TAMKANG JOURNAL OF MATHEMATICS
Volume 42, Number 4, 483-492, Winter 2011
doi:10.5556/j.tkjm.42.2011.483-492

Available online at http://journals.math.tku.edu.tw/

SOME INEQUALITIES IN PSEUDO-HILBERT SPACES

LOREDANA CIURDARIU

Abstract. The aim of this paper is to obtain new versions of the reverse of the generalized triangle inequalities given in [4], and [5] if the pair (a_i, x_i), $i \in \{1,\ldots,n\}$ from Theorem 1 of [4] belongs to $\mathbb{C}\times\mathscr{H}$, where $\mathscr{H}$ is a Loynes Z-space instead of $\mathbb{K}\times X$, X being a normed linear space and $\mathbb{K}$ is the field of scalars. By comparison, in [4] the pair (a_i, x_i), $i \in \{1,\ldots,n\}$ belongs to A^2, where A is a normed algebra over the real or complex number field $\mathbb{K}$. The results will be given in Theorem 1, Theorem 3, Remark 2 and Corollary 3 which represent other interesting variants of Theorem 2.1, Remark 2.2, Theorem 3.2 and Theorem 3.4., see [4].

1. Introduction

We start by presenting in Propositions 2 and 3 two inequalities concerning the monotone sequences of elements that belong to an arbitrary admissible space in the Loynes sense. We continue by giving in Proposition 5 and Corollary 2 other inequalities for the inner product of a monotone sequence of linear operators on $\mathscr{H}$, taking into account the form of the seminorms p and q_p. Also it is good to emphasize, in Proposition 4, that the classical Dunkl-Williams inequality, see [6], [11] and [10] remains true when the norm $\|\ \|$ is replaced with seminorm q_p or even with an arbitrary seminorm. Then several inequalities in pseudo-Hilbert spaces like the reverse of the generalized triangle inequality, using the seminorms $q_p(h) = (p([h,h]))^{1/2}$ on $\mathscr{H}$, are obtained in Theorem 1, Remark 2, Theorem 3 and Corollary 3 as generalizations of the Pecaric and Rajic inequality ([12]), see [5] and [4], Theorem 2.1, Remark 2.2, Theorem 3.2 and Theorem 3.4. In our case the pair (a_i, x_i), $i \in \{1,\ldots,n\}$ belongs to $\mathbb{C}\times\mathscr{H}$, where $\mathscr{H}$ is a Loynes Z-space instead of $\mathbb{K}\times X$, where X is a normed linear space and $\mathbb{K}$ is the field of scalars as in [5] or instead of A^2, where A is a normed algebra over the real or complex number field $\mathbb{K}$ as in [4]. Two interesting papers related to this subject are [1], where several new reverses of the triangle inequality in inner product spaces are presented and [7] where new versions of reverse triangle inequality in Hilbert C^*-modules are given.

2010 *Mathematics Subject Classification*. Primary 47A45; Secondary 42B10.
Key words and phrases. Pseudo-Hilbert spaces(Loynes spaces), seminorms, topology, admissible spaces.

We recall that a locally convex space Z is called admissible in the Loynes sense if the following conditions are satisfied:

Z is complete;

there is a closed convex cone in Z, denoted Z_+, that defines an order relation on Z (that is $z_1 \leq z_2$ if $z_2 - z_1 \in Z_+$);

there is an involution in Z, $Z \ni z \to z^* \in Z$ (that is $z^{**} = z$, $(\alpha z)^* = \overline{\alpha} z^*$, $(z_1 + z_2)^* = z_1^* + z_2^*$), so that $z \in Z_+$ implies $z^* = z$;

the topology of Z is compatible with the order (this means that a basis of convex solid neighbourhoods of the origin exists);

and any monotonously decreasing sequence in Z_+ is convergent.

We will say that a set $C \in Z$ is called *solid* if $0 \leq z' \leq z''$ and $z'' \in C$ implies $z' \in C$.

