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# CAUCHY-TYPE MEANS FOR POSITIVE LINEAR FUNCTIONALS

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**Abstract**. Some mean-value theorems of the Cauchy type, which are connected with Jensen's inequality, are given in [8] in discrete form and in [11] in integral form. Here we give the generalization of that result for positive linear functionals. Using that result, new means of Cauchy type for positive linear functionals are given. Monotonicity of these new means is also discussed.

## 1. Introduction

The well-known Jensen inequality asserts that for function $f$ holds

$$f\Big(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\Big) \le \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) \tag{1}$$

if and only if $f$ is convex function on interval $I \subseteq \mathbb{R}$, where $p_i$ are positive real numbers and $x_i \in I$ $(i = 1, \dots, n)$, while $P_n = \sum_{i=1}^{n} p_i$.

In [10, p.9] authors gave an estimation of the quotient of differences of the left and the right side of the Jensen inequality for different functions, assuming that $p_i$ and $x_i$ are as above and $P_n = 1$. Their result is a discrete version of a result previously given in [1].

**Theorem 1.1.** [10, p.9] *Let $p_i > 0$ $(i = 1, \dots, n)$ with $P_n = 1$, and $x_i \in I$ $(i = 1, \dots, n)$ are not all equal. Let $f, g : I \to \mathbb{R}$ be twice differentiable functions such that*

$$0 \le m \le f''(x) \le M \quad \text{and} \quad 0 < k \le g''(x) \le K \quad \text{for all } x \in I.$$

*Then*

$$\frac{m}{K} \le \frac{\sum_{i=1}^{n} p_i f(x_i) - f\big(\sum_{i=1}^{n} p_i x_i\big)}{\sum_{i=1}^{n} p_i g(x_i) - g\big(\sum_{i=1}^{n} p_i x_i\big)} \le \frac{M}{k}. \tag{2}$$

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Under the same conditions on $p_i$ and $x_i$, A. McD. Mercer in [7] and [8] gave the following two mean-value theorems of the Lagrange and Cauchy type.

**Theorem 1.2.** *Let $I$ be a compact real interval and $f, g : I \to \mathbb{R}$. Let $x_i \in I$ and $p_i > 0$ $(i = 1, \ldots, n)$ such that $P_n = 1$.*

(i) *If $f \in C^2(I)$, then*

$$\sum_{i=1}^{n} p_i f(x_i) - f\Big(\sum_{i=1}^{n} p_i x_i\Big) = \frac{1}{2} f''(\xi)\left(\sum_{i=1}^{n} p_i x_i^2 - \Big(\sum_{i=1}^{n} p_i x_i\Big)^2\right) \tag{3}$$

*holds for some $\xi \in I$.*

(ii) *If $f, g \in C^2(I)$, then*

$$\frac{\sum_{i=1}^{n} p_i f(x_i) - f\big(\sum_{i=1}^{n} p_i x_i\big)}{\sum_{i=1}^{n} p_i g(x_i) - g\big(\sum_{i=1}^{n} p_i x_i\big)} = \frac{f''(\xi)}{g''(\xi)} \tag{4}$$

*holds for some $\xi \in I$, provided that the denominator of the left-hand side is non-zero.*

**Remark 1.3.** We use $f \in C^2(I)$ to denote that function $f$ has continuous second derivative on $I$.

Furthermore, having in mind the integral Jensen inequality, the authors in [11] gave similar results in integral form.

**Theorem 1.4.** ([11]) *Let $I$ be a compact real interval and $\varphi, \psi : I \to \mathbb{R}$. Let $h$ be an integrable function with respect to a normalized weight $\omega$ on $[a, b] \subset \mathbb{R}$ such that the range of $h$ is a subset of $I$.*

(i) *If $\varphi \in C^2(I)$, then*

$$\int_a^b \varphi(h(x))\,\omega(x)\,dx - \varphi\left(\int_a^b h(x)\omega(x)\,dx\right) = \frac{1}{2}\varphi''(\xi)\left[\int_a^b (h(x))^2\,\omega(x)\,dx - \left(\int_a^b h(x)\omega(x)\,dx\right)^2\right] \tag{5}$$

*holds for some $\xi \in I$.*

(ii) *If $\varphi, \psi \in C^2(I)$, then*

$$\frac{\int_a^b \varphi(h(x))\,\omega(x)\,dx - \varphi\left(\int_a^b h(x)\omega(x)\,dx\right)}{\int_a^b \psi(h(x))\,\omega(x)\,dx - \psi\left(\int_a^b h(x)\omega(x)\,dx\right)} = \frac{\varphi''(\xi)}{\psi''(\xi)}, \tag{6}$$

*holds for some $\xi \in I$, provided that the denominator of the left-hand side is non-zero.*

In [13] the generalization of these results is given for the case when the function $h$ is defined on a convex set $\Omega \subseteq \mathbb{R}^n$ equipped with a probability measure.

The aim of our paper here is to give further generalization of (5) and (6), for positive linear functionals, and also to define adequate Cauchy-type means.

## 2. Mean-value theorems for positive linear functionals

Let $E$ be a nonempty set and $L$ be a linear class of real-valued functions $f : E \to \mathbb{R}$ having the properties:

(L1) $f, g \in L \Rightarrow (af + bg) \in L$ for all $a, b \in \mathbb{R}$

(L2) $1 \in L$, that is if $f(t) = 1$ for $t \in E$, then $f \in L$.

A positive (isotonic) linear functional is a functional $A : L \to \mathbb{R}$ having the properties:

(A1) $A(af + bg) = aA(f) + bA(g)$ for $f, g \in L,\ a, b \in \mathbb{R}$ ($A$ is linear);

(A2) $f \in L,\ f(t) \geq 0$ on $E \Rightarrow A(f) \geq 0$ ($A$ is positive or isotonic).

B. Jessen in [6] (see also [12, p.47]) gave the following generalization of Jensen's inequality for positive linear functionals.

**Theorem 2.1.** *Let $L$ satisfy properties L1, L2 on a nonempty set $E$ and let $\varphi : I \to \mathbb{R}$ be a continuous convex function on an interval $I \subset \mathbb{R}$. If $A$ is a positive linear functional on $L$ with $A(1) = 1$, then for all $g \in L$ such that $\varphi(g) \in L$ we have $A(g) \in I$ and*

$$\varphi(A(g)) \leq A(\varphi(g)).$$

Using this result we shall give two mean-value theorems for positive linear functionals.

The following result (see for example [12, p.4]) will be very useful.

**Lemma 2.2.** *Let $\varphi : I \to \mathbb{R}$, $I \subset \mathbb{R}$, be such that $\varphi \in C^2(I)$, $\varphi''$ is bounded and $m = \inf_{t \in I} \varphi''(t)$, $M = \sup_{t \in I} \varphi''(t)$.*

*Then the functions $\varphi_1, \varphi_2 : I \to \mathbb{R}$ defined by*

$$\begin{aligned} \varphi_1(t) &= \frac{M}{2}t^2 - \varphi(t) \\ \varphi_2(t) &= \varphi(t) - \frac{m}{2}t^2 \end{aligned} \tag{7}$$

*are convex functions.*

Now we give our mean-value theorems for positive linear functionals.

**Theorem 2.3.** *Let $L$ satisfy properties $L1, L2$ on a nonempty set $E$ and let $\varphi : I \to \mathbb{R}$, $\varphi \in C^2(I)$, where $I \subset \mathbb{R}$ is a compact real interval. If $A$ is a positive linear functional on $L$ with $A(1) = 1$, then for all $g \in L$ such that $g^2, \varphi(g) \in L$ there exists some $\xi \in I$ such that the following holds*

$$A(\varphi(g)) - \varphi(A(g)) = \frac{\varphi''(\xi)}{2}\left[A(g^2) - (A(g))^2\right]. \tag{8}$$

**Proof.** Denote $m = \min_{t\in I} \varphi''(t)$, $M = \max_{t\in I} \varphi''(t)$. The previous Lemma states that then the functions $\varphi_1, \varphi_2 : I \to \mathbb{R}$ defined by (7) are convex functions. As they are also continuous, we can apply Theorem 2.1 on them.

