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NEW SEVERAL INTEGRAL INEQUALITIES

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Abstract. Several new integral inequalities are presented.

1. Introduction

In [3] the following result was proved

Theorem 1.1. *If $f \geq 0$ is continuous function on $[a,b]$ such that*

$$\int_x^1 f(t)\,dt \geq \int_x^1 t\,dt, \qquad \forall x \in [0,1] \tag{1.1}$$

then

$$\int_0^1 f^{\alpha+1}(x)\,dx \geq \int_0^1 x^\alpha f(x)\,dx, \qquad \forall \alpha > 0, \tag{1.2}$$

and the following question was posed

If f satisfies the above assumption, under what additional assumption can one claim that

$$\int_0^1 f^{\alpha+\beta}(x)\,dx \geq \int_0^1 x^\alpha f^\beta(x)\,dx, \qquad \forall \alpha, \beta > 0. \tag{1.3}$$

The following result as well, was achieved in [2].

Theorem 1.2. *If $f \geq 0$ is a continuous function on $[0,b]$ satisfying*

$$\int_x^b f^\alpha(t)\,dt \geq \int_x^b t^\alpha\,dt, \quad b > 0, \quad \forall x \in [0,b] \tag{1.4}$$

then

$$\int_0^b f^{\alpha+\beta}(x)\,dx \geq \int_0^b x^\alpha f^\beta(x)\,dx, \quad \forall \beta > 0. \tag{1.5}$$

In his role, Hoang [1] generalized the previous results by introducing the following

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Theorem 1.3. *Suppose $f, g \in L^1[a,b]$, $f, g \geq 0$, g is non-decreasing. If*

$$\int_x^b f(t)dt \geq \int_x^b g(t)dt, \qquad \forall x \in [a,b], \tag{1.6}$$

then

$$\int_a^b f^{\alpha+\beta}(x)dx \geq \int_a^b f^{\alpha}(x)g^{\beta}(x)dx, \forall\ \alpha, \beta \geq 0,\ \alpha+\beta \geq 1. \tag{1.7}$$

2. Results

We prove the following

Theorem 2.1. *Let $f, g \geq 0$, and defined on $[a,b]$ .Define*

$$F^{\alpha}(x) = \int_a^x f^{\alpha}(t)dt, \quad G^{\alpha}(x) = \int_a^x g^{\alpha}(t)dt.$$

If

$$\int_x^b F^{\beta}(t)dt \geq \int_x^b G^{\beta}(t)dt, \forall\ x \in [a,b] \tag{2.1}$$

then

$$\int_a^b F^{\alpha+\beta}(x)dx \geq \int_a^b F^{\alpha}(x)G^{\beta}(x)dx, \tag{2.2}$$

for all $\alpha, \beta \geq 0$.

Proof. We have

$$\begin{aligned}
\int_a^b F^{\beta}(x)G^{\alpha}(x)dx &= \int_a^b F^{\beta}(x)\int_a^x g^{\alpha}(t)dtdx \\
&= \int_a^b g^{\alpha}(t)\int_t^b F^{\beta}(x)dxdt \\
&\geq \int_a^b g^{\alpha}(t)\int_t^b G^{\beta}(x)dxdt \\
&= \int_a^b G^{\beta}(x)\int_a^x g^{\alpha}(t)dtdx \\
&= \int_a^b G^{\alpha+\beta}(x)dx.
\end{aligned} \tag{2.3}$$

Applying the Arithmetic-geometric inequality, we have for $\beta, \alpha \geq 0$

$$F^{\alpha}(x)G^{\beta}(x) \leq \frac{\alpha}{\alpha+\beta}F^{\alpha+\beta}(x) + \frac{\beta}{\alpha+\beta}G^{\alpha+\beta}(x).$$

Integrating the above inequality and using 2.3 yields

$$\int_a^b F^{\alpha}(x)G^{\beta}(x)dx \leq \frac{\alpha}{\alpha+\beta}\int_a^b F^{\alpha+\beta}(x)dx + \frac{\beta}{\alpha+\beta}\int_a^b G^{\alpha+\beta}(x)dx$$

$$\leq \frac{\alpha}{\alpha+\beta}\int_a^b F^{\alpha+\beta}(x)\,dx + \frac{\beta}{\alpha+\beta}\int_a^b F^{\alpha}(x)G^{\beta}(x)\,dx,$$

and hence

$$\int_a^b F^{\alpha+\beta}(x)\,dx \geq \int_a^b F^{\alpha}(x)G^{\beta}(x)\,dx. \qquad \square$$