As an easy example we shall consider $Z = C$, a C^*–algebra with topology and natural involution.

Let Z be an admissible space in the Loynes sense. A linear topological space $\mathscr{H}$ is called pre-Loynes Z–space if it satisfies the following properties:

$\mathscr{H}$ is endowed with a Z–valued *inner product* (gramian), i.e. there exists an application $\mathscr{H} \times \mathscr{H} \ni (h,k) \to [h,k] \in Z$ having the properties: $[h,h] \geq 0$; $[h,h] = 0$ implies $h = 0$; $[h_1 + h_2, h] = [h_1, h] + [h_2, h]$; $[\lambda h, k] = \lambda[h,k]$; $[h,k]^* = [k,h]$;

for all $h, k, h_1, h_2 \in \mathscr{H}$ and $\lambda \in \mathbb{C}$.

The topology of $\mathscr{H}$ is the weakest locally convex topology on $\mathscr{H}$ for which the application $\mathscr{H} \ni h \to [h,h] \in Z$ is continuous. Moreover, if $\mathscr{H}$ is a complete space with this topology, then $\mathscr{H}$ is called Loynes Z–*space*.

Now, considering $Z = C$ as above, Z with $[z_1, z_2] = z_2^* z_1$ is a Loynes-Z space.

An important result which can be used below is given in the next statement, and it was proved in [9].

Let $\mathscr{H}$ and $\mathscr{K}$ be two Loynes Z-spaces. We recall that in [3, 8, 9] an operator $T \in \mathscr{L}(\mathscr{H}, \mathscr{K})$ is called gramian bounded, if there exists a constant $\mu > 0$ so that in the sense of order of Z holds

$$[Th, Th]_{\mathscr{K}} \leq \mu [h,h]_{\mathscr{H}}, \quad h \in \mathscr{H}. \tag{1.1}$$

We denote the class of such operators by $\mathscr{B}(\mathscr{H}, \mathscr{K})$, and $\mathscr{B}^*(\mathscr{H}, \mathscr{K}) = \mathscr{B}(\mathscr{H}, \mathscr{K}) \cap \mathscr{L}^*(\mathscr{H}, \mathscr{K})$.

We also denote the introduced norm by

$$\|T\| = \inf\{\sqrt{\mu},\ \mu > 0 \text{ and satisfies (1.1)}\}. \tag{1.2}$$

Corollary 1. *The space $\mathscr{B}^*(\mathscr{H},\mathscr{K})$ is a Banach space, and its involution $\mathscr{B}^*(\mathscr{H},\mathscr{K})$ in $\mathscr{B}^*(\mathscr{K},\mathscr{H})$ satisfies*

$$\|T^*T\| = \|T\|^2, \quad T \in \mathscr{B}^*(\mathscr{H},\mathscr{K}).$$

In particular $\mathscr{B}^*(\mathscr{H})$ is a C^*–algebra.

The following two results were presented in [3].

Lemma 1. *If p is a continuous and monotonous seminorm on Z, then $q_p(h) = (p([h,h]))^{1/2}$ is a continuous seminorm on $\mathscr{H}$.*

Proposition 1. *If $\mathscr{H}$ is a pre-Loynes Z-space and $\mathscr{P}$ is a set of monotonous (increasing) seminorms defining the topology of Z, then the topology of $\mathscr{H}$ is defined by the sufficient and directed set of seminorms $Q_{\mathscr{P}} = \{q_p \mid p \in \mathscr{P}\}$.*

Let Z be an admissible space in the Loynes sense.

