cl_\theta(\{x\}) \neq \beta Cl_\theta(\{y\})$.*

Definition 2. A topological space (X,τ) is a β-θ-R_0 space [3] if every β-θ-open set contains the β-θ-closure of each of its singletons.

Example 3.2. Let $X = \{a,b,c\}$ and $\tau = \{X,\emptyset,\{b\},\{c\},\{b,c\}\}$. We have $\beta O(X,\tau) = \{X,\emptyset,\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\}\}$. The β-θ-closed sets of (X,τ) are $\{X,\emptyset,\{a\},\{b\},\{c\},\{a,b\},\{a,c\}\}$. Then (X,τ) is a β-θ-R_0 space which is not R_0.

The following Theorem 3.3 coincides with the Theorem 3.5 in [3] and Theorem 3.4 is a special case of Theorem 3.6 in [3].

Theorem 3.3. *A topological space (X,τ) is a β-θ-R_0 space if and only if for any x and y in X, $\beta Cl_\theta(\{x\}) \neq \beta Cl_\theta(\{y\})$ implies $\beta Cl_\theta(\{x\}) \cap \beta Cl_\theta(\{y\}) = \emptyset$.*

Theorem 3.4. *A topological space (X,τ) is a β-θ-R_0 space if and only if for any points x and y in X , $\beta Ker_\theta(\{x\}) \neq \beta Ker_\theta(\{y\})$ implies $\beta Ker_\theta(\{x\}) \cap \beta Ker_\theta(\{y\}) = \emptyset$.*

Now, we give other characterizations of β-θ-R_0 spaces.

Theorem 3.5. *For a topological space (X,τ), the following properties are equivalent :*

(1) *(X,τ) is a β-θ-R_0 space;*

(2) *$x \in \beta Cl_\theta(\{y\})$ if and only if $y \in \beta Cl_\theta(\{x\})$.*

Proof. (1) $\Rightarrow$ (2) : Assume that X is β-θ-R_0. Let $x \in \beta Cl_\theta(\{y\})$. Then $\beta \mathrm{Cl}_\theta(\{x\}) \cap \beta \mathrm{Cl}_\theta(\{y\}) \neq \emptyset$. By Theorem 3.3 $\beta \mathrm{Cl}_\theta(\{x\}) = \beta \mathrm{Cl}_\theta(\{y\})$. Hence $y \in \beta \mathrm{Cl}_\theta(\{y\}) = \beta \mathrm{Cl}_\theta(\{x\})$. Therefore $y \in \beta Cl_\theta(\{x\})$. Similarly $y \in \beta Cl_\theta(\{x\})$ then $x \in \beta Cl_\theta(\{y\})$.
(2) $\Rightarrow$ (1) : Let U be a β-θ-open set and $x \in U$. If $y \notin U$, then $x \notin \beta Cl_\theta(\{y\})$ and hence $y \notin \beta Cl_\theta(\{x\})$. This implies that $\beta Cl_\theta(\{x\}) \subset U$. Hence (X,τ) is β-θ-R_0.

Theorem 3.6. *For a topological space (X,τ), the following properties are equivalent:*

(1) *(X,τ) is a β-θ-R_0 space;*

(2) *For any $F \in \beta\theta C(X,\tau)$, $x \notin F$ implies $F \subset U$ and $x \notin U$ for some $U \in \beta\theta O(X,\tau)$;*

(3) *For any $F \in \beta\theta C(X,\tau)$, $x \notin F$ implies $F \cap \beta Cl_\theta(\{x\}) = \emptyset$;*

(4) *For any distinct point x and y of X, either $\beta Cl_\theta(\{x\}) = \beta Cl_\theta(\{y\})$ or $\beta Cl_\theta(\{x\}) \cap \beta Cl_\theta(\{y\}) = \emptyset$.*

Proof. (1) $\Rightarrow$ (2) : Let $F \in \beta\theta C(X,\tau)$ and $x \notin F$. Then by (1) $\beta Cl_\theta(\{x\}) \subset X - F$. Set $U = X - \beta Cl_\theta(\{x\})$, then $U \in \beta\theta O(X,\tau)$, $F \subset U$ and $x \notin U$.
(2) $\Rightarrow$ (3) : Let $F \in \beta\theta C(X,\tau)$ and $x \notin F$. There exists $U \in \beta\theta O(X,\tau)$ such that $F \subset U$ and $x \notin U$. Since $U \in \beta\theta O(X,\tau)$, $U \cap \beta Cl_\theta(\{x\}) = \emptyset$ and

$F \cap \beta Cl_\theta(\{x\}) = \emptyset$.

