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ON CERTAIN CLASSES OF MEROMORPHICALLY P-VALENT STARLIKE AND P-VALENT QUASI-CONVEX FUNCTIONS

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Abstract. By making use of a familiar analogue of the generalized hypergeometric function, we introduce and investigate some inclusion properties of certain class of meromorphically p-valent starlike functions of order λ and certain class of meromorphically p-valent quasi-convex functions of order λ.

1. Introduction

Let Σ_p be the class of functions f of the form:

$$f(z) = z^{-p} + \sum_{k=1}^{\infty} a_k z^{k-p} \quad (p \in \mathbf{N} = \{1, 2, \ldots\}), \tag{1.1}$$

which are analytic in the punctured unit disc $\mathscr{U}^* = \{z : z \in C \text{ and } 0 < |z| < 1\} = \mathscr{U} \setminus \{0\}$. For a function $f(z) \in \Sigma_p$ given by (1.1) and $g(z) \in \Sigma_p$ a function given by

$$g(z) = z^{-p} + \sum_{k=1}^{\infty} b_k z^{k-p} \quad (z \in \mathscr{U}^*), \tag{1.2}$$

the Hadamard product (or convolution) of $f(z)$ and $g(z)$ is defined by

$$(f * g)(z) = z^{-p} + \sum_{k=1}^{\infty} a_k b_k z^{k-p} = (g * f)(z). \tag{1.3}$$

For complex parameters

$$\alpha_1, \ldots, \alpha_q \text{ and } \beta_1, \ldots, \beta_s \; (\beta_j \notin \mathbf{Z}_0 = \{0, -1, -2, \ldots\};\; j = 1, 2, \ldots, s),$$

we now define the generalized hypergeometric function ${}_qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z)$ by (see, for example, [15])

$${}_qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_q)_k}{(\beta_1)_k \cdots (\beta_s)_k k!} z^k \tag{1.4}$$

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$$(q \le s+1; q, s \in \mathbf{N}_0 = \mathbf{N} \cup \{0\}; z \in \mathcal{U}),$$

where $(\theta)_\nu$ is the Pochhammer symbol defined by

$$(\theta)_\nu = \begin{cases} 1 & (\nu = 0; \theta \in C \setminus \{0\}), \\ \theta(\theta+1)\cdots(\theta+\nu-1) & (\nu \in \mathbf{N}; \theta \in C). \end{cases} \tag{1.5}$$

Due to Dziok and Srivastava [7] (see also [8]) we consider a linear operator

$$H_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s) : \Sigma_p \to \Sigma_p, \tag{1.6}$$

which is defined by the following Hadamard product (or convolution):

$$H_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s) f(z) = \left\{ z^{-p} {}_qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) \right\} * f(z). \tag{1.7}$$

We observe that, for a function $f(z)$ of the form (1.1), we have

$$H_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s) f(z) = z^{-p} + \sum_{k=1}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_q)_k}{(\beta_1)_k \cdots (\beta_s)_k k!} a_k z^{k-p}. \tag{1.8}$$

If, for convenience, we write

$$H_{p,q,s}(\alpha_1) = H_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s), \tag{1.9}$$

then one can easily verify from the definition (1.7) that

$$z(H_{p,q,s}(\alpha_1) f(z))' = \alpha_1 H_{p,q,s}(\alpha_1+1) f(z) - (\alpha_1+p) H_{p,q,s}(\alpha_1) f(z). \tag{1.10}$$

The linear operator $H_{p,q,s}(\alpha_1)$ was investigated recently by Liu and Srivastava [13] and Aouf [2]. In particular, for $q=2,\ s=1$ and $\alpha_2 = 1$, we obtain the linear operator:

$$\ell_p(\alpha_1, \beta_1) f(z) = H_p(\alpha_1, 1; \beta_1) f(z),$$

which was introduced and studied by Liu and Srivastava [12]. Also for $q=2,\ s=1,\ \alpha_1 = n+p\ (n > -p; p \in \mathbf{N})$ and $\beta_1 = \alpha_2 = p$, we obtain the linear operator on Σ_p:

$$\begin{aligned} H_{p,2,1}(n+p, p; p) f(z) &= D^{n+p-1} f(z) \\ &= \frac{1}{z^p(1-z)^{n+p}} * f(z) \end{aligned}$$

or equivalently by

$$D^{n+p-1} f(z) = \frac{1}{z^p} \left(\frac{z^{n+2p-1} f(z)}{(n+p-1)!} \right)^{(n+p-1)} \quad (n > -p\,; p \in \mathbf{N}; f \in \Sigma_p).$$