Proposition 2. *If $a_1 \le a_2 \le \cdots \le a_n$, $a_i \in \mathbf{R}$, $i = \overline{1,n}$ and $b_1 \le b_2 \le \cdots \le b_n$, $b_i \in Z$, $i = \overline{1,n}$ then we have $a_1b_1 + a_2b_2 + \cdots + a_nb_n \ge \frac{1}{n}(a_1 + a_2 + \cdots + a_n)(b_1 + b_2 + \cdots + b_n)$;*

Proof. We shall use induction. □

Proposition 3. *Let $a_1 > a_2 > \cdots > a_n$, $a_i \in R$ and $b_1 > b_2 > \cdots > b_n$, $b_i \in Z$, Z being an admissible space in the Loynes sense. Then*

(i) $a_1b_1 + a_2b_2 + \cdots + a_nb_n > a_1b_n + a_2b_{n-1} + \cdots + a_nb_1$;

(ii) *If $\sigma : \{1,2,\ldots,n\} \longrightarrow \{1,2,\ldots,n\}$ is a bijective function then*

$$a_1b_1 + a_2b_2 + \cdots + a_nb_n \ge a_1b_{\sigma_1} + a_2b_{\sigma_2} + \cdots + a_nb_{\sigma_n} \ge a_1b_n + a_2b_{n-1} + \cdots + a_nb_1.$$

From the second assertion will easily result (i).

2. The main results

Proposition 4. *Let $\mathscr{H}$ be a Loynes Z-space and $x, y \in \mathscr{H} \setminus \{\theta\}$. We shall prove that*

$$q_p(x-y) \ge \frac{1}{4}(q_p(x) + q_p(y)) \cdot q_p\Big(\frac{x}{q_p(x)} - \frac{y}{q_p(y)}\Big),$$

where $(q_p)_p$ is a family of seminorms which generates the topology of $\mathscr{H}$.

Proof. Denoting the right side of the inequality by E, and assuming, for example, that $q_p(x) \geq q_p(y)$, we obtain the following increase

$$E \leq \frac{q_p(x)}{2} q_p\left(\frac{x}{q_p(x)} - \frac{y}{q_p(y)}\right).$$

Knowing that q_p is a seminorm, we first have $q_p(x) \geq 0$ and then the right side of the last inequality becomes

$$\frac{q_p(x)}{2} q_p\left(\frac{x}{q_p(x)} - \frac{y}{q_p(y)}\right) = \frac{1}{2} q_p\left(q_p(x)\frac{x}{q_p(x)} - q_p(x)\frac{y}{q_p(y)}\right) = \frac{1}{2} q_p\left(x - \frac{q_p(x)}{q_p(y)} y\right)$$

So now we only need to check the inequality,

$$\frac{1}{2} q_p\left(x - \frac{q_p(x)}{q_p(y)} y\right) \leq q_p(x-y).$$

But,

$$\frac{1}{2} q_p\left(x - \frac{q_p(x)}{q_p(y)} y\right) = \frac{1}{2} q_p\left(x - y + y - \frac{q_p(x)}{q_p(y)} y\right) \leq \frac{1}{2} q_p(x-y) + \frac{1}{2} q_p\left(y - \frac{q_p(x)}{q_p(y)} y\right),$$

so that it is sufficient to prove that

$$\frac{1}{2} q_p(x-y) + \frac{1}{2} q_p\left(y - \frac{q_p(x)}{q_p(y)} y\right) \leq q_p(x-y),$$

or

$$q_p\left(y - \frac{q_p(x)}{q_p(y)} y\right) \leq q_p(x-y)$$

since

$$\begin{aligned} q_p\left(y - \frac{q_p(x)}{q_p(y)} y\right) &= q_p\left(\left(1 - \frac{q_p(x)}{q_p(y)}\right) y\right) = \left| 1 - \frac{q_p(x)}{q_p(y)} \right| q_p(y) \\ &= | q_p(y) - q_p(x) | = | q_p(x) - q_p(y) | \leq q_p(x-y) \end{aligned}$$

which is obvious because

$$q_p(x) = q_p(x-y+y) \leq q_p(x-y) + q_p(y)$$

if we consider the modulus of the difference $q_p(x) - q_p(y)$. □

Remark 1. (a) We notice that no particular condition of seminorm q_p appears in the proof of the above Proposition 4, so that this property is true for every family of seminorms which defines the topology of the considered space.