(3) $\Rightarrow$ (4) :Suppose that $\beta Cl_\theta(\{x\}) \neq \beta Cl_\theta(\{y\})$ for distinct points $x, y \in X$. There exists $z \in \beta Cl_\theta(\{x\})$ such that $z \notin \beta Cl_\theta(\{y\})$ (or $z \in \beta Cl_\theta(\{y\})$ such that $z \notin \beta Cl_\theta(\{x\})$). There exists $V \in \beta\theta O(X,\tau)$ such that $y \notin V$ and $z \in V$; hence $x \in V$. Therefore, we have $x \notin \beta Cl_\theta(\{y\})$. By (3), we obtain $\beta Cl_\theta(\{x\}) \cap \beta Cl_\theta(\{y\}) = \emptyset$. The proof for the other case is similar.

(4) $\Rightarrow$ (1) : Let $V \in \beta\theta O(X,\tau)$ and $x \in V$. For each $y \notin V, x \neq y$ and $x \notin \beta Cl_\theta(\{y\})$. This shows that $\beta Cl_\theta(\{x\}) \neq \beta Cl_\theta(\{y\})$. By (4), $\beta Cl_\theta(\{x\}) \cap \beta Cl_\theta(\{y\}) = \emptyset$ for each $y \in X - V$ and hence $\beta Cl_\theta(\{x\}) \cap (\bigcup_{y\in X-V} \beta Cl_\theta(\{y\})) = \emptyset$. On the other hand, since $V \in \beta\theta O(X,\tau)$ and $y \in X - V$, we have $\beta Cl_\theta(\{y\}) \subset X - V$ and hence $X - V = \bigcup_{y\in X-V} \beta Cl_\theta(\{y\})$. Therefore, we obtain $(X - V) \cap \beta Cl_\theta(\{x\}) = \emptyset$ and $\beta Cl_\theta(\{x\}) \subset V$. This shows that (X,τ) is a β-θ-R_0 space.

Theorem 3.7. *For a topological space (X,τ), the following properties are equivalent :*

(1) *(X,τ) is a β-θ-R_0 space;*

(2) *For any nonempty set A and $G \in \beta\theta O(X,\tau)$ such that $A \cap G \neq \emptyset$, there exists $F \in \beta\theta C(X,\tau)$ such that $A \cap F \neq \emptyset$ and $F \subset G$;*

(3) *Any $G \in \beta\theta O(X,\tau)$, $G = \cup\{F \in \beta\theta C(X,\tau) \mid F \subset G\}$;*

(4) *Any $F \in \beta\theta C(X,\tau)$, $F = \cap\{G \in \beta\theta O(X,\tau) \mid F \subset G\}$;*

(5) *For any $x \in X, \beta Cl_\theta(\{x\}) \subset \beta Ker_\theta(\{x\})$.*

Proof. (1) $\Rightarrow$ (2) : Let A be a nonempty set of X and $G \in \beta\theta O(X,\tau)$ such that $A \cap G \neq \emptyset$. There exists $x \in A \cap G$. Since $x \in G \in \beta\theta O(X,\tau), \beta Cl_\theta(\{x\}) \subset G$. Set $F = \beta Cl_\theta(\{x\})$, then $F \in \beta\theta C(X,\tau), F \subset G$ and $A \cap F \neq \emptyset$.

(2) $\Rightarrow$ (3) : Let $G \in \beta\theta O(X,\tau)$, then $G \supset \cup\{F \in \beta\theta C(X,\tau) \mid F \subset G\}$. Let x be any point of G. There exists $F \in \beta\theta C(X,\tau)$ such that $x \in F$ and $F \subset G$. Therefore, we have $x \in F \subset \cup\{F \in \beta\theta C(X,\tau) \mid F \subset G\}$ and hence $G = \cup\{F \in \beta\theta C(X,\tau) \mid F \subset G\}$.

(3) $\Rightarrow$ (4) : This is obvious.