We get

$$A(\varphi(g)) - \varphi(A(g)) \leq \frac{M}{2}\left[A(g^2) - (A(g))^2\right] \tag{9}$$

and

$$A(\varphi(g)) - \varphi(A(g)) \geq \frac{m}{2}\left[A(g^2) - (A(g))^2\right]. \tag{10}$$

Now combining these two inequalities and since $\varphi''$ is continuous, there exists some $\xi \in I$ $(m \leq \varphi''(\xi) \leq M)$ such that

$$A(\varphi(g)) - \varphi(A(g)) = \frac{\varphi''(\xi)}{2}\left[A(g^2) - (A(g))^2\right]. \qquad \square$$

**Theorem 2.4.** *Let $L$ satisfy properties $L1, L2$ on a nonempty set $E$ and let $\varphi, \psi : I \to \mathbb{R}$, $\varphi, \psi \in C^2(I)$, where $I \subset \mathbb{R}$ is a compact real interval. If $A$ is a positive linear functional on $L$ with $A(1) = 1$, then for all $g \in L$ such that $g^2, \varphi(g), \psi(g) \in L$ and $A(g^2) - (A(g))^2 \neq 0$, there exists some $\xi \in I$ such that the following holds*

$$\frac{A(\varphi(g)) - \varphi(A(g))}{A(\psi(g)) - \psi(A(g))} = \frac{\varphi''(\xi)}{\psi''(\xi)}$$

*provided that the denominator of the left-hand side is non-zero.*

**Proof.** Consider the function $\chi$ defined by

$$\chi(t) = \left[A\big(\psi(g)\big) - \psi\big(A(g)\big)\right] \cdot \varphi(t) - \left[A\big(\varphi(g)\big) - \varphi\big(A(g)\big)\right] \cdot \psi(t).$$

Function $\chi$ is linear combination of functions $\varphi$ and $\psi$, so $\chi \in C^2(I)$ and $\chi(g) \in L$ for all $g \in L$. Now we can apply Theorem 2.3 on function $\chi$ and it follows that there exists some $\xi \in I$ such that the following holds

$$A\big(\chi(g)\big) - \chi\big(A(g)\big) = \frac{\chi''(\xi)}{2}\left[A(g^2) - (A(g))^2\right].$$

The left-hand side of this equation equals to zero, the term in the square brackets on the right-hand side is non-zero, so we have that

$$\chi''(\xi) = 0.$$

Now the assertion of our theorem follows directly. □

## 3. Cauchy-type means for positive linear functionals

Let $L$ satisfy properties $L1, L2$ on a nonempty set $E$. Let $A$ be a positive linear functional (i.e. $A$ satisfies the conditions $A1, A2$) on $L$ with $A(1) = 1$, and let $g \in L$.

Then for a strictly monotone continuous function $\alpha$ such that $\alpha \circ g \in L$, the generalized quasi-arithmetic mean, $M_\alpha(g, A)$, of $g$ with respect to the positive linear functional $A$ and the function $\alpha$ is defined by (see for example [12, p.107])

$$M_\alpha(g, A) = \alpha^{-1}(A(\alpha \circ g)). \tag{11}$$

The following theorem holds.

**Theorem 3.1.** *Let $g \in L$ be such that the image of $g$ is a compact real interval $I$, and let $\alpha, \beta, \gamma : I \to \mathbb{R}$ be strictly monotone functions, $\alpha, \beta, \gamma \in C^2(I)$, such that $\alpha \circ g, \beta \circ g, \gamma \circ g, (\gamma \circ g)^2 \in L$ and $A\left((\gamma \circ g)^2\right) - (A(\gamma \circ g))^2 \neq 0$.*

*Then*

$$\frac{\alpha(M_\alpha(g, A)) - \alpha(M_\gamma(g, A))}{\beta(M_\beta(g, A)) - \beta(M_\gamma(g, A))} = \frac{\alpha''(\eta) \cdot \gamma'(\eta) - \alpha'(\eta) \cdot \gamma''(\eta)}{\beta''(\eta) \cdot \gamma'(\eta) - \beta'(\eta) \cdot \gamma''(\eta)} \tag{12}$$

*holds for some $\eta$ in the image of $g$, provided that the denominator of the left-hand side is non-zero.*

**Proof.** If we apply Theorem 2.4 on the functions

$$\varphi = \alpha \circ \gamma^{-1}, \ \psi = \beta \circ \gamma^{-1}, \ g = \gamma \circ g,$$

we find that there exists some $\xi$ such that

$$\frac{\alpha(M_\alpha(g, A)) - \alpha(M_\gamma(g, A))}{\beta(M_\beta(g, A)) - \beta(M_\gamma(g, A))} = \frac{\alpha''(\gamma^{-1}(\xi)) \cdot \gamma'(\gamma^{-1}(\xi)) - \alpha'(\gamma^{-1}(\xi)) \cdot \gamma''(\gamma^{-1}(\xi))}{\beta''(\gamma^{-1}(\xi)) \cdot \gamma'(\gamma^{-1}(\xi)) - \beta'(\gamma^{-1}(\xi)) \cdot \gamma''(\gamma^{-1}(\xi))}. \tag{13}$$

Thus, by setting $\gamma^{-1}(\xi) = \eta$, we find that there exists some $\eta$ in the image of $g$ such that

$$\frac{\alpha(M_\alpha(g, A)) - \alpha(M_\gamma(g, A))}{\beta(M_\beta(g, A)) - \beta(M_\gamma(g, A))} = \frac{\alpha''(\eta) \cdot \gamma'(\eta) - \alpha'(\eta) \cdot \gamma''(\eta)}{\beta''(\eta) \cdot \gamma'(\eta) - \beta'(\eta) \cdot \gamma''(\eta)}$$

provided that the denominator of the left-hand side is non-zero. □

**Corollary 1.** *Let the conditions of* Theorem 3.1 hold.

Let

$$\chi(\eta)=\frac{\alpha''(\eta)\cdot\gamma'(\eta)-\alpha'(\eta)\cdot\gamma''(\eta)}{\beta''(\eta)\cdot\gamma'(\eta)-\beta'(\eta)\cdot\gamma''(\eta)}$$

be invertible function.

Then

$$\eta=\chi^{-1}\left(\frac{\alpha(M_\alpha(g,A))-\alpha(M_\gamma(g,A))}{\beta(M_\beta(g,A))-\beta(M_\gamma(g,A))}\right) \tag{14}$$

is a mean, provided that the denominator of the term in the brackets is non-zero.

**Proof.** Since $\eta$ is in the image of $g$, it follows that

$$\min_{t\in E} g(t)\leq\chi^{-1}\left(\frac{\alpha(M_\alpha(g,A))-\alpha(M_\gamma(g,A))}{\beta(M_\beta(g,A))-\beta(M_\gamma(g,A))}\right)\leq\max_{t\in E} g(t).$$

This shows that this is a mean. □

Now, from the results given above, we can deduce the corresponding results for the generalized power mean, $M_r(g,A)$, of $g$ with respect to the positive linear functional $A$ which is defined for $r\in\mathbb{R}$ by (see for example [12, p.108])

$$M_r(g,A)=\begin{cases}\left(A(g^r)\right)^{\frac{1}{r}}, & r\neq 0\\ \exp\left(A(\log g)\right), & r=0\end{cases} \tag{15}$$

where $g(t)>0$ for $t\in E$, $\log g\in L$ and $g^r\in L$ for $r\in\mathbb{R}\setminus\{0\}$.