(b) It is obvious that for every monotone seminorm $p \in \mathscr{P}_Z$, where Z is an admissible space, we have

$$\begin{aligned} &p([x,x]) + p([y,y]) + p([z,z]) \\ &\quad \geq p([x,x])^{\frac{1}{2}} p([y,y])^{\frac{1}{2}} + p([y,y])^{\frac{1}{2}} p([z,z])^{\frac{1}{2}} + p([z,z])^{\frac{1}{2}} p([x,x])^{\frac{1}{2}}, \end{aligned}$$

$(\forall)\ x, y, z \in \mathscr{H}$.

(c) We get

$$\frac{q_p(x) + q_p(y)}{q_p(z)} + \frac{q_p(y) + q_p(z)}{q_p(x)} + \frac{q_p(z) + q_p(x)}{q_p(y)} \geq 6,$$

$(\forall)\ x,\ y,\ z \in \mathscr{H}$ with $q_p(x) \neq 0;\ q_p(y) \neq 0;\ q_p(z \neq 0)$.

(d) For every $q_p \in \mathscr{P}_{\mathscr{H}}$ as before, the Jensen inequality holds:

$$q_p : \mathscr{H} \longrightarrow R;\ q_p(\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n) \leq \lambda_1 q_p(x_1) + \lambda_2 q_p(x_2) + \cdots + \lambda_n q_p(x_n),$$

$(\forall)\ x_1,\ x_2, \ldots, x_n \in \mathscr{H}$ with $\lambda_i \geq 0,\ i = \overline{1,n}$ and

$$\sum_{i=1}^{n} \lambda_i = 1.$$

(e) If p is increasing, then $p(\frac{[x,x]}{q_p^2(x)} + \frac{[y,y]}{q_p^2(y)}) \geq \frac{p([x,y]+[y,x])}{q_p(x) q_p(y)}$.

(f) Moreover, if p is increasing, it holds

$$q_p^2(\frac{x}{q_p(x)} - \frac{y}{q_p(y)}) \geq p(\frac{[x,x]}{q_p^2(x)} + \frac{[y,y]}{q_p^2(y)}) - 2.$$

Proposition 5. *Let $f : R \longrightarrow R,\ n \in N^*,\ i = \overline{1,n}$. Then the following inequalities are true:*

(a) *If f is increasing, then*

$$n \cdot \sum_{i=1}^{n} [A_i x, x] f(p([A_i x, x])) \geq \sum_{i=1}^{n} [A_i x, x] \sum_{j=1}^{n} f(p([A_j x, x])),$$

where $A_i \in \mathscr{L}(\mathscr{H})$ is an increasing sequence and $p \in \mathscr{P}_Z$ is a family of monotone seminorms on Z, $\mathscr{H}$ being a Loynes Z-space.

(b) *If f is decreasing, then*

$$n \cdot \sum_{i=1}^{n} [A_i x, x] f(p([A_i x, x])) \leq \sum_{i=1}^{n} [A_i x, x] \sum_{j=1}^{n} f(p([A_j x, x])),$$

(c) *If $A_i \geq 0,\ A_i \in \mathscr{L}(\mathscr{H})$, then*

$$n \sum_{i=1}^{n} [A_i x, x] f(q_p^2(A_i^{\frac{1}{2}} x)) \leq \sum_{i=1}^{n} [A_i x, x] \sum_{i=1}^{n} f(q_p^2(A_i^{\frac{1}{2}} x)),$$

(d) *Using the conditions from (c) for the particular case of $f(x) = x$ we obtain*

$$n\sum_{i=1}^{n}[A_i^{\frac{1}{2}}x, A_i^{\frac{1}{2}}x]q_p^2(A_i^{\frac{1}{2}}x) \geq \sum_{i=1}^{n}[A_i^{\frac{1}{2}}x, A_i^{\frac{1}{2}}x]\sum_{i=1}^{n}q_p^2(A_i^{\frac{1}{2}}x),$$