The operator $D^{n+p-1} f(z)\ (f \in \Sigma_p)$ was studied by Cho [5], Cho and Owa [6], Aouf [1], Aouf and Srivastava [3] and Uralegaddi and Somanatha [16].

Making use of the operator $H_{p,q,s}(\alpha_1)$, we define two classes of functions connected with the class of p–valent meromorphilally starlike and quasi–convex functions.

Definition 1. Let $p \in \mathbf{N}$, $0 \le \lambda < p$ and $\alpha_1 > 0$. A function $f(z) \in \Sigma_p$ is said to be in the class $\mathcal{J}_{p,q,s}(\alpha_1;\lambda)$ if it satisfies the following condition:

$$\Re\mathfrak{e}\left\{\frac{\alpha_1 H_{p,q,s}(\alpha_1+1)f(z)}{H_{p,q,s}(\alpha_1)f(z)} - (\alpha_1+p)\right\} < -\lambda \quad (z \in \mathcal{U}) \tag{1.11}$$

or, by using (1.10), if it satisfies the following condition:

$$\Re\mathfrak{e}\left\{\frac{z(H_{p,q,s}(\alpha_1)f(z))'}{H_{p,q,s}(\alpha_1)f(z)}\right\} < -\lambda \quad (z \in \mathcal{U}). \tag{1.12}$$

We note that for $q = 2$, $s = 1$, $\alpha_1 = n+p$ $(n > -p; p \in \mathbf{N})$ and $\alpha_2 = \beta_1 = p$, the class $\mathcal{J}_{p,2,1}(n+p,p;p,\lambda) = M_{n+p-1}(\lambda)$, $(n > -p; 0 \le \lambda < p; p \in \mathbf{N})$ was studied by Cho and Owa [6].

Definition 2. Let $p \in \mathbf{N}$, $0 \le \lambda < p$ and $\alpha_1 > 0$. A function $f(z) \in \Sigma_p$ is said to be in the class $\Omega_{p,q,s}(\alpha_1;\lambda)$ if and only if there exists a function $g \in \mathcal{J}_{p,q,s}(\alpha_1) := \mathcal{J}_{p,q,s}(\alpha_1;0)$ such that

$$\Re\mathfrak{e}\left\{\frac{z(H_{p,q,s}(\alpha_1)f(z))'}{H_{p,q,s}(\alpha_1)g(z)}\right\} < -\lambda \quad (z \in \mathcal{U}). \tag{1.13}$$

We note that for $q = 2$, $s = 1$, $\alpha_1 = n+p\,(n > -p; p \in \mathbf{N})$ and $\alpha_2 = \beta_1 = p$, the class $\Omega_{p,2,1}(n+p,p;p,\lambda) = \Omega_{n,p}(\lambda)\,(n > -p; 0 \le \lambda < p; p \in \mathbf{N})$ was studied by Kulkarni et al. [11]. If $H_{p,q,s}(\alpha_1)f(z) = f(z)$, $z \in \mathcal{U}$, then the classes $\mathcal{J}_{p,q,s}(\alpha_1;\lambda)$ and $\Omega_{p,q,s}(\alpha_1;\lambda)$ become the classes of p-valent meromorphically starlike or quasi-convex of order λ, respectively.

2. Main results

In proving our main results, we shall need the following lemma due to Jack [10].

Lemma 1 ([10])**.** *Let a function $w(z)$ be analytic in $\mathcal{U}$ with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point $z_0 \in \mathcal{U}$, then*

$$z_0 w'(z_0) = \zeta w(z_0),$$

where ζ is a real number and $\zeta \ge 1$.