(4) $\Rightarrow$ (5) : Let x be any point of X and $y \notin \beta Ker_\theta(\{x\})$. There exists $V \in \beta\theta O(X,\tau)$ such that $x \in V$ and $y \notin V$; hence $\beta Cl_\theta(\{y\}) \cap V = \emptyset$. By (4) $(\cap\{G \in \beta\theta O(X,\tau) \mid \beta Cl_\theta(\{y\}) \subset G\}) \cap V = \emptyset$ and there exists $G \in \beta\theta O(X,\tau)$ such that $x \notin G$ and $\beta Cl_\theta(\{y\}) \subset G$. Therefore, $\beta Cl_\theta(\{x\}) \cap G = \emptyset$ and $y \notin \beta Cl_\theta(\{x\})$. Consequently, we obtain $\beta Cl_\theta(\{x\}) \subset \beta Ker_\theta(\{x\})$.

(5) $\Rightarrow$ (1) : Let $G \in \beta\theta O(X,\tau)$ and $x \in G$. Let $y \in \beta Ker_\theta(\{x\})$, then $x \in \beta Cl_\theta(\{y\})$ and $y \in G$. This implies that $\beta Ker_\theta(\{x\}) \subset G$. Therefore, we obtain $x \in \beta Cl_\theta(\{x\}) \subset \beta Ker_\theta(\{x\}) \subset G$. This shows that (X,τ) is a β-θ-R_0 space.

Theorem 3.8. *For a topological space (X,τ), the following properties are equivalent :*

(1) (X,τ) *is a* β-θ-R_0 *space;*

(2) *If* F *is* β-θ-*closed, then* $F = \beta Ker_\theta(F)$;

(3) *If* F *is* β-θ-*closed and* $x \in F$, *then* $\beta Ker_\theta(\{x\}) \subset F$;

(4) *If* $x \in X$, *then* $\beta Ker_\theta(\{x\}) \subset \beta Cl_\theta(\{x\})$.

Proof. (1) $\Rightarrow$ (2) : Let F be β-θ-closed and $x \notin F$. Thus $X - F$ is β-θ-open and contains x. Since (X,τ) is β-θ-R_0, $\beta Cl_\theta(\{x\}) \subset X - F$. Thus $\beta Cl_\theta(\{x\}) \cap F = \emptyset$ and by Lemma 3.1 $x \notin \beta Ker_\theta(F)$. Therefore $\beta Ker_\theta(F) = F$.

(2) $\Rightarrow$ (3) : In general, $A \subset B$ implies $\beta Ker_\theta(A) \subset \beta Ker_\theta(B)$. Therefore, it follows from (2) that $\beta Ker_\theta(\{x\}) \subset \beta Ker_\theta(F) = F$.

(3) $\Rightarrow$ (4) : Since $x \in \beta Cl_\theta(\{x\})$ and $\beta Cl_\theta(\{x\})$ is β-θ-closed, by (3) $\beta Ker_\theta(\{x\}) \subset \beta Cl_\theta(\{x\})$.

(4) $\Rightarrow$ (1) : We show the implication by using Theorem 3.5. Let $x \in \beta Cl_\theta(\{y\})$. Then by Lemma 3.1 $y \in \beta Ker_\theta(\{x\})$. By (4) we obtain $y \in \beta Ker_\theta(\{x\}) \subset \beta Cl_\theta(\{x\})$. Therefore $x \in \beta Cl_\theta(\{y\})$ implies $y \in \beta Cl_\theta(\{x\})$. The converse is obvious and (X,τ) is β-θ-R_0.

Corollary 3.9. *For a topological space* (X,τ), *the following properties are equivalent :*

(1) (X,τ) *is a* β-θ-R_0 *space;*

(2) $\beta Cl_\theta(\{x\}) = \beta Ker_\theta(\{x\})$ *for all* $x \in X$.

Proof. This is obvious by Theorem 3.7 and Theorem 3.8.

Definition 3. Let (X,τ) be a topological space, $x \in X$ and $\{x_\alpha\}_{\alpha\in\Lambda}$ be a net of X. We say that the net $\{x_\alpha\}_{\alpha\in\Lambda}$ $\beta\theta$-converges to x if for each β-θ-open set U containing x there exists an element $\alpha_0 \in \Lambda$ such that $\alpha \geq \alpha_0$ implies $x_\alpha \in U$.