(e) *If $f(x) = x$, then*

$$n\sum_{i=1}^{n}[A_ix,x]p([A_ix,x]) \geq \sum_{i=1}^{n}[A_ix,x]\sum_{i=1}^{n}p([A_ix,x]).$$

Proof. (a) From $A_i, (i, j \in \overline{1,n})$ increasing, it follows that

$$[A_ix,x] \leq [A_jx,x],\ (\forall)\ x \in \mathscr{H},\ i \leq j$$

so considering p and f are increasing we have, $f(p([A_ix,x])) \leq f(p([A_jx,x]))$.

This fact implies

$$([A_ix,x]-[A_jx,x])(f(p([A_ix,x]))-f(p([A_jx,x]))) \geq 0,\ (\forall)\ x \in \mathscr{H},\ i,j \in \overline{1,n}.$$

Summing up, we find

$$\sum_{i=1}^{n}\sum_{j=1}^{n}([A_ix,x]-[A_jx,x])[f(p([A_ix,x]))-f(p([A_jx,x]))] \geq 0,\ (\forall)\ x \in \mathscr{H},\ i,j \in \overline{1,n}$$

or

$$\sum_{i=1}^{n}\sum_{j=1}^{n}([A_ix,x]f(p([A_ix,x]))-[A_jx,x]f(p([A_ix,x]))-[A_ix,x]f(p([A_jx,x])) \\ +[A_jx,x]f(p([A_ix,x]))) \geq 0$$

and this is equivalent to

$$\begin{aligned} 2n\sum_{i=1}^{n}[A_ix,x]f(p([A_ix,x])) &\geq 2\sum_{i=1}^{n}\sum_{j=1}^{n}[A_jx,x]f(p([A_ix,x])) \\ &= 2\sum_{j=1}^{n}[A_jx,x]\sum_{i=1}^{n}f(p([A_ix,x])). \end{aligned}$$

Thus, we have

$$n\sum_{i=1}^{n}[A_ix,x]f(p([A_ix,x])) \geq \sum_{i=1}^{n}[A_ix,x]\sum_{i=1}^{n}f(p([A_ix,x])).$$

Similarly, we deduce (b), (c), (d), (e). □

Corollary 2. *If $A_n \in \mathscr{L}(\mathscr{H})$ is increasing with $A_n \geq 0$ then*

$$\sum_{i=1}^{n}[A_ix,x]\sum_{i=1}^{n}\frac{1}{p[A_ix,x]} \geq n\sum_{i=1}^{n}\frac{[A_ix,x]}{p([A_{n-i+1}x,x])}$$

or

$$\sum_{i=1}^{n}[A_i^{\frac{1}{2}}x, A_i^{\frac{1}{2}}x]\sum_{i=1}^{n}\frac{1}{q_p^2(A_i^{\frac{1}{2}}x)} \geq n\sum_{i=1}^{n}\frac{[A_i^{\frac{1}{2}}x, A_i^{\frac{1}{2}}x]}{q_p^2(A_{n-i+1}^{\frac{1}{2}}x)}.$$

Proof. Let us consider the function $f:\{x_1,x_2,\dots,x_n\}\longrightarrow\{\frac{1}{x_1},\dots,\frac{1}{x_n}\}$,

$$f(x_i)=\frac{1}{x_{n-i+1}},\ i=\overline{1,n}.$$

It is obvious that f is decreasing. If we take $x_i=p([A_ix,x])\in R_+$ we see that

$$\sum_{i=1}^n[A_ix,x]\sum_{i=1}^n\frac{1}{p([A_{n-i+1}x,x])}\geq n\sum_{i=1}^n\frac{[A_ix,x]}{p([A_{n-i+1}x,x])}$$

or

$$\sum_{i=1}^n[A_ix,x]\sum_{i=1}^n\frac{1}{p([A_ix,x])}\geq n\sum_{i=1}^n\frac{[A_ix,x]}{p([A_{n-i+1}x,x])}\qquad\square$$

The following results are some generalizations of the Pecaric-Rajic inequality in normed spaces, see [4].