Theorem 1. *A function f is in the class $\mathcal{J}_{p,q,s}(\alpha_1;\lambda)$ if and only if there exists an analytic function w, $|w(z)| < 1)$, $w(0) = 0$, such that*

$$H_{p,q,s}(\alpha_1)f(z) = z^{-p}\exp\int_0^z \frac{2(p-\lambda)w(t)}{t[1+w(t)]}\,\mathrm{dt}. \tag{2.1}$$

Proof. If f is in the class $\mathcal{J}_{p,q,s}(\alpha_1;\lambda)$, then there exists a function w, $|w(z)|<1)$, $w(0)=0$, such that

$$\frac{(H_{p,q,s}(\alpha_1)f(z))'}{H_{p,q,s}(\alpha_1)f(z)}=-\frac{p+(2\lambda-p)w(z)}{z[1+w(z)]}.$$

Therefore we obtain

$$\frac{(H_{p,q,s}(\alpha_1)f(z))'}{H_{p,q,s}(\alpha_1)f(z)}+\frac{p}{z}=-\frac{p+(2\lambda-p)w(z)}{z[1+w(z)]}+\frac{p}{z}. \tag{2.2}$$

Integrating (2.2) we get

$$\log\left[z^pH_{p,q,s}(\alpha_1)f(z)\right]=\int_0^z\frac{2(p-\lambda)w(t)}{t[1+w(t)]}\,\mathrm{d}t, \tag{2.3}$$

which gives (2.1). If, conversely, the function $H_{p,q,s}(\alpha_1)f(z)$ is given by (2.1) with an analytic function w, $|w(z)|<1)$, $w(0)=0$, then differentiating (2.1) logarithmically we can obtain (1.12) thus f is in the class $\mathcal{J}_{p,q,s}(\alpha_1;\lambda)$ □

Theorem 2. *If $p\in\mathbf{N}$, $0\le\lambda<p$ and $\alpha_1>0$, then*

$$\mathcal{J}_{p,q,s}(\alpha_1+1;\lambda)\subset\mathcal{J}_{p,q,s}(\alpha_1;\lambda). \tag{2.4}$$

Proof. For $f(z)\in\mathcal{J}_{p,q,s}(\alpha_1+1;\lambda)$, we find from (1.12) that

$$\Re\mathfrak{e}\left\{\frac{z(H_{p,q,s}(\alpha_1+1)f(z))'}{H_{p,q,s}(\alpha_1+1)f(z)}\right\}<-\lambda. \tag{2.5}$$

In order to show that (2.5) implies the inequality (1.12), we define $w(z)$ in $\mathcal{U}$ by

$$\frac{z(H_{p,q,s}(\alpha_1)f(z))'}{H_{p,q,s}(\alpha_1)f(z)}=-\frac{p+(2\lambda-p)w(z)}{1+w(z)}. \tag{2.6}$$

Clearly, $w(z)$ is regular in $\mathcal{U}$ with $w(0)=0$. Using the identity (1.10), the equation (2.6) may be written as

$$\alpha_1\frac{H_{p,q,s}(\alpha_1+1)f(z)}{H_{p,q,s}(\alpha_1)f(z)}=\frac{\alpha_1+(\alpha_1+2p-2\lambda)w(z)}{1+w(z)}. \tag{2.7}$$

Differentiating (2.7) logarithmically with respect to z, we obtain

$$\frac{z(H_{p,q,s}(\alpha_1+1)f(z))'}{H_{p,q,s}(\alpha_1+1)f(z)}=-\frac{p+(2\lambda-p)w(z)}{1+w(z)}+\frac{(\alpha_1+2p-2\lambda)zw'(z)}{\alpha_1+(\alpha_1+2p-2\lambda)w(z)}-\frac{zw'(z)}{1+w(z)}. \tag{2.8}$$

We claim that $|w(z)|<1$ in $\mathcal{U}$. For, otherwise (by Jack's lemma) there exists z_0 in $\mathcal{U}$ such that

$$z_0w'(z_0)=\zeta w(z_0),\ \ |w(z_0)|=1, \tag{2.9}$$

where $\zeta \geq 1$. From (2.8) and (2.9) we obtain

$$\frac{z_0(H_{p,q,s}(\alpha_1+1)f(z_0))'}{H_{p,q,s}(\alpha_1+1)f(z_0)} = -\frac{p+(2\lambda-p)w(z_0)}{1+w(z_0)} + \frac{\left(\alpha_1+2p-2\lambda\right)zw'(z_0)}{\alpha_1+(\alpha_1+2p-2\lambda)w(z_0)} - \frac{zw'(z_0)}{1+w(z_0)}. \tag{2.10}$$