Lemma 3.10. *Let* (X,τ) *be a topological space and let* x *and* y *be any two points in* X *such that every net in* X $\beta\theta$-*converging to* y $\beta\theta$-*converges to* x. *Then* $x \in \beta Cl_\theta(\{y\})$.

Proof. Suppose that $x_n = y$ for each $n \in \mathbf{N}$. Then $\{x_n\}_{n\in\mathbf{N}}$ is a net in $\beta Cl_\theta(\{y\})$. By the fact that $\{x_n\}_{n\in\mathbf{N}}$ $\beta\theta$-converges to y , then $\{x_n\}_{n\in\mathbf{N}}$ $\beta\theta$-converges to x and this means that $x \in \beta Cl_\theta(\{y\})$.

Theorem 3.11. *For a topological space* (X,τ), *the following statements are equivalent :*

(1) (X,τ) *is a* β-θ-R_0 *space;*

(2) *If* $x, y \in X$, *then* $y \in \beta Cl_\theta(\{x\})$ *if and only if every net in* X $\beta\theta$ *-converging to* y $\beta\theta$-*converges to* x.

Proof. (1) $\Rightarrow$ (2) : Let $x, y \in X$ such that $y \in \beta Cl_\theta(\{x\})$. Let $\{x_\alpha\}_{\alpha\in\Lambda}$ be a net in X such that $\{x_\alpha\}_{\alpha\in\Lambda}$ $\beta\theta$-converges to y. Since $y \in \beta Cl_\theta(\{x\})$, by Theorem 3.3 we have $\beta Cl_\theta(\{x\}) = \beta Cl_\theta(\{y\})$. Therefore $x \in \beta Cl_\theta(\{y\})$. This means that $\{x_\alpha\}_{\alpha\in\Lambda}$ $\beta\theta$-converges to x. Conversely, let $x, y \in X$ such that every net in X $\beta\theta$-converging to y $\beta\theta$-converges to x. Then $x \in \beta Cl_\theta(\{y\})$ by Lemma 3.1. From Theorem 3.3, we have $\beta Cl_\theta(\{x\}) = \beta Cl_\theta(\{y\})$. Therefore $y \in \beta Cl_\theta(\{x\})$.

(2) $\Rightarrow$ (1) : Assume that x and y are any two points of X such that $\beta Cl_\theta(\{x\}) \cap \beta Cl_\theta(\{y\}) \neq \emptyset$. Let $z \in \beta Cl_\theta(\{x\}) \cap \beta Cl_\theta(\{y\})$. So there exists a net $\{x_\alpha\}_{\alpha\in\Lambda}$ in $\beta Cl_\theta(\{x\})$ such that $\{x_\alpha\}_{\alpha\in\Lambda}$ $\beta\theta$-converges to z. Since $z \in \beta Cl_\theta(\{y\})$, then $\{x_\alpha\}_{\alpha\in\Lambda}$ $\beta\theta$-converges to y. It follows that $y \in \beta Cl_\theta(\{x\})$. By the same token we obtain $x \in \beta Cl_\theta(\{y\})$. Therefore $\beta Cl_\theta(\{x\}) = \beta Cl_\theta(\{y\})$ and by Theorem 3.3 (X,τ) is β-θ-R_0.

Recall that a topological space (X,τ) is said to be:

(1) β-θ-T_0 (resp. β-θ-T_1) if for any distinct pair of points x and y in X, there is a β-θ-open U in X containing x but not y or (resp. and) a β-θ-open set V in X containing y but not x.

(2) β-θ-T_2 [5] (resp. β-T_2 [11]) if for every pair of distinct points x and y, there exist two β-θ-open (resp. β-open) sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

From the definitions above, we obtain the following diagram:

$$\beta\text{-}\theta\text{-}T_2 \Rightarrow \beta\text{-}\theta\text{-}T_1 \Rightarrow \beta\text{-}\theta\text{-}T_0.$$

Theorem 3.12. *If (X,τ) is β-θ-T_0, then (X,τ) is β-θ-T_2.*

Proof. For any points $x \neq y$ let V be a β-θ-open set that $x \in V$ and $y \notin V$. Then, there exists $U \in \beta\theta O(X,\tau)$ such that $x \in U \subset \beta Cl_\theta(U) \subset V$. By Lemma 2.1 $\beta Cl_\theta(U) \in \beta R(X,\tau)$. Then $\beta Cl_\theta(U)$ is β-θ-open and also $X - \beta Cl_\theta(U)$ is a β-θ-open set containing y. Therefore, X is β-θ-T_2.