Theorem 1. *If $(a_i,x_i)\in\mathbb{C}\times\mathscr{H}$, $i\in\{1,\dots,n\}$, where $\mathscr{H}$ is a Loynes Z-space, then*

$$\begin{aligned}\max_{k\in\{1,\dots,n\}}&\{|a_k|q_p(\sum_{j=1}^n x_j)-\sum_{j=1}^n|a_j-a_k|q_p(x_j)\}\\&\leq q_p(\sum_{j=1}^n a_jx_j)\\&\leq\min_{k\in\{1,\dots,n\}}\{|a_k|q_p(\sum_{j=1}^n x_j)+\sum_{j=1}^n|a_j-a_k|q_p(x_j)\},\end{aligned}\qquad\square$$

for every $q_p\in Q_{\mathscr{P}}$.

Proof. We notice that for any $k\in\{1,\dots,n\}$ we have the equality

$$\sum_{j=1}^n a_jx_j=a_k(\sum_{j=1}^n x_j)+\sum_{j=1}^n(a_j-a_k)x_j.$$

Taking into consideration the seminorm and using the triangle inequality and a seminorm property, we deduce that

$$\begin{aligned}q_p(\sum_{j=1}^n a_jx_j)&\leq q_p(a_k(\sum_{j=1}^n x_j))+q_p(\sum_{j=1}^n(a_j-a_k)x_j)\\&\leq|a_k|q_p(\sum_{j=1}^n x_j)+\sum_{j=1}^n q_p((a_j-a_k)x_j)\\&=|a_k|q_p(\sum_{j=1}^n x_j)+\sum_{j=1}^n|a_j-a_k|q_p(x_j),\end{aligned}$$

for any $k\in\{1,\dots,n\}$, so that the second part of (*) holds.

Considering
$$\sum_{j=1}^{n} a_j x_j = a_k(\sum_{j=1}^{n} x_j) - \sum_{j=1}^{n} (a_k - a_j) x_j$$
we shall obtain
$$\begin{aligned}
q_p(\sum_{j=1}^{n} a_j x_j) &= q_p(a_k(\sum_{j=1}^{n} x_j) - \sum_{j=1}^{n} (a_k - a_j) x_j) \\
&\geq |q_p(a_k(\sum_{j=1}^{n} x_j)) - q_p(\sum_{j=1}^{n} (a_k - a_j) x_j)| \\
&\geq q_p(a_k(\sum_{j=1}^{n} x_j)) - q_p(\sum_{j=1}^{n} (a_k - a_j) x_j) \\
&\geq |a_k| q_p(\sum_{j=1}^{n} x_j) - \sum_{j=1}^{n} q_p((a_k - a_j) x_j) \\
&= |a_k| q_p(\sum_{j=1}^{n} x_j) - \sum_{j=1}^{n} |a_k - a_j| q_p(x_j)
\end{aligned}$$
for any $k \in \{1, \dots, n\}$, which proves the first part of the inequality. □

Theorem 2. *In fact this inequality is true for every seminorm p from a family of seminorms which defines the topology of the linear space considered.*

Remark 2. If there exists a $r > 0$ so that $q_p(a_j - a_k) \leq r\ q_p(a_k)$ for any $j, k \in \{1, \dots, n\}$, then we obtain
$$q_p(\sum_{j=1}^{n} a_j x_j) \leq \min_{k \in \{1,\dots,n\}} |a_k| [q_p(\sum_{j=1}^{n} x_j) + r \sum_{j=1}^{n} q_p(x_j)].$$