**Corollary 2.** *Let $g\in L$ be such that the image of $g$ is a compact real interval $I$. Let $r,l,s\in\mathbb{R}\setminus\{0\}$, $r\neq l,s$; $l\neq s$, such that $g^r,g^l,g^s,g^{2s}\in L$ and $A(g^{2s})-(A(g^s))^2\neq 0$. Then*

$$\frac{M_r^r(g,A)-M_s^r(g,A)}{M_l^l(g,A)-M_s^l(g,A)}=\frac{r(r-s)}{l(l-s)}\cdot\eta^{r-l} \tag{16}$$

*holds for some $\eta$ in the image of $g$, provided that the denominator of the left-hand side is non-zero.*

**Proof.** If we set

$$\alpha(t)=t^r,\ \beta(t)=t^l,\ \gamma(t)=t^s,$$

in Theorem 3.1, we get the assertion (16). □

Since $\eta$ is in the image of $g$, (16) suggests a new mean as it is

$$\min_{t\in E} g(t)\leq\left(\frac{l(l-s)}{r(r-s)}\frac{M_r^r(g,A)-M_s^r(g,A)}{M_l^l(g,A)-M_s^l(g,A)}\right)^{\frac{1}{r-l}}\leq\max_{t\in E} g(t)$$

for $r, l, s \in \mathbb{R} \setminus \{0\}$, $r \neq l, s$; $l \neq s$; $A(g^{2s}) - (A(g^s))^2 \neq 0$.

From (16) it follows that we can define a new mean $M_{r,l}^{[s]}(g, A)$ as follows

$$M_{r,l}^{[s]}(g, A) = \left( \frac{l(l-s)}{r(r-s)} \frac{M_r^r(g, A) - M_s^r(g, A)}{M_l^l(g, A) - M_s^l(g, A)} \right)^{\frac{1}{r-l}},$$

for $r, l, s \in \mathbb{R} \setminus \{0\}$, $r \neq l, s$; $l \neq s$; $A(g^{2s}) - (A(g^s))^2 \neq 0$.

Similarly, we can calculate some other cases for $r, l, s \in \mathbb{R}$ and we get the following definition of $M_{r,l}^{[s]}(g, A)$.

**Definition 3.2.** Let $L$ satisfy properties $L1, L2$ on a nonempty set $E$ and let $A$ be a positive linear functional on $L$ with $A(1) = 1$. Let $r, l, s \in \mathbb{R}$ and let $g \in L$ be such that the image of $g$ is a compact real interval $I \subset \mathbb{R}^+$, and $g^r, g^l, g^s, g^{2s}, \log g, \log^2 g \in L$ for $r, l, s \in \mathbb{R} \setminus \{0\}$.

Then we define the Cauchy-type mean $M_{r,l}^{[s]}(g, A)$ of $g$ with respect to the positive linear functional $A$ by

$$M_{r,l}^{[s]}(g, A) = \left( \frac{l(l-s)}{r(r-s)} \cdot \frac{M_r^r(g, A) - M_s^r(g, A)}{M_l^l(g, A) - M_s^l(g, A)} \right)^{\frac{1}{r-l}},$$

for $r, l, s \neq 0$; $r \neq l, s$; $l \neq s$; $A(g^{2s}) - (A(g^s))^2 \neq 0$;

$$M_{r,0}^{[s]}(g, A) = M_{0,r}^{[s]}(g, A) = \left( -\frac{s}{r(r-s)} \cdot \frac{M_r^r(g, A) - M_s^r(g, A)}{\log(M_0(g, A)) - \log(M_s(g, A))} \right)^{\frac{1}{r}},$$

for $r, s \neq 0$; $r \neq s$; $A(g^{2s}) - (A(g^s))^2 \neq 0$;

$$M_{r,l}^{[0]}(g, A) = \left( \frac{l^2}{r^2} \cdot \frac{M_r^r(g, A) - M_0^r(g, A)}{M_l^l(g, A) - M_0^l(g, A)} \right)^{\frac{1}{r-l}},$$

for $r, l \neq 0$; $r \neq l$; $A(\log^2 g) - (A(\log g))^2 \neq 0$;

where we suppose that all expressions are well defined.

## 4. Monotonicity of Cauchy-type means for positive linear functionals

The following Lemma is valid.

**Lemma 4.1.** *Define the function* $\varphi_s : \mathbb{R}^+ \to \mathbb{R}$ *by*

$$\varphi_s(x) = \begin{cases} \frac{x^s}{s(s-1)}, & s \neq 0, 1 \\ -\log x, & s = 0 \\ x \log x, & s = 1. \end{cases} \tag{17}$$

*It is* $\varphi_s''(x) = x^{s-2}$*, so* $\varphi_s$ *is convex function.*

We shall use the following result from [2].

**Theorem 4.2.** ([2]) *Let $L$ satisfy properties $L1, L2$ on a nonempty set $E$ and let $A$ be a positive linear functional on $L$ with $A(1) = 1$. Let a positive function $g \in L$ be such that $g^r \in L$ for $r \in J \setminus \{0,1\}$, ($J \subset \mathbb{R}$), $\log g \in L$ if $r = 0 \in J$ and $g \log g \in L$ if $r = 1 \in J$.*

*Let us define*

$$\Lambda_t(g, A) = A(\varphi_t(g)) - \varphi_t(A(g)). \tag{18}$$

(i) *Then for all $s, t \in J$ we have*

$$\Lambda^2_{\frac{s+t}{2}} \le \Lambda_s \cdot \Lambda_t.$$

*That is, $t \mapsto \Lambda_t$ is log-convex in Jensen's sense.*

(ii) *If $t \mapsto \Lambda_t$ is continuous on $J$, then it is also log-convex. That is, for $r < s < t$ ($r, s, t \in J$) we have that*

$$(\Lambda_s)^{t-r} \le (\Lambda_r)^{t-s} \cdot (\Lambda_t)^{s-r}.$$

We can represent the function $\Lambda_t$ defined by (18), where the function $\varphi_t$ is defined by (17), using generalized power means defined by (15), as

$$\Lambda_t(g, A) = \begin{cases} \frac{1}{t(t-1)} \left[ M_t^t(g, A) - M_1^t(g, A) \right], & t \neq 0, 1 \\ -\log M_0(g, A) + \log M_1(g, A), & t = 0 \\ M_1(g \log g, A) - M_1(g, A) \cdot \log\big(M_1(g, A)\big), & t = 1. \end{cases} \tag{19}$$

Note that, as it follows from the previous Theorem, that if so defined function $t \mapsto \Lambda_t$ is continuous on $J$, then it is also log-convex.

The following Lemma is also valid.

**Lemma 4.3.** *Define the function $\psi_s : \mathbb{R} \to \mathbb{R}$ by*

$$\psi_s(x) = \begin{cases} \frac{1}{s^2} e^{sx}, & s \neq 0 \\ \frac{1}{2} x^2, & s = 0. \end{cases} \tag{20}$$

*It is $\psi_s''(x) = e^{sx}$, so $\psi_s$ is convex function.*

**Remark 4.4.** The authors in [2] proved that Theorem 4.2 also holds if we define function $\Lambda_t$ by

$$\Lambda_t(g, A) = A(\psi_t(g)) - \psi_t(A(g)), \tag{21}$$

where the function $\psi_t$ is defined by (20).

Now we can also represent our newly defined function $\Lambda_t$, which is defined by (21) where the function $\psi_t$ is defined by (20), using generalized power means, as follows

$$\Lambda_t = \begin{cases} \frac{1}{t^2}\left[M_t^t(g,A) - M_0^t(g,A)\right], & t \neq 0 \\ \frac{1}{2}\left[M_2^2(\log g, A) - M_1^2(\log g, A)\right], & t = 0. \end{cases} \tag{22}$$

Also note that if so defined function $t \mapsto \Lambda_t$ is continuous on $J$, then it is also log-convex.

The following result gives us the useful property for log-convex functions.