Thus, if $w(z_0) = e^{i\theta}$, then

$$\begin{aligned}\Re\left\{\frac{z_0\left(H_{p,q,s}(\alpha_1+1)f(z_0)\right)'}{H_{p,q,s}(\alpha_1+1)f(z_0)}\right\} &\geq -\lambda+\zeta\Re\left(\frac{\left(\alpha_1+2p-2\lambda\right)e^{i\theta}}{\alpha_1+(\alpha_1+2p-2\lambda)e^{i\theta}} - \frac{e^{i\theta}}{1+e^{i\theta}}\right)\\ &> -\lambda+\zeta\left(\frac{\alpha_1+2p-2\lambda}{\alpha_1+(\alpha_1+2p-2\lambda)} - \frac{1}{2}\right) > -\lambda,\end{aligned}$$

which obviously contradicts (2.5). Hence $|w(z)|<1$ in $\mathscr{U}$ and it follows from (2.6) that $f(z) \in \mathscr{J}_{p,q,s}(\alpha_1;\lambda)$. This completes the proof of Theorem 2. □

Theorem 3. *Let $f(z) \in \Sigma_p$ satisfy the condition*

$$\Re\left\{\frac{z(H_{p,q,s}(\alpha_1)f(z))'}{H_{p,q,s}(\alpha_1)f(z)}\right\} < -\lambda + \frac{p-\lambda}{2(c+p-\lambda)} \quad (z\in\mathscr{U}), \tag{2.11}$$

for $c>0$. Then

$$(F_{c,p}f)(z) = \frac{c}{z^{c+p}}\int_0^z t^{c+p-1}f(t)\ \mathrm{d}t \tag{2.12}$$

belongs to the class $\mathscr{J}_{p,q,s}(\alpha_1;\lambda)$.

Proof. Let $f(z) \in \mathscr{J}_{p,q,s}(\alpha_1;\lambda)f(z)$. Define $w(z)$ in $\mathscr{U}$ by

$$\frac{z(H_{p,q,s}(\alpha_1)(F_{c,p}f)(z))'}{H_{p,q,s}(\alpha_1)(F_{c,p}f)(z)} = -\frac{p+(2\lambda-p)w(z)}{1+w(z)}. \tag{2.13}$$

Clearly, $w(z)$ is regular and $w(0)=0$. Using the identity

$$z(H_{p,q,s}(\alpha_1)(F_{c,p}f)(z))' = cH_{p,q,s}(\alpha_1)f(z) - (c+p)H_{p,q,s}(\alpha_1)(F_{c,p}f)(z), \tag{2.14}$$

the equation (2.13) may be written as

$$\frac{H_{p,q,s}(\alpha_1)f(z)}{H_{p,q,s}(\alpha_1)(F_{c,p}f)(z)} = \frac{c+(c+2p-2\lambda)w(z)}{c(1+w(z))}. \tag{2.15}$$

Differentiating (2.15) logarithmically with respect to z, we obtain

$$\frac{z(H_{p,q,s}(\alpha_1)f(z))'}{H_{p,q,s}(\alpha_1)f(z)} = -\frac{p+(2\lambda-p)w(z)}{1+w(z)} + \frac{\left(c+2p-2\lambda\right)zw'(z)}{c+(c+2p-2\lambda)w(z)} - \frac{zw'(z)}{1+w(z)}. \tag{2.16}$$