Remark 3.13. For a topological space (X,τ) the three properties in the diagram are equivalent.

Theorem 3.14. *A topological space (X,τ) is β-θ-T_2 if and only if the singletons are β-θ-closed sets.*

Proof. Suppose that (X,τ) is β-θ-T_2 and $x \in X$. Let $y \in \{x\}^c$. Then $x \neq y$ and so there exists a β-θ-open set U_y such that $y \in U_y$ but $x \notin U_y$. Consequently $y \in U_y \subset \{x\}^c$ i.e., $\{x\}^c = \bigcup\{U_y / y \in \{x\}^c\}$ which is β-θ-open.

Conversely. Suppose that $\{p\}$ is β-θ-closed for every $p \in X$. Let $x, y \in X$ with $x \neq y$. Now $x \neq y$ implies that $y \in \{x\}^c$. Hence$\{x\}^c$ is a β-θ-open set containing y but not x. Similarly $\{y\}^c$ is a β-θ-open set containing x but not y. From Remark 3.13, X is a β-θ-T_2 space.

Theorem 3.15. *For a topological space (X,τ), the following properties are equivalent:*

(1) *For every pair of distinct points* $x, y \in X$*, there exist* $U \in \beta\theta O(X, x)$ *and* $V \in \beta\theta O(X, y)$ *such that* $\beta Cl_\theta(U) \cap \beta Cl_\theta(V) = \emptyset$*;*

(2) (X, τ) *is* β-θ-T_2*;*

(3) (X, τ) *is* β-T_2*;*

(4) *For every pair of distinct points* $x, y \in X$*, there exist* $U, V \in \beta O(X)$ *such that* $x \in U$*,* $y \in V$ *and* $\beta Cl(U) \cap \beta Cl(V) = \emptyset$*;*

(5) *For every pair of distinct points* $x, y \in X$*, there exist* $U, V \in \beta R(X)$ *such that* $x \in U$*,* $y \in V$ *and* $U \cap V = \emptyset$*.*

Proof. (1) $\Rightarrow$ (2): This is obvious.

(2) $\Rightarrow$ (3): Since $\beta\theta O(X) \subset \beta O(X)$, the proof is obvious.

(3) $\Rightarrow$ (4): This follows from Lemma 5.2 of [12].

(4) $\Rightarrow$ (5): By Lemma 2.1, $\beta Cl(U) \in \beta R(X)$ for every $U \in \beta O(X)$ and the proof immediately follows.

(5) $\Rightarrow$ (1): By Lemma 2.1, every β-regular set is β-θ-open and β-θ-closed. Hence the proof is obvious.

Theorem 3.16. *A topological space* (X, τ) *is* β-θ-R_0 *if it is* β-θ-T_2.

Proof. Let U be any β-θ-open set of X and $x \in U$. Since $\{x\}$ is β-θ-closed (by Theorem 3.14), $\beta Cl_\theta(\{x\}) = \{x\} \subset U$. Therefore (X, τ) is β-θ-R_0.

Example 3.17. Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a\}, \{a, b\}\}$. Then the topological space (X, τ) is a β-θ-R_0 space which is not β-θ-T_2.

Theorem 3.18. *Suppose that in every topological space, each singleton is* β-θ*-open or* β-θ*-closed. Then* (X, τ) *is* β-θ-R_0 *if and only if it is* β-θ-T_2.

Proof. Necessity. Suppose that (X, τ) is β-θ-R_0. For each point $x \in X$, we have by hypothesi that $\{x\}$ is β-θ-open or β-θ-closed in X. If $\{x\}$ is β-θ-open, then $\beta Cl_\theta(\{x\}) \subset \{x\}$ and hence $\{x\}$ is β-θ-closed. By Theorem 3.14 (X, τ) is β-θ-T_2.
Sufficiency. Theorem 3.16.