Theorem 2.2. *Let $f, g \geq 0$, and defined on $[a,b]$. Define*

$$F^{\alpha}(x) = \int_a^x f^{\alpha}(t)\,dt, \quad G^{\alpha}(x) = \int_a^x g^{\alpha}(t)\,dt.$$

If

$$\int_x^b F(t)\,dt \geq \int_x^b G(t)\,dt, \quad \forall x \in [a,b] \tag{2.4}$$

then

$$\int_a^b F^{\alpha+\beta}(x)\,dx \geq \int_a^b F^{\alpha}(x)G^{\beta}(x)\,dx, \tag{2.5}$$

for all $\alpha \geq 1$, $\beta \geq 0$.

Proof. Making use of 2.3 with $\beta = 1$, gives

$$\int_a^b F(x)G^{\alpha}(x)\,dx \geq \int_a^b G^{\alpha+1}(x)\,dx. \tag{2.6}$$

By the AG inequality, we have for $\alpha \geq 1$,

$$\frac{1}{\alpha}F^{\alpha}(x) + \frac{\alpha-1}{\alpha}G^{\alpha}(x) \geq F(x)G^{\alpha-1}(x).$$

Integrating the above inequality

$$\frac{1}{\alpha}\int_a^b F^{\alpha}(x)\,dx + \frac{\alpha-1}{\alpha}\int_a^b G^{\alpha}(x)\,dx \geq \int_a^b F(x)G^{\alpha-1}(x)\,dx,$$

which implies with the use of 2.6

$$\begin{aligned}\frac{1}{\alpha}\int_a^b F^{\alpha}(x)\,dx &\geq \int_a^b F(x)G^{\alpha-1}(x)\,dx - \frac{\alpha-1}{\alpha}\int_a^b G^{\alpha}(x)\,dx \\ &\geq \int_a^b G^{\alpha}(x)\,dx - \frac{\alpha-1}{\alpha}\int_a^b G^{\alpha}(x)\,dx \\ &= \frac{1}{\alpha}\int_a^b G^{\alpha}(x)\,dx, \quad \alpha \geq 1.\end{aligned} \tag{2.7}$$

Again, by the AG inequality,

$$F^{\alpha}(x)G^{\beta}(x) \leq \frac{\alpha}{\alpha+\beta}F^{\alpha+\beta}(x) + \frac{\beta}{\alpha+\beta}G^{\alpha+\beta}(x). \tag{2.8}$$

Since, by 2.7, we have for $\alpha \geq 1$, $\beta > 0$,

$$\int_a^b F^{\alpha+\beta}(x)\,dx \geq \int_a^b G^{\alpha+\beta}(x)\,dx, \tag{2.9}$$

the result follows by integrating 2.8 and making use of 2.9. $\square$

Corollary 2.3. *Let $f, g \geq 0$, and defined on $[a,b]$. Define*

$$F^{\alpha}(x) = \int_a^x f^{\alpha}(t)dt, \quad G^{\alpha}(x) = \int_a^x g^{\alpha}(t)dt.$$

If 2.4 is satisfied, then

$$2\int_a^b F^{\alpha}(x)G^{\beta}(x)dx \leq \int_a^b \left(F^{2\alpha}(x) + F^{2\beta}(x)\right)dx, \forall \beta \geq 1/2. \tag{2.10}$$

In particular, if $F^{\beta}(x) \leq F^{\alpha}(x)$, then

$$\int_a^b F^{\alpha}(x)G^{\beta}(x)dx \leq \int_a^b F^{2\alpha}(x)dx. \tag{2.11}$$

Proof. As

$$\left(F^{\alpha}(x) - G^{\beta}(x)\right)^2 \geq 0,$$

opening and integrating gives

$$2\int_a^b F^{\alpha}(x)G^{\beta}(x)dx \leq \int_a^b \left(F^{2\alpha}(x) + G^{2\beta}(x)\right)dx.$$