We claim that $|w(z)| < 1$ in $\mathscr{U}$. For otherwise (by Jack's lemma) there exists z_0 in $\mathscr{U}$ such that

$$z_0 w'(z_0) = \zeta w(z_0), \tag{2.17}$$

where $|w(z_0)| = 1$ and $\zeta \geq 1$. Combining (2.16) and (2.17), we obtain

$$\mathfrak{Re}\, \frac{z_0 (H_{p,q,s}(\alpha_1) f(z_0))'}{H_{p,q,s}(\alpha_1) f(z_0)} \geq -\lambda + \zeta \frac{p-\lambda}{2(c+p-\lambda)} > -\lambda, \tag{2.18}$$

which contradicts (2.11). Hence $|w(z)| < 1$ in $\mathscr{U}$ and from (2.13) it follows that $(F_{c,p} f)(z) \in \mathscr{J}_{p,q,s}(\alpha_1;\lambda)$. This completes the proof of Theorem 3. □

Similarly, from Theorem 3, we have

Corollary 1. *Let $f(z) \in \mathscr{J}_{p,q,s}(\alpha_1;\lambda)$. Then $(F_{c,p} f)(z)$ defined by (2.12) belongs to the class $\mathscr{J}_{p,q,s}(\alpha_1;\lambda)$.*

Remark 1. (i) For $p = 1$, $q = s = 1$, $\alpha_1 = \beta_1 = 1$, $c = 1$ and $\lambda = 0$, we note that Theorem 3 extends a result of Bajpai [4].

(ii) For $p = 1$, $q = s = 1$, $\alpha_1 = \beta_1 = 1$ and $\lambda = 0$, we note that Theorem 3 extends a result of Goel and Sohi [9].

For the function $(F_{\alpha_1,p} f)(z)$ defined by (2.12), we have

$$H_{p,q,s}(\alpha_1) f(z) = H_{p,q,s}(\alpha_1+1)(F_{\alpha_1,p} f)(z). \tag{2.19}$$

Thus we obtain following theorem:

Theorem 4. *Let $f(z) \in \mathscr{J}_{p,q,s}(\alpha_1;\lambda)$. Then $(F_{\alpha_1,p} f)(z)$ defined by (2.12) belongs to the class $\mathscr{J}_{p,q,s}(\alpha_1+1;\lambda)$.*

Theorem 5. *Let $(F_{c,p} f)(z) \in \mathscr{J}_{p,q,s}(\alpha_1;\lambda)$ and let $f(z)$ be defined as (2.12). Then*

$$\mathfrak{Re}\left\{\frac{\alpha_1 H_{p,q,s}(\alpha_1+1) f(z)}{H_{p,q,s}(\alpha_1) f(z)} - (\alpha_1 + p)\right\} < -\lambda \quad (|z| < R_c), \tag{2.20}$$

where

$$R_c = \frac{-(p-\lambda+1) + \sqrt{(p-\lambda+1)^2 + c[c+2(p-\lambda)]}}{c+2(p-\lambda)}. \tag{2.21}$$

Proof. Since $(F_{c,p} f)(z) \in \mathscr{J}_{p,q,s}(\alpha_1;\lambda)$, we can write

$$\frac{z(H_{p,q,s}(\alpha_1)(F_{c,p} f)(z))'}{H_{p,q,s}(\alpha_1)(F_{c,p} f)(z)} = -\left[\lambda + (p-\lambda) u(z)\right], \tag{2.22}$$

where $u(z) \in \mathscr{P}$, the class of functions with positive real part in $\mathscr{U}$ and normalized by $u(0) = 1$. Using the equation (2.13) and differentiating (2.22) logarithmically with respect to z, we obtain

$$-\frac{\frac{z(H_{p,q,s}(\alpha_1)f(z))'}{H_{p,q,s}(\alpha_1)f(z)}+\lambda}{p-\lambda} = u(z) + \frac{zu'(z)}{\{(c+p)-[\lambda+(p-\lambda)u(z)]\}}. \tag{2.23}$$