**Lemma 4.5.** ([4]) *Let $f : J \to \mathbb{R}^+$ $(J \subset \mathbb{R})$ be log-convex function and $x_1, x_2, y_1, y_2 \in J$ such that $x_1 \leq y_1$, $x_2 \leq y_2$, $x_1 \neq x_2$, $y_1 \neq y_2$. Then the following inequality holds*

$$\left(\frac{f(x_2)}{f(x_1)}\right)^{\frac{1}{x_2 - x_1}} \leq \left(\frac{f(y_2)}{f(y_1)}\right)^{\frac{1}{y_2 - y_1}}. \tag{23}$$

Now in the next theorem we give the comparison for our newly defined Cauchy-type means, $M_{r,l}^{[s]}$, from the previous section.

We shall suppose that $g$ and $A$ are such that both defined functions $\Lambda_t$ (in (18) and in (21)) are continuous functions.

**Theorem 4.6.** *Let $a, b, c, d, s \in \mathbb{R}$ be such that $a \leq c$, $b \leq d$, $a \neq b$, $c \neq d$ and $s \neq a, b, c, d$.*

*Then, for $M_{r,l}^{[s]}$ defined in* Definition 3.2 we have that

$$M_{b,a}^{[s]} \leq M_{d,c}^{[s]}. \tag{24}$$

**Proof.** Case I.:

Let us consider the function $\Lambda_t$ defined by (18). As $\Lambda_t$ is log-convex, previous Lemma implies that for $a, b, c, d \in \mathbb{R}$ such that $a \leq c$, $b \leq d$, $a \neq b$, $c \neq d$, the following is valid

$$\left(\frac{\Lambda_b}{\Lambda_a}\right)^{\frac{1}{b-a}} \leq \left(\frac{\Lambda_d}{\Lambda_c}\right)^{\frac{1}{d-c}}. \tag{25}$$

For $s > 0$, by substituting: $g = f^s$, $a = \frac{a}{s}$, $b = \frac{b}{s}$, $c = \frac{c}{s}$, $d = \frac{d}{s}$, such that $\frac{a}{s} \leq \frac{c}{s}$, $\frac{b}{s} \leq \frac{d}{s}$, $a \neq b$, $c \neq d$, in $\Lambda_b$, we get

$$\Lambda_{b,s}(f,A) = \begin{cases} \frac{s^2}{b(b-s)}\left[M_b^b(f,A) - M_s^b(f,A)\right], & b \neq 0, s \\ s\left[-\log M_0(f,A) + \log M_s(f,A)\right], & b = 0 \\ s\left[M_1(f^s \log f, A) - M_s^s(f,A) \cdot \log\left(M_s(f,A)\right)\right], & b = s, \end{cases} \tag{26}$$

and (25) becomes

$$\left(\frac{\Lambda_{b,s}}{\Lambda_{a,s}}\right)^{\frac{s}{b-a}} \leq \left(\frac{\Lambda_{d,s}}{\Lambda_{c,s}}\right)^{\frac{s}{d-c}}. \tag{27}$$

As it is $s > 0$, the statement (24) follows directly.

For $s < 0$, by substituting: $g = f^s$, $a = \frac{a}{s}$, $b = \frac{b}{s}$, $c = \frac{c}{s}$, $d = \frac{d}{s}$, such that $\frac{a}{s} \geq \frac{c}{s}$, $\frac{b}{s} \geq \frac{d}{s}$, $a \neq b$, $c \neq d$, in $\Lambda_b$, we get again (26), but (25) becomes

$$\left(\frac{\Lambda_{b,s}}{\Lambda_{a,s}}\right)^{\frac{s}{b-a}} \leq \left(\frac{\Lambda_{d,s}}{\Lambda_{c,s}}\right)^{\frac{s}{d-c}}. \tag{28}$$

Because it is $s < 0$, we have that

$$\left(\frac{\Lambda_{b,s}}{\Lambda_{a,s}}\right)^{\frac{1}{b-a}} \geq \left(\frac{\Lambda_{d,s}}{\Lambda_{c,s}}\right)^{\frac{1}{d-c}} \tag{29}$$

where $a \geq c$, $b \geq d$, $a \neq b$, $c \neq d$ and we get our required result.

Case II.:

Let us consider the function $\Lambda_t$ defined by (21). As $\Lambda_t$ is log-convex, previous Lemma implies that for $a, b, c, d \in \mathbb{R}$ such that $a \leq c$, $b \leq d$, $a \neq b$, $c \neq d$, inequality (25) holds.

Therefore we have for $a, b, c, d \in \mathbb{R}$ such that $a \leq c$, $b \leq d$, $a \neq b$, $c \neq d$, that it holds

$$M_{b,a}^{[0]} \leq M_{d,c}^{[0]}. \tag{30}$$

□

**Remark 4.7.** If we set the functional $A$ of the function $g$ to be

$$A(g) = \int_{\Omega} g(u)d\mu(u),$$

where $\Omega \subseteq \mathbb{R}^n$ is a convex set equipped with a probability measure $\mu$, we get the means of Cauchy-type given in [4] and also adequate results.

## 5. Some related results

In this section we will give related results of Mercer's and Aczél's type.

### 5.1. Means of the Mercer type

In [5] the following version of Jessen's inequality is given.

**Theorem 5.2.** *Let $L$ satisfy properties $L1, L2$ on a nonempty set $E$ and let $\varphi : I \to \mathbb{R}$ be a continuous convex function on an interval $I = [m, M]$ $(-\infty < m < M < \infty)$. If $A$ is a positive linear functional on $L$ with $A(1) = 1$, then for all $g \in L$ such that $\varphi(g), \varphi(m + M - g) \in L$ (so that $m \le g(t) \le M$ for all $t \in E$), we have the following variant of Jessen's inequality*

$$\varphi\left(m + M - A(g)\right) \le \varphi(m) + \varphi(M) - A\left(\varphi(g)\right). \tag{31}$$

Using this result we shall give two mean-value theorems for positive linear functionals.

**Theorem 5.3.** *Let $L$ satisfy properties $L1, L2$ on a nonempty set $E$ and let $\varphi : I \to \mathbb{R}$, $\varphi \in C^2(I)$, where $I = [m, M]$ $(-\infty < m < M < \infty)$. If $A$ is a positive linear functional on $L$ with $A(1) = 1$, then for all $g \in L$ such that $g^2, \varphi(g), \varphi(m + M - g) \in L$ (so that $m \le g(t) \le M$ for all $t \in E$) there exists some $\xi \in I$ such that the following holds*

$$\begin{aligned}\varphi(m) + \varphi(M) - A\left(\varphi(g)\right) &- \varphi\left(m + M - A(g)\right) \\ &= \frac{\varphi''(\xi)}{2}\left[m^2 + M^2 - A(g^2) - \left(m + M - A(g)\right)^2\right].\end{aligned} \tag{32}$$

**Proof.** The proof is similar to the proof of Theorem 2.3 using Theorem 5.2 instead of Theorem 2.1. □

**Theorem 5.4.** *Let $L$ satisfy properties $L1, L2$ on a nonempty set $E$ and let $\varphi, \psi : I \to \mathbb{R}$, $\varphi, \psi \in C^2(I)$, where $I = [m, M]$ $(-\infty < m < M < \infty)$. If $A$ is a positive linear functional on $L$ with $A(1) = 1$, then for all $g \in L$ such that $g^2$, $\varphi(g)$, $\varphi(m + M - g)$, $\psi(g)$, $\psi(m + M - g) \in L$ (so that $m \le g(t) \le M$ for all $t \in E$) and $m^2 + M^2 - A(g^2) - \left(m + M - A(g)\right)^2 \neq 0$, there exists some $\xi \in I$ such that the following holds*

$$\frac{\varphi(m) + \varphi(M) - A\left(\varphi(g)\right) - \varphi\left(m + M - A(g)\right)}{\psi(m) + \psi(M) - A\left(\psi(g)\right) - \psi\left(m + M - A(g)\right)} = \frac{\varphi''(\xi)}{\psi''(\xi)}, \tag{33}$$

*provided that the denominator of the left-hand side is non-zero.*

**Proof.**The proof is similar to the proof of Theorem 2.4 using Theorem 5.3 instead of Theorem 2.3. □

Let $L$ satisfy properties $L1, L2$ on a nonempty set $E$. Let $A$ be positive linear functional on $L$ with $A(1) = 1$ and let $g \in L$. Then for a strictly monotone continuous function $\alpha$ such that $\alpha \circ g \in L$, the generalized quasi-arithmetic mean of the Mercer type, $\mathcal{M}_\alpha(g, A)$, of $g$ with respect to the positive linear functional $A$ and the function $\alpha$ is defined by (see [5])

$$\mathcal{M}_\alpha(g, A) = \alpha^{-1}\left(\alpha(m) + \alpha(M) - A(\alpha \circ g)\right). \tag{34}$$

The following theorem holds.