For $a_k = \frac{1}{q_p(x_k)}$, with $q_p(x_k) \neq 0,\ k \in \{1, \dots, n\}$ the inequality from (*) produces the following results established by Pecaric and Rajic in [12] for norms:
$$\begin{aligned}
&\max_{k \in \{1,\dots,n\}} \{\frac{1}{q_p(x_k)} [q_p(\sum_{j=1}^{n} x_j) - \sum_{j=1}^{n} |q_p(x_j) - q_p(x_k)|]\} \\
&\leq q_p(\sum_{j=1}^{n} \frac{x_j}{q_p(x_j)}) \\
&\leq \min_{k \in \{1,\dots,n\}} \{\frac{1}{q_p(x_k)} [q_p(\sum_{j=1}^{n} x_j) + \sum_{j=1}^{n} |q_p(x_j) - q_p(x_k)|]\},
\end{aligned}$$
and then we get the following reverse of the generalized triangle inequality of M. Kato:
$$\min_{k \in \{1,\dots,n\}} \{q_p(x_k)\} [n - q_p(\sum_{j=1}^{n} \frac{x_j}{q_p(x_j)})] \leq \sum_{j=1}^{n} q_p(x_j) - q_p(\sum_{j=1}^{n} x_j)$$

$$\leq \max_{k\in\{1,\ldots,n\}}\{q_p(x_k)\}[n-q_p(\sum_{j=1}^{n}\frac{x_j}{q_p(x_j)})].$$

If we now take $a_k = q_p(x_k),\ k \in \{1,\ldots,n\}$ then from $(*)$ we shall have,

$$\max_{k\in\{1,\ldots,n\}}\{q_p(x_k)q_p(\sum_{j=1}^{n}x_j)-\sum_{j=1}^{n}|q_p(x_j)-q_p(x_k)|q_p(x_j)\}]$$
$$\leq q_p(\sum_{j=1}^{n}q_p(x_j)x_j)\leq \min_{k\in\{1,\ldots,n\}}\{q_p(x_k)q_p(\sum_{j=1}^{n}x_j)+\sum_{j=1}^{n}|q_p(x_j)-q_p(x_k)|q_p(x_j)\}.$$

Remark 3. We notice that in these inequalities no special properties of the seminorm q_p were used, so the inequalities are true for every arbitrary seminorm.

Lemma 2. *If we take $a, b \in \mathbb{C}$ and $x, y \in \mathscr{H}$, then*

$$\max\{|a|q_p(x\pm y)-|(b-a)|q_p(y),\ |b|q_p(x\pm y)-|b-a|q_p(x)\}$$
$$\leq q_p(ax\pm by)\leq \min\{|a|q_p(x\pm y)+|(b-a)|q_p(y),\ |b|q_p(x\pm y)+|b-a|q_p(x)\}.$$

Proof. We use inequality $(*)$ with $n=2,\ a_1=a,\ a_2=b,\ x_1=x$ and $x_2=\pm y$. □

Like in [4], we can obtain the following properties in Loynes spaces:

Theorem 3. *If $a, b \in \mathbb{C}$ and $x, y \in \mathscr{H}$ then,*

$$q_p(ax\pm by)\leq q_p(x\pm y)\min\{|a|,|b|\}+|b-a|\max\{q_p(x),q_p(y)\}$$

and

$$q_p(ax\pm by)\leq q_p(x\pm y)\max\{|a|,|b|\}+|b-a|\min\{q_p(x),q_p(y)\}.$$

Proof. The check will be like in [4]. For the first inequality for example, we use the last part of Lemma 2 and

$$\min\{|a|q_p(x\pm y)+|(b-a)|q_p(y),\ |b|q_p(x\pm y)+|b-a|q_p(x)\}$$
$$\leq q_p(x\pm y)\min\{|a|,|b|\}+|b-a|\max\{q_p(x),q_p(y)\}$$ □

Remark 4. Lemma 2 and Theorem 3 are true if instead of the particular seminorm q_p we take every arbitrary seminorm of a family of seminorms which defines the topology of the considered space.

