TAMKANG JOURNAL OF MATHEMATICS
Volume 42, Number 4, 531-536, Winter 2011
doi:10.5556/j.tkjm.42.2011.531-536
Available online at http://journals.math.tku.edu.tw/

NUMBER OF ZEROS OF A POLYNOMIAL IN A GIVEN DOMAIN

A. EBADIAN, M. BIDKHAM AND M. ESHAGHI GORDJI

Abstract. In this paper, we obtain results concerning the bound for the number of zeros for the polynomial $p(z)$ which generalize earlier well-known result due to Bidkham and Dewan [*On the zeros of a polynomial*, Numerical Methods and Approximation Theory III, Niš (1987), 121–128] and Mohammad [*On the zeros of polynomials*, Amer. Math. Monthly, 72 (1965), 631-633]. We also obtain result for location of zeros of polynomial $p(z)=\sum\limits_{i=0}^{m}\frac{a_i}{(i!)^{\lambda}}z^i+a_nz^n$, $a_n\neq 0$, $0\le m\le n-1$, $\lambda\ge 0$.

1. Introduction and Statement of Results

The problems in the analytic theory of polynomials concerning the number of zeros in a circle have been frequently investigated.

Over many decades, a large number of research papers, e.g., [2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 16, 17, 19, 20] have been published.

For the number of zeros in a circle we have the following result due to Singh [18].

Theorem A. *Let $p(z)=\sum\limits_{i=0}^{n}a_iz^i$ be a polynomial of degree n such that $\min\limits_{0\le i\le n}|a_i|\ge 1$ and $\max\limits_{0\le i\le n-1}|a_i|\le |a_n|$, then*

$$n\left(\frac{R}{k}\right)\le \frac{2\log\{(n+1)|a_n|R^n\}}{\log k}\quad (k>1) \tag{1.1}$$

where $n(r)$ is the number of zeros of $p(z)$ in $|z|\le r$ and

$$R=\max\left\{|\frac{a_{n-1}}{a_n}|,|\frac{a_{n-2}}{a_n}|^{\frac{1}{2}},|\frac{a_{n-3}}{a_n}|^{\frac{1}{3}},\ldots\right\}.$$

The above theorem was later improved by Rahman [15].

For the class of polynomial with real coefficients we have the following theorem due to Mohammad [13].

Corresponding author: A. Ebadian.
2010 *Mathematics Subject Classification*. 26C10, 12D10, 30D15.
Key words and phrases. Polynomials, zeros of polynomials.

Theorem B. *Let* $p(z)=\sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree* n *such that* $a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0 > 0$*, then the number of zeros of* $p(z)$ *in* $|z| \leq \frac{1}{2}$ *does not exceed*

$$1+\frac{1}{\log 2}\log\frac{a_n}{a_0}. \tag{1.2}$$

Bidkham and Dewan [3] generalized the above theorem for different classes of polynomials and proved the following.

Theorem C. *Let* $p(z)=\sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree* n *such that* $a_n \leq a_{n-1} \leq \cdots \leq a_{k+1} \leq a_k \geq a_{k-1} \geq \cdots \geq a_0$ *for some* k, $0 \leq k \leq n$, *then the number of zeros of* $p(z)$ *in* $|z| \leq \frac{1}{2}$ *does not exceed*

$$\frac{1}{\log 2}\left\{\log\frac{|a_n|+|a_0|-a_n-a_0+2a_k}{|a_0|}\right\}. \tag{1.3}$$

Theorem D. *Let* $p(z)=\sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree* n *with complex coefficients. If for some real* β, $|\arg a_i-\beta| \leq \alpha \leq \frac{\pi}{2}$, $0 \leq i \leq n$ *and for some* $0<t\leq 1$,

$$|a_0| \leq t|a_1| \leq \cdots \leq t^k|a_k| \geq t^{k+1}|a_{k+1}| \geq \cdots \geq t^n|a_n|,$$

$0 \leq k \leq n$, *then the number of zeros of* $p(z)$ *in* $|z| \leq \frac{1}{2}$ *does not exceed*

$$\frac{1}{\log 2}\log\left\{\frac{2t^{k+1}|a_k|\cos\alpha+2t\sin\alpha\sum_{i=0}^{n} t^i|a_i|-t^{n+1}|a_n|(\cos\alpha+\sin\alpha-1)}{t|a_0|}\right\}. \tag{1.4}$$

In this paper, we generalize Theorems B,C and D for polynomials with real and complex coefficients. More precisely, we prove.