Using the well-known estimates [14]

$$\frac{|zu'(z)|}{\Re\mathfrak{e}\, u(z)} \le \frac{2r}{1-r^2} \quad (|z|=r) \quad \text{and} \quad \Re\mathfrak{e}\, u(z) \le \frac{1+r}{1-r} \quad (|z|=r),$$

the equation (2.23) yields

$$\begin{aligned}\Re\mathfrak{e}\left\{-\frac{\frac{z(H_{p,q,s}(\alpha_1)f(z))'}{H_{p,q,s}(\alpha_1)f(z)}+\lambda}{p-\lambda}\right\} &\ge \Re\mathfrak{e}\left\{u(z) - \left|\frac{zu'(u)}{(c+p)-[\lambda+(p-\lambda)u(z)]}\right|\right\} \\ &\ge \Re\mathfrak{e}\left\{u(z) - \left|\frac{zu'(u)}{(c+p)-[\lambda+(p-\lambda)u(z)]}\right|\right\} \\ &\ge \Re\mathfrak{e}\, u(z)\left\{1-\frac{\frac{|zu'(z)|}{\Re\mathfrak{e}\, u(z)}}{(c+p)-[\lambda+(p-\lambda)\Re\mathfrak{e}\, u(z)]}\right\} \\ &\ge \Re\mathfrak{e}\, u(z)\left\{1-\frac{2r}{(1-r^2)\{c+p-[\lambda+(p-\lambda)\frac{1+r}{1-r}]\}}\right\}. \end{aligned} \tag{2.24}$$

Now the right hand side of (2.24) is positive provided $r < R_c$, where R_c is given by (2.21). Hence we have (2.20). This completes the proof of Theorem 5. •

We shall prove an inclusion property of the general class $\Omega_{p,q,s}(\alpha_1;\lambda)$, associated with the integral operator $(F_{\alpha_1,p}f)(z)$ defined by (2.12), which is stated as :

Theorem 6. *If $f(z) \in \Omega_{p,q,s}(\alpha_1;\lambda)$, then*

$$F(z) = (F_{\alpha_1,p}f)(z) \in \Omega_{p,q,s}(\alpha_1+1;\lambda). \tag{2.25}$$

Proof. Let

$$G(z) = (F_{\alpha_1,p}g)(z) \quad (g \in \mathscr{J}_{p,q,s}(\alpha_1;\lambda)). \tag{2.26}$$

Then, by using Theorem 3, we have

$$g \in \mathscr{J}_{p,q,s}(\alpha_1;\lambda) \Rightarrow G \in \mathscr{J}_{p,q,s}(\alpha_1+1;\lambda) \ (\alpha_1>0;\ 0\le\lambda<p). \tag{2.27}$$

In view of (1.8) and (2.12), the definitions in (2.25) and (2.26) yield

$$z(H_{p,q,s}(\alpha_1)F(z))' = \alpha_1 H_{p,q,s}(\alpha_1)f(z) - (\alpha_1+p)H_{p,q,s}(\alpha_1)F(z) \tag{2.28}$$

and

$$z(H_{p,q,s}(\alpha_1)G(z))' = \alpha_1 H_{p,q,s}(\alpha_1)g(z) - (\alpha_1 + p)H_{p,q,s}(\alpha_1)G(z), \tag{2.29}$$

respectively. Upon expressing the first members of (2.28) and (2.29) by means of the identity (1.10), and then comparing the corresponding right-hand sides with the second members of (2.28) and (2.29), respectively, we obtain (see (2.19))

$$H_{p,q,s}(\alpha_1+1)F(z) = H_{p,q,s}(\alpha_1)f(z) \tag{2.30}$$

and

$$H_{p,q,s}(\alpha_1+1)G(z) = H_{p,q,s}(\alpha_1)g(z). \tag{2.31}$$

Finally, since $f(z) \in \Omega_{p,q,s}(\alpha_1;\lambda)$, we have

$$\mathfrak{Re}\left\{\frac{z(H_{p,q,s}(\alpha_1)f(z))'}{H_{p,q,s}(\alpha_1)g(z)}\right\} < -\lambda \quad (g \in \Omega_{p,q,s}(\alpha_1); z \in \mathcal{U}),$$

which, by virtue of (2.30), (2.31) and (2.27), yields

$$\mathfrak{Re}\left\{\frac{z(H_{p,q,s}(\alpha_1+1)F(z))'}{H_{p,q,s}(\alpha_1+1)G(z)}\right\} < -\lambda \quad (G \in \Omega_{p,q,s}(\alpha_1); z \in \mathcal{U}),$$

and the proof of the assertion (2.25) of Theorem 6 is completed. □

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