Corollary 3. (i) *If $a,\ b \in \mathbb{C}$ and $x,\ y \in \mathscr{H}$ then*

$$q_p(ax\pm by)\leq \frac{|a|+|b|}{2}q_p(x\pm y)+|b-a|\frac{q_p(x)+q_p(y)}{2}.$$

(ii) *If the above conditions are satisfied for* a, b, x, y *then*

$$q_p(x \pm y)\max\{|a|,|b|\} - |b-a|\max\{q_p(x), q_p(y)\} \le q_p(ax \pm by)$$

and

$$\min\{|a|,|b|\}|q_p(x) - q_p(y)| - |b-a|\min\{q_p(x), q_p(y)\} \le q_p(ax \pm by).$$

(iii) *Adding the above inequalities (ii), we find*

$$\frac{1}{2}\cdot(|a|+|b|)|q_p(x) - q_p(y)| - |b-a|\cdot\frac{q_p(x)+q_p(y)}{2} \le q_p(ax \pm by),$$

by using the property that $\min\{\alpha,\beta\} + \max\{\alpha,\beta\} = \alpha+\beta,\ \alpha,\ \beta \in R$.

The proof of this corollary will use the same techniques as in [4].

Remark 5. The above inequalities are true for any arbitrary seminorm in a family of seminorms which defines the topology of the considered space.

References

[1] A. H. Ansari and M. S. Moslehian, *Refinements of reverse triangle inequalities in inner product spaces*, J. Inequal. Pure Appl. Math., **6**(2005), art 64, 12 pp.

[2] S. A. Chobanyan and A. Weron, *Banach-space-valued stationary processes and their linear prediction*, Dissertations Math., **125**, (1975), 1–45.

[3] L. Ciurdariu, Classes of linear operators on pseudo-Hilbert spaces and applications, Part I, Monografii matematice, Tipografia Universitatii de Vest din Timişoara, 2006.

[4] S. S. Dragomir, *On some inequalities in normed algebras*, Journal of inequalities in pure and applied mathematics, **9** (2008), Issue 1, Art. 5, 10 pp.

[5] S. S. Dragomir, *A generalization of the Pecaric-Rajic inequality in normed linear spaces*, Preprint, RGMIA Res. Rep. Coll., **10**(2007), Art.3, [ONLINE:http://rmgia.vu.edu.au/v10n3.html].

[6] C. F. Dunkl and K. S. Willams, *A simple norm inequality*, Amer. Math. Monthly, **71** (1964), 53–54.

[7] M. Khosravi, H. Mahyar and M. S. Moslehian, *Reverse triangle inequality in Hilbert* C^* *-modules*, J. Inequal. Pure Appl. Math., **10** (2009), art. 110, 11 pp.

[8] R. M. Loynes, *Linear operators in* VH*-spaces*, Trans. American Math. Soc., **116**(1965), 167–180.

[9] R. M. Loynes, *On generalized positive definite functions*, Proc. London Math. Soc, **3** (1965), 373–384.

[10] L. Maligranda, *Simple norm inequalities*, Amer. Math. Monthly, **113** (2006), 256–260.

[11] P. R. Mercer, *The Dunkl-Williams inequality in an inner-product space*, Math. Inequal. Appl., **10**(2007), 447–450.

[12] J. Pecaric and R. Rajic, *The Dunkl-Wiliams inequality with n elements in normed linear spaces*, Math. Ineq. Appl., **10** (2007), 461–470.

[13] A. Weron and S. A. Chobanyan, *Stochastic processes on pseudo-Hilbert spaces* (russian), Bull. Acad. Polon., Ser. Math. Astr. Phys., tom XX1, **9** (1973), 847–854.

[14] A. Weron, *Prediction theory in Banach spaces*, Proc. of Winter School on Probability, Karpacz, Springer Verlag, London, 1975, 207-228.

Department of Mathematics,"Politehnica" University of Timisoara, P-ta. Victoriei, No.2, 300006-Timisoara, Romania.

E-mail: cloredana43@yahoo.com