Theorem 1. *Let* $p(z)=\sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree* n *such that* $a_n \leq a_{n-1} \leq \cdots \leq a_{k+1} \leq a_k \geq a_{k-1} \geq \cdots \geq a_0$ *for some* k, $0 \leq k \leq n$, *then the number of zeros of* $p(z)$ *in* $|z| \leq \frac{R}{2}$ $(R>0)$ *does not exceed*

$$\frac{1}{\log 2}\left\{\log\frac{|a_n|R^{n+1}+|a_0|+R^k(a_k-a_0)+R^n(a_k-a_n)}{|a_0|}\right\} \quad for \quad R\geq 1$$

and

$$\frac{1}{\log 2}\left\{\log\frac{|a_n|R^{n+1}+|a_0|+R(a_k-a_0)+R^n(a_k-a_n)}{|a_0|}\right\} \quad for \quad R\leq 1. \tag{1.5}$$

For $R = 1$, Theorem 1 reduces to inequality (1.3). For $R = 1$, $k = n$ and $a_0 > 0$ it reduces to inequality (1.2).

Theorem 2. *Let* $p(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree* n *with complex coefficients. If for some real* β, $|\arg a_i - \beta| \le \alpha \le \frac{\pi}{2}$, $0 \le i \le n$, *and for some* $R > 0$,

$$|a_0| \le R|a_1| \le \cdots \le R^k|a_k| \ge \cdots \ge R^n|a_n|,$$

$0 \le k \le n$, *then the number of zeros of* $p(z)$ *in* $|z| \le \frac{R}{2} (R > 0)$ *does not exceed*

$$\frac{1}{\log 2}\left\{\log \frac{2R^{k+1}|a_k|\cos\alpha + 2R\sin\alpha \sum_{i=0}^{n} R^i|a_i| - R^{n+1}|a_n|(\cos\alpha + \sin\alpha - 1)}{R|a_0|}\right\}$$

For $R = 1$, $k = n$ and $\alpha = \beta = 0$, the above theorem reduces to inequality (1.2).

Regarding the bound for zeros of the polynomial $p(z) = \sum_{i=0}^{m} \frac{a_i}{(i!)^\lambda} z^i + a_n z^n$, $a_n \ne 0$, $0 \le m \le n-1$, $\lambda \ge 0$, we have able to prove the following.

Theorem 3. *Let* $p(z) = \sum_{i=0}^{m} \frac{a_i}{(i!)^\lambda} z^i + \frac{a_n}{(n!)^\lambda} z^n$, $0 \le m \le n-1$, $\lambda \ge 0$ *be a polynomial of degree* n *with complex coefficients such that* $|a_i| \le (i!)^\lambda |a_n|$, $0 \le i \le m$, *then all zeros of* $p(z)$ *lie in* $|z| < r$, *where* r *is unique positive root of equation*

$$|z|^{n-m} - |z|^{n-m-1} = (n!)^\lambda \tag{1.6}$$

2. Lemmas

The following lemma is well–known Jensen's theorem (see page 208 of [1]).

Lemma 2.1. *Assume that* f *is analytic in a disk* $|z| \le R$, *but not identically zero. Assume also* $f(0) \ne 0$. *Let* f *have zeros* $\{a_k\}$, $k = 1, 2, \ldots, n$ *in* $|z| \le R$, *then*

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|d\theta - \log|f(0)| = \sum_{i=1}^{n} \log\frac{R}{|a_i|}.$$

One get easily from Lemma 2.1, the following.

Lemma 2.2. *If* $f(z)$ *is regular,* $f(0) \ne 0$ *and* $|f(z)| \le M(r)$ *in* $|z| \le r$, *then the number of zeros of* $f(z)$ *in* $|z| \le \frac{r}{2}$ *does not exceed*

$$\frac{1}{\log 2}\left(\log\frac{M(r)}{|f(0)|}\right).$$

Lemma 2.3. *Let* $p(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree* n *such that* $|\arg_i - \beta| \le \alpha \le \frac{\pi}{2}$ *for* $i = 0, 1, \ldots, n$ *for some real* β*, then for some* $t > 0$

$$|ta_i - a_{i-1}| \le |t|a_i| - |a_{i-1}||\cos\alpha + (t|a_i| + |a_{i-1}|)\sin\alpha.$$

The proof is omitted as it follows from the inequality (6) in [4].