By 2.7,

$$\int_a^b F^{2\beta}(x)dx \geq \int_a^b G^{2\beta}(x)dx,$$

then, the corollary follows. □

The other way round direction follows from the coming result

Theorem 2.4. *Let $f, g \geq 0$, and defined on $[a,b]$.Define*

$$F^{\alpha}(x) = \int_a^x f^{\alpha}(t)dt, \quad G^{\alpha}(x) = \int_a^x g^{\alpha}(t)dt, \quad \alpha \in R.$$

If

$$\int_a^b F^{\beta}(x)dx \leq \int_a^b G^{\beta}(x)dx, \quad \forall x \in [a,b] \tag{2.12}$$

then

$$\int_a^b F^{\beta-\alpha}(x)dx \leq \int_a^b F^{\beta}(x)G^{-\alpha}(x)dx, \tag{2.13}$$

for all $\beta > \alpha > 0$.

Proof. We have

$$\int_a^b F^{\beta}(x)G^{-\alpha}(x)dx = \int_a^b F^{\beta}(x)\int_a^x g^{-\alpha}(t)dtdx$$

$$= \int_a^b g^{-\alpha}(t) \int_t^b F^{\beta}(x)\,dx\,dt$$

$$\le \int_a^b g^{-\alpha}(t) \int_t^b G^{\beta}(x)\,dx\,dt$$
$$= \int_a^b G^{\beta}(x) \int_a^x g^{-\alpha}(t)\,dt\,dx$$
$$= \int_a^b G^{\beta-\alpha}(x)\,dx. \tag{2.14}$$

Applying the AG inequality, we have for $0 < \alpha < \beta$,

$$F^{\beta}(x)G^{-\alpha}(x) \ge \frac{\beta}{\beta-\alpha}F^{\beta-\alpha}(x) - \frac{\alpha}{\beta-\alpha}G^{\beta-\alpha}(x).$$

Integrating the above inequality, and using 2.14, we obtain

$$\int_a^b F^{\beta}(x)G^{-\alpha}(x)\,dx \ge \frac{\beta}{\beta-\alpha}\int_a^b F^{\beta-\alpha}(x)\,dx - \frac{\alpha}{\beta-\alpha}\int_a^b G^{\beta-\alpha}(x)\,dx$$
$$\ge \frac{\beta}{\beta-\alpha}\int_a^b F^{\beta-\alpha}(x)\,dx - \frac{\alpha}{\beta-\alpha}\int_a^b F^{\beta}(x)G^{-\alpha}(x)\,dx,$$

and hence 2.13 follows. □

Theorem 2.5. *Let $f, g \ge 0$, and defined on $[a,b]$ .Define*

$$F^{\alpha}(x) = \int_x^b f^{\alpha}(t)\,dt, \quad G^{\alpha}(x) = \int_x^b g^{\alpha}(t)\,dt.$$

If

$$\int_a^x F^{\beta}(t)\,dt \ge \int_a^x G^{\beta}(t)\,dt, \forall\ x \in [a,b] \tag{2.15}$$

then

$$\int_a^b F^{\alpha+\beta}(x)\,dx \ge \int_a^b F^{\alpha}(x)G^{\beta}(x)\,dx, \tag{2.16}$$

for all $\alpha, \beta \ge 0$.

Proof. We have

$$\int_a^b F^{\beta}(x)G^{\alpha}(x)\,dx = \int_a^b F^{\beta}(x) \int_x^b g^{\alpha}(t)\,dt\,dx$$
$$= \int_a^b g^{\alpha}(t) \int_a^t F^{\beta}(x)\,dx\,dt$$
$$\ge \int_a^b g^{\alpha}(t) \int_a^t G^{\beta}(x)\,dx\,dt$$
$$= \int_a^b G^{\beta}(x) \int_x^b g^{\alpha}(t)\,dt\,dx$$

$$= \int_a^b G^{\alpha+\beta}(x)\,dx. \tag{2.17}$$

□

The rest can be achieved exactly as it has been done in theorem 2.1, and therefore it is omitted.

References

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