**Theorem 5.5.** *Let $g \in L$ be such that the image of $g$ is a compact real interval $I = [m, M]$ and let $\alpha, \beta, \gamma : I \to \mathbb{R}$ be strictly monotone functions, $\alpha, \beta, \gamma \in C^2(I)$, such that $\alpha \circ g, \beta \circ g, \gamma \circ g, (\gamma \circ g)^2 \in L$ and $(\gamma(m))^2 + (\gamma(M))^2 - A((\gamma \circ g)^2) - \big(\gamma(m) + \gamma(M) - A(\gamma \circ g)\big)^2 \neq 0$.*

*Then*

$$\frac{\alpha(\mathcal{M}_\alpha(g, A)) - \alpha(\mathcal{M}_\gamma(g, A))}{\beta(\mathcal{M}_\beta(g, A)) - \beta(\mathcal{M}_\gamma(g, A))} = \frac{\alpha''(\eta) \cdot \gamma'(\eta) - \alpha'(\eta) \cdot \gamma''(\eta)}{\beta''(\eta) \cdot \gamma'(\eta) - \beta'(\eta) \cdot \gamma''(\eta)} \tag{35}$$

*holds for some $\eta$ in the image of $g$, provided that the denominator of the left-hand side is non-zero.*

**Proof.**The proof is similar to the proof of Theorem 3.1 using Theorem 5.4 instead of Theorem 2.4 with substitution

$$\varphi = \alpha \circ \gamma^{-1},\ \psi = \beta \circ \gamma^{-1},\ g = \gamma \circ g,\ m = \gamma(m),\ M = \gamma(M).$$

□

**Corollary 3.** *Let the conditions of* Theorem 5.5 hold.

Let

$$\chi(\eta) = \frac{\alpha''(\eta) \cdot \gamma'(\eta) - \alpha'(\eta) \cdot \gamma''(\eta)}{\beta''(\eta) \cdot \gamma'(\eta) - \beta'(\eta) \cdot \gamma''(\eta)}$$

be invertible function.

Then

$$\eta = \chi^{-1}\left(\frac{\alpha(\mathcal{M}_\alpha(g, A)) - \alpha(\mathcal{M}_\gamma(g, A))}{\beta(\mathcal{M}_\beta(g, A)) - \beta(\mathcal{M}_\gamma(g, A))}\right) \tag{36}$$

is a mean, provided that the denominator of the term in the brackets is non-zero.

**Proof.** Since $\eta$ is in the image of $g$, it follows that

$$\min_{t\in E} g(t) \leq \chi^{-1}\left(\frac{\alpha(\mathcal{M}_\alpha(g, A)) - \alpha(\mathcal{M}_\gamma(g, A))}{\beta(\mathcal{M}_\beta(g, A)) - \beta(\mathcal{M}_\gamma(g, A))}\right) \leq \max_{t\in E} g(t).$$

This shows that this is a mean. □

Now, from the results given above, we can deduce corresponding results for the generalized power mean of the Mercer type, $Q_r(g, A)$, of $g$ with respect to the positive linear functional $A$ which is defined for $r \in \mathbb{R}$ by (see [9])

$$Q_r(g, A) = \begin{cases} \big(m^r + M^r - A(g^r)\big)^{\frac{1}{r}}, & r \neq 0 \\ \dfrac{mM}{\exp\big(A(\log g)\big)}, & r = 0, \end{cases} \tag{37}$$

where $g(t) > 0$ for $t \in E$, $\log g \in L$ and $g^r \in L$ for $r \in \mathbb{R} \setminus \{0\}$.

**Corollary 4.** *Let $g \in L$ be such that the image of $g$ is a compact real interval $I = [m, M]$. Let $r, l, s \in \mathbb{R}\setminus\{0\}$, $r \neq l, s$; $l \neq s$, such that $g^r, g^l, g^s, g^{2s} \in L$ and $m^{2s}+M^{2s}-A(g^{2s})-\left(m^s+M^s-A(g^s)\right)^2 \neq 0$. Then*

$$\frac{Q_r^r(g,A)-Q_s^r(g,A)}{Q_l^l(g,A)-Q_s^l(g,A)} = \frac{r(r-s)}{l(l-s)} \cdot \eta^{r-l} \tag{38}$$

*holds for some $\eta$ in the image of $g$, provided that the denominator of the left-hand side is non-zero.*

**Proof.** If we set

$$\alpha(t) = t^r\ ,\ \beta(t) = t^l\ ,\ \gamma(t) = t^s\ ,$$

in Theorem 5.5, we get the assertion (38). □

Since $\eta$ is in the image of $g$, (38) suggests a new mean as it is

$$\min_{t\in E} g(t) \leq \left(\frac{l(l-s)}{r(r-s)}\frac{Q_r^r(g,A)-Q_s^r(g,A)}{Q_l^l(g,A)-Q_s^l(g,A)}\right)^{\frac{1}{r-l}} \leq \max_{t\in E} g(t)$$

for $r, l, s \in \mathbb{R}\setminus\{0\}$, $r \neq l, s$; $l \neq s$; $m^{2s}+M^{2s}-A(g^{2s})-\left(m^s+M^s-A(g^s)\right)^2 \neq 0$.

From (38) it follows that we can define a new mean $\mathrm{M}_{r,l}^{[s]}(g,A)$ as follows

$$\mathrm{M}_{r,l}^{[s]}(g,A) = \left(\frac{l(l-s)}{r(r-s)}\frac{Q_r^r(g,A)-Q_s^r(g,A)}{Q_l^l(g,A)-Q_s^l(g,A)}\right)^{\frac{1}{r-l}},$$

for $r, l, s \in \mathbb{R}\setminus\{0\}$, $r \neq l, s$; $l \neq s$; $m^{2s}+M^{2s}-A(g^{2s})-\left(m^s+M^s-A(g^s)\right)^2 \neq 0$.

Similarly, we can calculate some other cases for $r, l, s \in \mathbb{R}$ and we get the following definition of $\mathrm{M}_{r,l}^{[s]}(g,A)$.

**Definition 5.6.** Let $L$ satisfy properties $L1, L2$ on a nonempty set $E$ and let $A$ be a positive linear functional on $L$ with $A(1) = 1$. Let $r, l, s \in \mathbb{R}$ and let $g \in L$ be such that the image of $g$ is a compact real interval $I = [m, M] \subset \mathbb{R}^+$, and $g^r, g^l, g^s, g^{2s}, \log g, \log^2 g \in L$ for $r, l, s \in \mathbb{R}\setminus\{0\}$.