3. Proof of the Theorems

Proof of Theorem 1. Consider the polynomial

$$\begin{aligned} g(z) &= (1-z)p(z) \\ &= -a_n z^{n+1} + a_0 + \sum_{i=1}^{n} (a_i - a_{i-1}) z^i. \end{aligned}$$

For $|z| \le R$, we have

$$|g(z)| \le |a_n| R^{n+1} + |a_0| + \sum_{i=1}^{k} (a_i - a_{i-1}) R^i + \sum_{i=k+1}^{n} (a_{i-1} - a_i) R^i$$

which gives

$$|g(z)| \le |a_n| R^{n+1} + |a_0| + R^k (a_k - a_0) + R^n (a_k - a_n) \quad \text{for} \quad R \ge 1$$

and

$$|g(z)| \le |a_n| R^{n+1} + |a_0| + R(a_k - a_0) + R^k (a_k - a_n) \quad \text{for} \quad R \le 1$$

which further imply

$$|\frac{g(z)}{g(0)}| \le \frac{|a_n| R^{n+1} + |a_0| + R^k (a_k - a_0) + R^n (a_k - a_n)}{|a_0|} \quad \text{for} R \ge 1$$

and

$$|\frac{g(z)}{g(0)}| \le \frac{|a_n| R^{n+1} + |a_0| + R(a_k - a_0) + R^k (a_k - a_n)}{|a_0|} \quad \text{for} \quad R \le 1.$$

Applying Lemma 2.2 to $g(z)$, we get the number of zeros of $g(z)$ in $|z| \le \frac{R}{2}$ does not exceed

$$\frac{1}{\log 2} \left\{ \log \frac{|a_n| R^{n+1} + |a_0| + R^k (a_k - a_0) + R^n (a_k - a_n)}{|a_0|} \right\} \quad \text{for} \quad R \ge 1$$

and

$$\frac{1}{\log 2} \left\{ \log \frac{|a_n| R^{n+1} + |a_0| + R(a_k - a_0) + R^k (a_k - a_n)}{|a_0|} \right\} \quad \text{for} \quad R \le 1.$$

As the number of zeros of $p(z)$ in $|z| \le \frac{R}{2}$ does not exceed the number of zeros of $g(z)$ in $|z| \le \frac{R}{2}$, the theorem follows. □

Proof of Theorem 2. Consider

$$F(z) = (R-z)p(z) = -a_n z^{n+1} + Ra_0 + \sum_{i=1}^{n}(Ra_i - a_{i-1})z^i.$$

For $|z| \le R$, we have

$$|F(z)| \le |a_n|R^{n+1} + R|a_0| + \sum_{i=1}^{n}|Ra_i - a_{i-1}|R^i.$$

Using Lemma 2.3, for $|z| \le R$, we get

$$\begin{aligned}
|F(z)| \le &|a_n|R^{n+1} + R|a_0| + \sum_{i=1}^{n}|R|a_i| - |a_{i-1}||R^i\cos\alpha + \sum_{i=1}^{n}(R|a_i| + |a_{i-1}|)R^i\sin\alpha \\
=&|a_n|R^{n+1} + R|a_0| + \sum_{i=1}^{k}(R|a_i| - |a_{i-1}|)R^i\cos\alpha + \sum_{i=k+1}^{n}(|a_{i-1}| - R|a_i|)R^i\cos\alpha \\
&+\sum_{i=1}^{n}(R|a_i| + |a_{i-1}|)R^i\sin\alpha \\
=&2R^{k+1}|a_k|\cos\alpha + 2R\sin\alpha\sum_{i=0}^{n}R^i|a_i| - R|a_0|(\cos\alpha + \sin\alpha - 1) - R^{n+1}|a_n|(\cos\alpha + \sin\alpha - 1) \\
\le&2R^{k+1}|a_k|\cos\alpha + 2R\sin\alpha\sum_{i=0}^{n}R^i|a_i| - R^{n+1}|a_n|(\cos\alpha + \sin\alpha - 1).
\end{aligned}$$

Further, proceeding on the same lines of the proof of Theorem 1, the proof of Theorem 2 can be completed. □

Proof of Theorem 3. For $|z| \le 1$, the conclusion of Theorem 3 is evident. If $|z| > 1$, then

$$\begin{aligned}
|p(z)| &= \left|\frac{a_n}{(n!)^\lambda}z^n + \frac{a_m}{(m!)^\lambda}z^m + \cdots + a_1 z + a_0\right| \\
&\ge \frac{|a_n||z|^n}{(n!)^\lambda} - \left\{\frac{|a_m|}{(m!)^\lambda}|z|^m + \cdots + |a_1||z| + |a_0|\right\} \\
&= |a_n||z^n|\left\{\frac{1}{(n!)^\lambda} - \left[\frac{1}{(m!)^\lambda}\frac{|a_m|}{|a_n|}\frac{1}{|z|^{n-m}} + \cdots + \left|\frac{a_1}{a_n}\right|\frac{1}{|z|^{n-1}} + \frac{|a_0|}{|a_n|}\frac{1}{|z|^n}\right]\right\}
\end{aligned}$$