4. β-θ-R_1 spaces

Definition 4. A topological space (X, τ) is said to be β-θ-R_1 if for x, y in X with $\beta Cl_\theta(\{x\}) \neq \beta Cl_\theta(\{y\})$, there exist disjoint β-θ-open sets U and V such that $\beta Cl_\theta(\{x\})$ is a subset of U and $\beta Cl_\theta(\{y\})$ is a subset of V.

Clearly every β-θ-R_1 space is β-θ-R_0. Indeed let U be a β-θ-open set such that $x \in U$. If $y \notin U$, then since $x \notin \beta Cl_\theta(\{y\})$, $\beta Cl_\theta(\{x\}) \neq \beta Cl_\theta(\{y\})$. Hence, there exists a β-θ-open set V_y such that $\beta Cl_\theta(\{y\}) \subset V_y$ and $x \notin V_y$, which implies $y \notin \beta Cl_\theta(\{x\})$. Thus $\beta Cl_\theta(\{x\}) \subset U$. Therefore (X,τ) is β-θ-R_0.

Example 4.1. Let $X = \{a,b,c\}$ and $\tau = \{X, \emptyset, \{a,b\}\}$. Then the topological space (X,τ) is a β-θ-R_1 space.

Theorem 4.2. *If a topological space (X,τ) is β-θ-T_2 then (X,τ) is β-θ-R_1.*

Proof. Let X be a β-θ-T_2 space. If $x, y \in X$ such that $\beta Cl_\theta(\{x\}) \neq \beta Cl_\theta(\{y\})$, then $x \neq y$. Therefore there exists disjoint β-θ-open sets U and V such that $x \in U$ and $y \in V$; hence $\beta Cl_\theta(\{x\}) = \{x\} \subset U$ and $\beta Cl_\theta(\{y\}) = \{y\} \subset V$. Hence X is β-θ-R_1.

Theorem 4.3. *Suppose that in every topological space, each singleton is β-θ-open or β-θ-closed. Then (X,τ) is β-θ-R_1 if and only if it is β-θ-R_0.*

Proof. It follows from the observation above, plus Theorem 4.2 and Theorem 3.18.

Theorem 4.4. *For a topological space (X,τ), the following statements are equivalent :*
(1) (X,τ) is β-θ-R_1 ,
(2) If $x, y \in X$ such that $\beta Cl_\theta(\{x\}) \neq \beta Cl_\theta(\{y\})$, then there exist β-θ-closed sets F_1 and F_2 such that $x \in F_1$, $y \notin F_1$, $y \in F_2$, $x \notin F_2$ and $X = F_1 \cup F_2$.

Proof. (1) $\Rightarrow$ (2) : Let $x, y \in X$ such that $\beta Cl_\theta(\{x\}) \neq \beta Cl_\theta(\{y\})$. Therefore, there exist disjoint β-θ-open sets U_1 and U_2 such that $x \in U_1$ and $y \in U_2$. Then $F_1 = X - U_2$ and $F_2 = X - U_1$ are β-θ-closed sets such that $x \in F_1$, $y \notin F_1$, $y \in F_2$, $x \notin F_2$ and $X = F_1 \cup F_2$.

(2) $\Rightarrow$ (1) : Suppose that x and y are distinct points of X such that $\beta Cl_\theta(\{x\}) \neq \beta Cl_\theta(\{y\})$. Therefore there exist β-θ-closed sets F_1 and F_2 such that $x \in F_1$, $y \notin F_1$, $y \in F_2$, $x \notin F_2$ and $X = F_1 \cup F_2$. Now, we set $U_1 = X - F_2$ and $U_2 = X - F_1$, so that we obtain that $x \in U_1$, $y \in U_2$, $U_1 \cap U_2 = \emptyset$ and U_1, U_2 are β-θ-open. This shows that (X,τ) is β-θ-T_2. It follows from Theorem 4.2 that (X,τ) is β-θ-R_1.

Theorem 4.5. *A topological space (X,τ) is β-θ-R_1 if and only if for $x, y \in X, \beta Ker_\theta(\{x\}) \neq \beta Ker_\theta(\{y\})$, there exist disjoint β-θ-open sets U and V such that $\beta Cl_\theta(\{x\}) \subset U$ and $\beta Cl_\theta(\{y\}) \subset V$.*

Proof. It follows from Lemma 3.1(3).

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