Then we define the Cauchy-type mean $\mathrm{M}_{r,l}^{[s]}(g,A)$ of $g$ with respect to the positive linear functional $A$ by

$$\mathrm{M}_{r,l}^{[s]}(g,A) = \left(\frac{l(l-s)}{r(r-s)}\cdot\frac{Q_r^r(g,A)-Q_s^r(g,A)}{Q_l^l(g,A)-Q_s^l(g,A)}\right)^{\frac{1}{r-l}},$$

for $r, l, s \neq 0$; $r \neq l, s$; $l \neq s$; $m^{2s}+M^{2s}-A(g^{2s})-\left(m^s+M^s-A(g^s)\right)^2 \neq 0$;

$$\mathrm{M}_{r,0}^{[s]}(g,A) = \mathrm{M}_{0,r}^{[s]}(g,A) = \left(-\frac{s}{r(r-s)}\cdot\frac{Q_r^r(g,A)-Q_s^r(g,A)}{\log(Q_0(g,A))-\log(Q_s(g,A))}\right)^{\frac{1}{r}},$$

for $r,s\neq 0$; $r\neq s$; $m^{2s}+M^{2s}-A(g^{2s})-\left(m^s+M^s-A(g^s)\right)^2\neq 0$;

$$\mathrm{M}_{r,l}^{[0]}(g,A)=\left(\frac{l^2}{r^2}\cdot\frac{Q_r^r(g,A)-Q_0^r(g,A)}{Q_l^l(g,A)-Q_0^l(g,A)}\right)^{\frac{1}{r-l}},$$

for $r,l\neq 0$; $r\neq l$; $\log^2 m+\log^2 M-A(\log^2 g)-\left(\log m+\log M-A(\log g)\right)^2\neq 0$;

where we suppose that all expressions are well defined.

**Theorem 5.7.** *Let $a,b,c,d,s\in\mathbb{R}$ be such that $a\leq c$, $b\leq d$, $a\neq b$, $c\neq d$ and $s\neq a,b,c,d$.*

*Then, for $\mathrm{M}_{r,l}^{[s]}$ defined in* Definition 5.6 we have that

$$\mathrm{M}_{b,a}^{[s]}\leq \mathrm{M}_{d,c}^{[s]}. \tag{39}$$

**Proof.** The proof is similar to the proof of Theorem 4.6, using adequate results from [2]. ☐

**Remark 5.8.** If we set the functional $A$ of the function $g$ to be

$$A(g)=\int_\Omega g(u)\,d\mu(u),$$

where $\Omega\subseteq\mathbb{R}^n$ is a convex set equipped with a probability measure $\mu$, we get the means of Cauchy-type given in [3] and also adequate results.

### 5.2. Means of the Aczél type

The following version of Jensen's inequality is valid too ([12, p.124-125]).

**Theorem 5.9.** *Let $L$ satisfy properties $L1, L2$ on a nonempty set $E$ and let $A$ be a positive linear functional on $L$. Let $\varphi: I\to\mathbb{R}$ be a continuous convex function on an interval $I\subset\mathbb{R}$. Suppose that $w\in L$ with $w\geq 0$ on $E$ and $0<A(w)<u\in\mathbb{R}$, $\frac{ua-A(wg)}{u-A(w)}\in I$ $(a\in I)$, where $g: E\to\mathbb{R}$ is such that $wg\in L$ and $w\varphi(g)\in L$. Then*

$$\varphi\left(\frac{ua-A(wg)}{u-A(w)}\right)\geq\frac{u\varphi(a)-A(w\varphi(g))}{u-A(w)}. \tag{40}$$

Using this result we shall give two mean-value theorems for positive linear functionals.

**Theorem 5.10.** *Let $L$ satisfy properties $L1, L2$ on a nonempty set $E$ and let $A$ be a positive linear functional on $L$. Let $\varphi: I\to\mathbb{R}$, $\varphi\in C^2(I)$, where $I\subset\mathbb{R}$ is a compact real interval. Suppose that $w\in L$ with $w\geq 0$ on $E$ and $0<A(w)<u\in\mathbb{R}$, $\frac{ua-A(wg)}{u-A(w)}\in I$ $(a\in I)$, where $g: E\to\mathbb{R}$ is such that $wg, wg^2, w\varphi(g)\in L$. Then there exists some $\xi\in I$ such that the following holds*

$$\varphi\left(\frac{ua-A(wg)}{u-A(w)}\right)-\frac{u\varphi(a)-A(w\varphi(g))}{u-A(w)}=\frac{\varphi''(\xi)}{2}\left[\left(\frac{ua-A(wg)}{u-A(w)}\right)^2-\frac{ua^2-A(wg^2)}{u-A(w)}\right]. \tag{41}$$

**Proof.** The proof is similar to the proof of Theorem 2.3 using Theorem 5.9 instead of Theorem 2.1. ☐

**Theorem 5.11.** *Let $L$ satisfy properties $L1, L2$ on a nonempty set $E$ and let $A$ be a positive linear functional on $L$. Let $\varphi, \psi : I \to \mathbb{R}$, $\varphi, \psi \in C^2(I)$, where $I \subset \mathbb{R}$ is a compact real interval. Suppose that $w \in L$ with $w \geq 0$ on $E$ and $0 < A(w) < u \in \mathbb{R}$, $\frac{ua - A(wg)}{u - A(w)} \in I$ $(a \in I)$, where $g : E \to \mathbb{R}$ is such that $wg, wg^2, w\varphi(g), w\psi(g) \in L$ and $\left(\frac{ua - A(wg)}{u - A(w)}\right)^2 - \frac{ua^2 - A(wg^2)}{u - A(w)} \neq 0$. Then there exists some $\xi \in I$ such that the following holds*

$$\frac{\varphi\left(\frac{ua - A(wg)}{u - A(w)}\right) - \frac{u\varphi(a) - A(w\varphi(g))}{u - A(w)}}{\psi\left(\frac{ua - A(wg)}{u - A(w)}\right) - \frac{u\psi(a) - A(w\psi(g))}{u - A(w)}} = \frac{\varphi''(\xi)}{\psi''(\xi)}, \tag{42}$$

*provided that the denominator of the left-hand side is non-zero.*

**Proof.** The proof is similar to the proof of Theorem 2.4 using Theorem 5.10 instead of Theorem 2.3. ☐

Let $L$ satisfy properties $L1, L2$ on a nonempty set $E$ and let $A$ be positive linear functional on $L$. Let $w \in L$ with $w \geq 0$ on $E$, $0 < A(w) < u \in \mathbb{R}$ and $a \in \mathbb{R}$. Let $g : E \to \mathbb{R}$ be such that $wg \in L$.

Then for a strictly monotone continuous function $\alpha : I \to \mathbb{R}$ $(I \subseteq \mathbb{R},\ a \in I)$ such that $w\alpha(g) \in L$, the generalized quasi-arithmetic mean of the Aczél type, $\mathfrak{M}_\alpha(g, A)$, of $g$ with respect to the positive linear functional $A$ and the function $\alpha$ is defined by

$$\mathfrak{M}_\alpha(g, A) = \alpha^{-1}\left(\frac{u\alpha(a) - A(w\alpha(g))}{u - A(w)}\right). \tag{43}$$

The following theorem holds.

**Theorem 5.12.** *Let $g : E \to \mathbb{R}$ be such that the image of $g$ is a compact real interval $I$ and let $\alpha, \beta, \gamma : I \to \mathbb{R}$ be strictly monotone functions, $\alpha, \beta, \gamma \in C^2(I)$, such that $w\alpha(g), w\beta(g), w\gamma(g), w(\gamma \circ g)^2 \in L$ and $\left(\frac{u\gamma(a) - A(w\gamma(g))}{u - A(w)}\right)^2 - \frac{u(\gamma(a))^2 - A(w(\gamma\circ g)^2)}{u - A(w)} \neq 0$.*

*Then*

$$\frac{\alpha(\mathfrak{M}_\gamma(g, A)) - \alpha(\mathfrak{M}_\alpha(g, A))}{\beta(\mathfrak{M}_\gamma(g, A)) - \beta(\mathfrak{M}_\beta(g, A))} = \frac{\alpha''(\eta) \cdot \gamma'(\eta) - \alpha'(\eta) \cdot \gamma''(\eta)}{\beta''(\eta) \cdot \gamma'(\eta) - \beta'(\eta) \cdot \gamma''(\eta)} \tag{44}$$

*holds for some $\eta$ in the image of $g$, provided that the denominator of the left-hand side is non-zero.*

**Proof.** The proof is similar to the proof of Theorem 3.1 using Theorem 5.11 instead of Theorem 2.4 with substitution

$$\varphi = \alpha \circ \gamma^{-1},\ \psi = \beta \circ \gamma^{-1},\ g = \gamma \circ g,\ \ a = \gamma(a).$$

☐

**Corollary 5.** *Let the conditions of* Theorem 5.12 hold.

Let

$$\chi(\eta)=\frac{\alpha''(\eta)\cdot\gamma'(\eta)-\alpha'(\eta)\cdot\gamma''(\eta)}{\beta''(\eta)\cdot\gamma'(\eta)-\beta'(\eta)\cdot\gamma''(\eta)}$$

be invertible function.