Now if $|a_i| \le (i!)^\lambda |a_n|$, $i = 0,\dots,m$ is assumed, we conclude

$$\begin{aligned}
|p(z)| &\ge |a_n||z|^n\left\{\frac{1}{(n!)^\lambda} - \left[\frac{1}{|z|^{n-m}} + \cdots + \frac{1}{|z|^n}\right]\right\} \\
&> |a_n||z|^n\left\{\frac{1}{(n!)^\lambda} - \sum_{i=0}^{\infty}\frac{1}{|z|^i} + \left(1 + \frac{1}{|z|} + \cdots + \frac{1}{|z|^{n-m-1}}\right)\right\} \\
&= |a_n||z|^n\left\{\frac{1}{(n!)^\lambda} - \frac{|z|}{|z|-1} + \frac{|z|}{|z|-1} - \frac{\frac{1}{|z|^{n-m}}}{1-\frac{1}{|z|}}\right\}
\end{aligned}$$

$$
\begin{aligned}
&= |a_n||z|^n \left\{ \frac{1}{(n!)^\lambda} - \frac{1}{(|z|-1)|z|^{n-m-1}} \right\} \\
&= \frac{|a_n||z|^n}{(n!)^\lambda |z|^{n-m-1}(|z|-1)} \left\{ (|z|-1)|z|^{n-m-1} - (n!)^\lambda \right\} \\
&= \frac{|a_n||z|^n}{(n!)^\lambda |z|^{n-m-1}(|z|-1)} H(|z|) \\
&\geq 0 \quad \text{for} \quad |z| \geq r.
\end{aligned}
$$

Hereby proving Theorem 3. □

References

[1] Lars V. Ahlfors, Complex Analysis, 3rd ed., McGraw–Hill.

[2] H. Alzer, *Bounds for the zeros of polynomials*, Complex Variables, Theory Appl., **8**(2001), 143–150.

[3] M. Bidkham and K. K. Dewan, *On the zeros of a polynomial*, Numerical Methods and Approximation Theory III, Niš (1987), 121–128.

[4] M. Dehmer, *On the location of zeros of complex polynomials*, Jipam, **4**(2006), 1–13.

[5] M. Dehmer, Schranken fr die Nullstellen komplexwertiger Polynome, Die Wurzel, **8** (2000), 182–185.

[6] N. K. Govil and Q. I. Rahman, *On the Eneström–Kakeya Theoryem*, Tôhoku Math. J., **20**(1968), 126–136.

[7] F. Heigl, Neuere Entwicklungen in der Geometrie der Polynome, Jber. Deutsch. Math.–Verein., **65** (1962/63), 97–142.

[8] P. Henrich, *Applied and computational complex analysis*, Volume I, Wiley Classics Library Edition, (1988).

[9] F. Kittaneh, *Bounds for the zeros of polynomials from matrix inequalities*, Arch. Math., **81**(2003), 601–608.

[10] S. Lipka, *ber die Abgrenzung der Wurzeln von algebraischen Gleichungen*, Acta. Litt. Sci. (Szeged), **3** (1927), 73–80.

[11] M. Marden, *Geometry of polynomials*, Mathematical Surveys. Amer. Math. Soc., Rhode Island, **3** (1966).

[12] Q. G. Mohammad, *Location of the zeros of polynomials*, Amer. Math. Monthly, **74** (1967), 290–292.

[13] Q. G. Mohammad, *On the zeros of polynomials*, Amer. Math. Monthly, **72** (1965), 631–633.

[14] N. Obreschkoff, Verteilung und Berechnung der Nullstellen reeller Polynome, Hochschulbcher fr Mathematik. 55. Berlin, VEB Deutscher Verlag der Wissenschaften, 1963.

[15] Q. I. Rahman, *On the zeros of a class of polynomials*, Proc. Nat. Inst. Sci. India, **22**(1956), 137–139.

[16] Q. I. Rahman and G. Schmeisser, Analytic Theory of Polynomials. Critical Points, Zeros and Extremal Properties, Clarendon Press, 2002.

[17] D. M. Simeunovic', *On the location of the zeros of polynomials*, Math. Morav., **2** (1998), 91–96.

[18] S. K. Singh, *On the zeros of a class of polynomials*, Proc. Nat. Inst. Sci. India, **19**(1953), 601–603.

[19] W. Specht, *Die Lage der Nullstellen eines Polynoms*, Mathematische Nachrichten, **15** (1956), 353–374.

[20] E. C. Westerfield, *A new bound for the zeros of polynomials*, Amer. Math. Monthly, **40** (1933), 18–23. J. Inequal. Pure and Appl. Math., **7**(1) Art. 26, 2006.

Department of Mathematics, Faculty of Sciences, Urmia University, Urmia, Iran.

E-mail: a.ebadian@urmia.ac.ir

Department of Mathematics, Faculty of Natural Sciences, Semnan University, Semnan, Iran.

E-mail: mbidkham@semnan.ac.ir

Department of Mathematics, Faculty of Natural Sciences, Semnan University, Semnan, Iran.

E-mail: majid.eshaghi@gmail.com