Then

$$\eta=\chi^{-1}\left(\frac{\alpha(\mathfrak{M}_\gamma(g,A))-\alpha(\mathfrak{M}_\alpha(g,A))}{\beta(\mathfrak{M}_\gamma(g,A))-\beta(\mathfrak{M}_\beta(g,A))}\right) \tag{45}$$

is a mean, provided that the denominator of the term in the brackets is non-zero.

**Proof.** Since $\eta$ is in the image of $g$, it follows that

$$\min_{t\in E} g(t)\le\chi^{-1}\left(\frac{\alpha(\mathfrak{M}_\gamma(g,A))-\alpha(\mathfrak{M}_\alpha(g,A))}{\beta(\mathfrak{M}_\gamma(g,A))-\beta(\mathfrak{M}_\beta(g,A))}\right)\le\max_{t\in E} g(t).$$

This shows that this is a mean.

□

Now, from the results given above, we can deduce corresponding results for the generalized power mean of Aczél's type, $\mathcal{Q}_r(g,A)$, of $g$ with respect to the positive linear functional $A$, which is defined for $r\in\mathbb{R}$ by

$$\mathcal{Q}_r(g,A)=\begin{cases}\left(\frac{ua^r-A(wg^r)}{u-A(w)}\right)^{\frac{1}{r}}, & r\neq 0\\ \\ \exp\left(\frac{u\log a-A(w\log g)}{u-A(w)}\right), & r=0,\end{cases} \tag{46}$$

where $g(t)>0$ for $t\in E$, $w\log g\in L$ and $wg^r\in L$ for $r\in\mathbb{R}\setminus\{0\}$.

**Corollary 6.** *Let $g:E\to\mathbb{R}$ be such that the image of $g$ is a compact real interval $I$. Let $r,l,s\in\mathbb{R}\setminus\{0\}$, $r\neq l,s$; $l\neq s$, such that $wg^r, wg^l, wg^s, wg^{2s}\in L$ and $\left(\frac{ua^s-A(wg^s)}{u-A(w)}\right)^2-\frac{ua^{2s}-A(wg^{2s})}{u-A(w)}\neq 0$. Then*

$$\frac{\mathcal{Q}_s^r(g,A)-\mathcal{Q}_r^r(g,A)}{\mathcal{Q}_s^l(g,A)-\mathcal{Q}_l^l(g,A)}=\frac{r(r-s)}{l(l-s)}\cdot\eta^{r-l} \tag{47}$$

*holds for some $\eta$ in the image of $g$, provided that the denominator of the left-hand side is non-zero.*

**Proof.** If we set

$$\alpha(t)=t^r\ ,\ \beta(t)=t^l\ ,\ \gamma(t)=t^s\ ,$$

in Theorem 5.12, we get the assertion (47). □

Since $\eta$ is in the image of $g$, (47) suggests a new mean as it is

$$\min_{t\in E} g(t) \le \left(\frac{l(l-s)}{r(r-s)}\,\frac{\mathcal{Q}_r^r(g,A)-\mathcal{Q}_s^r(g,A)}{\mathcal{Q}_l^l(g,A)-\mathcal{Q}_s^l(g,A)}\right)^{\frac{1}{r-l}} \le \max_{t\in E} g(t)$$

for $r,l,s\in\mathbb{R}\setminus\{0\}$, $r\neq l,s$; $l\neq s$; $\left(\frac{ua^s-A(wg^s)}{u-A(w)}\right)^2-\frac{ua^{2s}-A(wg^{2s})}{u-A(w)}\neq 0$.

From (47) it follows that we can define a new mean $\mathrm{N}_{r,l}^{[s]}(g,A)$ as follows

$$\mathrm{N}_{r,l}^{[s]}(g,A)=\left(\frac{l(l-s)}{r(r-s)}\,\frac{\mathcal{Q}_s^r(g,A)-\mathcal{Q}_r^r(g,A)}{\mathcal{Q}_s^l(g,A)-\mathcal{Q}_l^l(g,A)}\right)^{\frac{1}{r-l}},$$

for $r,l,s\in\mathbb{R}\setminus\{0\}$, $r\neq l,s$; $l\neq s$; $\left(\frac{ua^s-A(wg^s)}{u-A(w)}\right)^2-\frac{ua^{2s}-A(wg^{2s})}{u-A(w)}\neq 0$.

Similarly, we can calculate some other cases for $r,l,s\in\mathbb{R}$ and we get the following definition of $\mathrm{N}_{r,l}^{[s]}(g,A)$.

**Definition 5.13.** Let $L$ satisfy properties $L1, L2$ on a nonempty set $E$ and let $A$ be a positive linear functional on $L$. Let $w\in L$ with $w\geq 0$ on $E$, $0<A(w)<u\in\mathbb{R}$ and $a\in\mathbb{R}$. Let $r,l,s\in\mathbb{R}$ and let $g:E\to\mathbb{R}$ be such that the image of $g$ is a compact real interval $I\subset\mathbb{R}^+$ ($a\in I$), and $wg, wg^r, wg^l, wg^s, wg^{2s}, w\log g, w\log^2 g\in L$ for $r,l,s\in\mathbb{R}\setminus\{0\}$.

Then we define the Cauchy-type mean $\mathrm{N}_{r,l}^{[s]}(g,A)$ of $g$ with respect to the positive linear functional $A$ by

$$\mathrm{N}_{r,l}^{[s]}(g,A)=\left(\frac{l(l-s)}{r(r-s)}\cdot\frac{\mathcal{Q}_s^r(g,A)-\mathcal{Q}_r^r(g,A)}{\mathcal{Q}_s^l(g,A)-\mathcal{Q}_l^l(g,A)}\right)^{\frac{1}{r-l}},$$

for $r,l,s\neq 0$; $r\neq l,s$; $l\neq s$; $\left(\frac{ua^s-A(wg^s)}{u-A(w)}\right)^2-\frac{ua^{2s}-A(wg^{2s})}{u-A(w)}\neq 0$.

$$\mathrm{N}_{r,0}^{[s]}(g,A)=\mathrm{N}_{0,r}^{[s]}(g,A)=\left(-\frac{s}{r(r-s)}\cdot\frac{\mathcal{Q}_s^r(g,A)-\mathcal{Q}_r^r(g,A)}{\log(\mathcal{Q}_s(g,A))-\log(\mathcal{Q}_0(g,A))}\right)^{\frac{1}{r}},$$

for $r,s\neq 0$; $r\neq s$; $\left(\frac{ua^s-A(wg^s)}{u-A(w)}\right)^2-\frac{ua^{2s}-A(wg^{2s})}{u-A(w)}\neq 0$.

$$\mathrm{N}_{r,l}^{[0]}(g,A)=\left(\frac{l^2}{r^2}\cdot\frac{\mathcal{Q}_0^r(g,A)-\mathcal{Q}_r^r(g,A)}{\mathcal{Q}_0^l(g,A)-\mathcal{Q}_l^l(g,A)}\right)^{\frac{1}{r-l}},$$

for $r,l\neq 0$; $r\neq l$; $\left(\frac{u\log a-A(w\log g)}{u-A(w)}\right)^2-\frac{u\log^2 a-A(w\log^2 g)}{u-A(w)}\neq 0$;

where we suppose that all expressions are well defined.

**Theorem 5.14.** *Let $a, b, c, d, s \in \mathbb{R}$ be such that $a \le c$, $b \le d$, $a \ne b$, $c \ne d$ and $s \ne a, b, c, d$.*

*Then, for* $\mathrm{N}_{r,l}^{[s]}$ *defined in* Definition 5.13 we have that

$$\mathrm{N}_{b,a}^{[s]} \le \mathrm{N}_{d,c}^{[s]}. \tag{48}$$

**Proof.** The proof is similar to the proof of Theorem 4.6, using adequate results from [2]. □

### 5.3. Means of the Aczél type for discrete case

Let $x = (x_1, \dots, x_n) \in I^n \subset \mathbb{R}^n$ and $p = (p_1, \dots, p_n)$ be the real $n$-tuples such that

$$p_1 > 0,\ p_2, \dots, p_n \le 0,\ \ P_n > 0,$$

where $P_n = \sum_{i=1}^n p_i$ and $\frac{1}{P_n} \sum_{i=1}^n p_i x_i \in I$.

Setting in Theorem 5.9 that $E = \{2, 3, \dots, n\}$, $u = p_1$, $a = x_1$, $w(i) = -p_i$ and $g(i) = x_i$ for $i \in E$, $A(w) = \sum_{i \in E} w(i)$, then for continuous convex function $f : I \to \mathbb{R}$ we get the reverse Jensen inequality

$$f\left(\frac{1}{P_n} \sum_{i=1}^n p_i x_i\right) \ge \frac{1}{P_n} \sum_{i=1}^n p_i f(x_i). \tag{49}$$

Now we can define similar results as in the previous subsection, and also the new means of the Cauchy type. Firstly, let us define the adequate power means.

**Definition 5.15.** Let $r \in \mathbb{R}$ and let $x \in \mathbb{R}_+^n$ and $p$ be the real $n$-tuples such that

$$p_1 > 0,\ p_2, \dots, p_n \le 0,\ \ P_n > 0,$$

where $\sum_{i=1}^n p_i x_i^r > 0$ for $r \in \mathbb{R} \setminus \{0\}$.

Then the power means of order $r \in \mathbb{R}$ are defined by

$$\bar{M}_r(x, p) = \begin{cases} \left(\frac{1}{P_n} \sum_{i=1}^n p_i x_i^r\right)^{\frac{1}{r}}, & r \ne 0; \\ \left(\prod_{i=1}^n x_i^{p_i}\right)^{\frac{1}{P_n}}, & r = 0. \end{cases} \tag{50}$$

The definition of the means of the Cauchy type follows.

**Definition 5.16.** Let $r, l, s \in \mathbb{R}$. Let $x = (x_1, \dots, x_n)$ be a positive real $n-$ tuple such that not all $x_i$ are equal and let $p$ be a real $n-$tuple such that

$$p_1 > 0,\ p_2, \dots, p_n \le 0,\ \ P_n > 0,$$

where $\sum_{i=1}^n p_i x_i^r, \sum_{i=1}^n p_i x_i^l, \sum_{i=1}^n p_i x_i^s > 0$ for $r, l, s \in \mathbb{R} \setminus \{0\}$.

Then we define the Cauchy-type mean $\mathbf{M}_{r,l}^{[s]}(x,p)$ in which we suppose that all expressions are well defined.

$$\begin{aligned}
&\mathbf{M}_{r,l}^{[s]}(x,p)=\left(\frac{l(l-s)}{r(r-s)}\frac{\bar{M}_r^r(x,p)-\bar{M}_s^r(x,p)}{\bar{M}_l^l(x,p)-\bar{M}_s^l(x,p)}\right)^{\frac{1}{r-l}}, && r,l,s\neq 0; r\neq l,s; l\neq s;\\
&\mathbf{M}_{r,0}^{[s]}(x,p)=\mathbf{M}_{0,r}^{[s]}(x,p)=\left(\frac{s[\bar{M}_r^r(x,p)-\bar{M}_s^r(x,p)]}{r(r-s)[\log\bar{M}_s(x,p)-\log\bar{M}_0(x,p)]}\right)^{\frac{1}{r}}, && r,s\neq 0; r\neq s;\\
&\mathbf{M}_{s,l}^{[s]}(x,p)=\mathbf{M}_{l,s}^{[s]}(x,p)=\left(\frac{l(l-s)}{s}\frac{\sum_{i=1}^n p_i x_i^s\log x_i-\bar{M}_s^s(x,p)\log\bar{M}_s(x,p)}{\bar{M}_l^l(x,p)-\bar{M}_s^l(x,p)}\right)^{\frac{1}{s-l}}, && l,s\neq 0; l\neq s;\\
&\mathbf{M}_{s,0}^{[s]}(x,p)=\mathbf{M}_{0,s}^{[s]}(x,p)=\left(\frac{\sum_{i=1}^n p_i x_i^s\log x_i-\bar{M}_s^s(x,p)\log\bar{M}_s(x,p)}{\log\bar{M}_s(x,p)-\log\bar{M}_0(x,p)}\right)^{\frac{1}{s}}, && s\neq 0;\\
&\mathbf{M}_{r,l}^{[0]}(x,p)=\left(\frac{l^2(\bar{M}_r^r(x,p)-\bar{M}_0^r(x,p))}{r^2(\bar{M}_l^l(x,p)-\bar{M}_0^l(x,p))}\right)^{\frac{1}{r-l}}, && l,r\neq 0; r\neq l;\\
&\mathbf{M}_{r,0}^{[0]}(x,p)=\mathbf{M}_{0,r}^{[0]}(x,p)=\left(\frac{2[\bar{M}_r^r(x,p)-\bar{M}_0^r(x,p)]}{r^2[\bar{M}_2^2(\log x,p)-\bar{M}_1^2(\log x,p)]}\right)^{\frac{1}{r}}, && r\neq 0.\\
&\mathbf{M}_{r,r}^{[s]}(x,p)=\exp\left(-\frac{2r-s}{r(r-s)}+\frac{\sum_{i=1}^n p_i x_i^r\log x_i-\bar{M}_s^r(x,p)\log\bar{M}_s(x,p)}{\bar{M}_r^r(x,p)-\bar{M}_s^r(x,p)}\right), && r,s\neq 0; r\neq s;\\
&\mathbf{M}_{r,r}^{[0]}(x,p)=\exp\left(-\frac{2}{r}+\frac{\sum_{i=1}^n p_i x_i^r\log x_i-\bar{M}_0^r(x,p)\log\bar{M}_0(x,p)}{\bar{M}_r^r(x,p)-\bar{M}_0^r(x,p)}\right), && r\neq 0;\\
&\mathbf{M}_{0,0}^{[0]}(x,p)=\exp\left(\frac{1}{3}\frac{\sum_{i=1}^n p_i(\log x_i)^3-(\log\bar{M}_0(x,p))^3}{\sum_{i=1}^n p_i(\log x_i)^2-(\log\bar{M}_0(x,p))^2}\right);\\
&\mathbf{M}_{s,s}^{[s]}(x,p)=\exp\left(-\frac{1}{s}+\frac{\sum_{i=1}^n p_i x_i^s(\log x_i)^2-\bar{M}_s^s(x,p)(\log\bar{M}_s(x,p))^2}{2(\sum_{i=1}^n p_i x_i^s\log x_i-(\bar{M}_s^s(x,p)\log\bar{M}_s(x,p)))}\right), && s\neq 0;\\
&\mathbf{M}_{0,0}^{[s]}(x,p)=\exp\left(\frac{1}{s}+\frac{\sum_{i=1}^n p_i(\log x_i)^2-(\log\bar{M}_s(x,p))^2}{2(\sum_{i=1}^n p_i\log x_i-\log\bar{M}_s(x,p)))}\right), && s\neq 0.
\end{aligned}\tag{51}$$

**Theorem 5.17.** *Let $a,b,c,d,s\in\mathbb{R}$ be such that $a\leq c$, $b\leq d$. Then, for $\mathbf{M}_{r,l}^{[s]}$ defined in Definition 5.16 we have that*

$$\mathrm{M}_{b,a}^{[s]}\leq\mathrm{M}_{d,c}^{[s]}.\tag{52}$$

**Proof.** The proof is similar to the proof of Theorem 4.6. □

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